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# CONVERGENCE THEOREMS FOR NEWTON-LIKE METHODS USING DATA FROM A SET OR A SINGLE POINT AND OUTER INVERSES

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**Abstract.** Semilocal convergence theorems for Newton-like methods are provided using information from a set or a single point and outer inverses.

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**Key Words and Phrases.** Newton-like methods, Banach space, semilocal convergence theorem, outer inverse.

## 1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^*$ of the equation

$$LF(x) = 0, \tag{1}$$

where $F$ is a Fréchet-differentiable operator defined on an open convex subset $D$ of a Banach space $X$ with values in a Banach space $Y$, $L \in L(X, Y)$ the space of bounded linear operators from $X$ into $Y$.

We consider Newton-like methods of the form

$$x_{n+1} = x_n - A(x_n)^{\#} F(x_n) \quad (n \geq 0) \tag{2}$$

for generating a sequence converging to $x^*$. Here $A(x_n) \in L(X, Y)$ $(n \geq 0)$ is an approximation of the Fréchet-derivative $F'(x_n)$ $(n \geq 0)$ and $A(x_n)^{\#}$ denotes an outer inverse of $A(x_n)$, i.e., $A(x_n)^{\#} A(x_n) A(x_n)^{\#} = A(x_n)^{\#}$. This setting includes generalized Newton methods for undetermined systems [1], [3].

Semilocal convergence theorems for Newton-like methods of the form (2) are provided using information from a set or a single point.

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## 2. Convergence Analysis

Define parameters $r_i$, $i = 0, 1, \dots, 5$, $h_0$ and $\gamma$ by

$$
\begin{aligned}
& r_0 = \frac{1}{\gamma}\left(1 - \sqrt[3]{\frac{2\gamma}{M+L}}\right), \quad M + L \neq 0, \quad r_1 = \frac{1-\ell}{L}, \quad L \neq 0, \\
& r_2 = \frac{1}{\gamma}, \quad \gamma \neq 0, \quad r_3 = \min\{r_0, r_1, r_2\}, \\
& r_3 = r_0 \text{ if } L = \gamma = 0, \quad r_3 = \min\{r_0, r_2\} \text{ if } L = 0, \\
& r_3 = \min\{r_0, r_1\} \text{ if } \gamma = 0, \quad r_4 = \min\{r_3, r^{**}\}, \\
& r_5 = \min\{r_1, r_2\}, \quad r_5 = r_1 \text{ if } \gamma = 0, \\
& r_5 = r_2 \text{ if } L = 0, \quad r_5 = \infty \text{ if } L = \gamma = 0, \\
& h_0 = (M+L)\eta - \frac{1}{2}(1-b)^2, \text{ for } L \geq 0, \ M \geq 0 \\
& \gamma = \sup_{k>1} \left\| \frac{1}{k!} A^{\#} F^{(k)}(x_0) \right\|, \quad x_0 \in X \quad \text{(if it exists)}
\end{aligned}
$$

where $A^{\#}$ is a bounded outer inverse of $A$ $(= A(x_0))$ and functions $f$, $g$, $f_1$, $f_2$ by

$$
\begin{aligned}
f(r) &= \frac{\sigma}{2} r^2 - (1-b)r + \eta, \quad \sigma = \max\{K, M+L\}, \\
g(r) &= Lr + \ell - 1, \\
f_1(r) &= \frac{M+L}{2} r^2 - (1-b)r + \eta, \\
f_2(r) &= \frac{\gamma r^2}{(1-\gamma r)^3} - (1-b)r + \eta,
\end{aligned}
$$

where $r^*$, $r^{**}$ are the zeros of function $f_1$.

Assume:

(C) $r_0 \geq 0$, $h_0 \leq 0$ and $r^* \leq r_3$

Note that under condition (C), $0 \leq r^* \leq r^{**}$, $f_1(r) \geq 0$, $g(r) < 0$ and $\frac{2\gamma}{(1-\gamma r)^3} \equiv K_0 \leq M + L$ on $[0, r^*]$.

or

($C_1$) $r_0 \leq 0$ (or $M + L = 0$) and there exists $r_6 \in (0, r_5)$ or $r_6 \in [r_0, r_5))$ such that $f_2(r_6) \leq 0$.

Denote by $r_7$ the zero of function $f_2$ in $(0, r_6)$. (Such a zero exists by the intermediate value theorem.) Note that under condition ($C_1$), $f_2(r) \geq 0$, $g(r) < 0$ and $M + L \leq K_0$ on $[0, r_6]$.

We can state and prove the following semilocal convergence theorem for Newton-like methods using information from a set and outer inverses.

**Theorem 1.** *Let $F\colon D \subseteq X \to Y$ be twice-Fréchet-differentiable and let $A(x) \in L(X,Y)$ $(x \in D)$ be an approximation of $F'(x)$. Assume:*

(a) *there exists an open convex subset $D_0$ of $D$, $x_0 \in D_0$, a bounded outer inverse $A^{\#}$ of $A$ $(= A(x_0))$ and constants $\eta$, $K > 0$, $M$, $L$, $\mu$, $\ell \geq 0$ such that for al $x \in D_0$*

*the following conditions hold*:

$$\|A^{\#}F(x_0)\| \le \eta, \tag{3}$$

$$\|A^{\#}F''(x)\| \le K, \tag{4}$$

$$\|A^{\#}(F'(x) - A(x))\| \le M\|x - x_0\| + \mu \tag{5}$$

$$\|A^{\#}(A(x) - A)\| \le L\|x - x_0\| + \ell, \tag{6}$$

$$b = \mu + \ell < 1, \tag{7}$$

$$h = \sigma\eta \le \frac{1}{2}(1-b)^2 \tag{8}$$

*and*

$$\overline{U}(x_0, t^*) \subseteq D_0, \tag{9}$$

*where, $t^*$ is the small zero of function $f$.*

*Then,*

(1) *Newton-like method $\{x_n\}$ $(n \ge 0)$ generated by (2) is well defined, remains in $U(x_0, t^*)$ for all $n \ge 0$ and converges to a solution $x^* \in \overline{U}(x_0, t^*)$ of equation $A^{\#}F(x) = 0$, with*

$$A(x_n)^{\#} = (I + A^{\#}(A(x_n) - A))^{-1}A^{\#};$$

(2) *equation $A^{\#}F(x) = 0$ has a unique solution $x^*$ in $\widetilde{U} \cap \{R(A^{\#}) + x_0\}$, where*

$$\widetilde{U} = \begin{cases} \overline{U}(x_0, t^*) \cap D_0, & \text{if } h = \frac{1}{2}(1-b)^2 \\ U(x_0, t^{**}) \cap D_0, & \text{if } h < \frac{1}{2}(1-b)^2, \end{cases}$$

*where $t^{**}$ is the large zero of function $f$ and*

$$R(A^{\#}) + x_0 = \{x + x_0; x \in R(A^{\#})\};$$

(3) *the following error bounds hold for all $n \ge 0$*

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n$$

*and*

$$\|x_n - x^*\| \le t^* - t_n,$$

*where,*

$$t_0 = 0, \quad t_{n+1} = t_n - \frac{f(t_n)}{g(t_n)}.$$

**Proof.** Lipschitz condition (3.2) in [3] follows from (4). The rest of the theorem follows exactly as Theorem 3.1 in [3].

That completes the proof of the theorem.

We can show the following semilocal result for Newton-like methods using information from a set of a single point and outer inverses.

**Theorem 2.** *Let $F: X \to Y$ be analytic on some open convex subset $D$ of $X$ and consider $A(x) \in L(X,Y)$ $(x \in D)$ to be an approximation of $F'(x)$. Assume*:

(a) *there exists an open convex subset $D_0$ of $D$, $x_0 \in D_0$, a bounded outer inverse $A\#$ of $A$ $(= A(x_0))$ and constants $\eta > 0$, $M$, $L$, $\mu$, $\ell \geq 0$ such that for all $x \in D_0$ conditions (3), (5), (6), and (7) hold*;

(b) *conditions (C) (or $(C_1)$) hold*;

(c) $\overline{U}(x_0, r^*) \subseteq D_0$ *(or $\overline{U}(x_0, r_1) \subseteq D_0$).*

*Then,*

(1) *Newton's method $\{x_n\}$ $(n \geq 0)$ generated by (2) is well defined, remains in $U(x_0, r^*)$ (or $U(x_0, r_7)$) for all $n \geq 0$ and converges to a solution $x^* \in \overline{U}(x_0, r^*)$ (or $\overline{U}(x_0, r_7)$) of equation $A^{\#}F(x) = 0$ with $A(x_n)^{\#} = (I + A^{\#}(A(x_n) - A))^{-1}A^{\#}$;*

(2) *the equation $A^{\#}F(x) = 0$ has a unique solution in $\widetilde{U}_1 \cap \{R(A^{\#}) + x_0\}$ (or $\widetilde{U}_2 \cap \{R(A^{\#} + x_0\})$, where*

$$\widetilde{U}_1 = \begin{cases} \overline{U}(x_0, r^*) \cap D_0 & \text{if } h_0 = 0 \\ U(x_0, r_4) \cap D_0 & \text{if } h_0 < 0, \end{cases}$$

$$\widetilde{U}_2 = \begin{cases} \overline{U}(x_0, r_5) \cap D_0 & r_6 < r_5 \\ U(x_0, r_6) \cap D_0 & r_5 = r_6; \end{cases}$$

(3) *the following error bound hold for all $n \geq 0$:*

$$\|x_{n+1} - x_n\| \leq r_{n+1} - r_n \quad (\text{or } r^1_{n+1} - r^1_n), \tag{10}$$

$$\|x_n - x^*\| \leq r^* - r_n \quad (\text{or } r_6 - r^1_n), \tag{11}$$

*where,*

$$r_0 = 0, \quad r_{n+1} = r_n - \frac{f_1(r_n)}{g(r_n)}$$

*and*

$$r^1_0 = 0, \quad r^1_{n+1} = r^1_n - \frac{f_2(r^1_n)}{g(r^1_n)}.$$

**Proof.** Since $\gamma r^* < 1$ under (C) (or $\gamma r_6 < 1$ under ($\mathrm{C}_1$)), we find for all $x \in U(x_0, r^*)$ (or $r_6$) that

$$\begin{aligned} \|A^{\#}F''(x)\| &= \left\| \sum_{n=0}^{\infty} \frac{1}{k!} A^{\#}F^{(k+2)}(x_0)(x - x_0)^k \right\| \\ &\leq \gamma \sum_{k=0}^{\infty} (k+2)(k+2)(\|x - x_0\|)^k \leq K_0. \end{aligned}$$

Estimates (10) and (11) are shown exactly as in Theorem 1 since under condition (C) $r^* \leq r^{**}$, $\frac{2\gamma}{(1-\gamma r)^3} = K_0 \leq M + L$, $f_1(r) \geq 0$, $g(r) < 0$ on $[0, r^*]$ (or under ($\mathrm{C}_1$): $f_2(r) \geq 0$, $g(r) < 0$ and $M + L < K_0$ on $[0, r_6]$).

The rest follows exactly as in Theorem 1.

That completes the proof of the theorem.

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