Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 2, December 2001, pp. 7–18.
Submitted: July 18, 2001. Published: December 31 2001.

STABILITY FOR NONLINEAR 2-D MULTIRESOLUTION: THE APPROACH OF A.HARTEN

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Abstract. In this paper a class of multiresolution representation based on nonlinear prediction in 2-D is presented. In contrast with the wavelet framework these representations cannot be consider as a change of basis, and the stability theory requires different considerations. Some error-control algorithms in order to obtain stability are proposed.

1991 A.M.S. (MOS) Subject Classification Codes. 41A05, 41A15

Key Words and Phrases. Multiscale decomposition, nonlinearity, stability

1. Introduction

Multiresolution representations are a powerful tool for several applications like data compression, numerical simulation of PDE's and statistical estimation. Their best property is to obtain high accuracy approximations of functions by a very small set of coefficients. In wavelet basis decompositions the accuracy is limited by the presence of singularities. Nonlinear multiscale representations are a possible track for such improvements. We will consider the multiresolution framework introduced by Harten (see e.g. [9] [10]). Formally speaking, the multiresolution of Harten has a similar form than the original wavelet representations: a sequence $\bar{f}^L$ of data sampled at the finest resolution level is encoded into $(\bar{f}^0, e^1, \ldots, e^L)$, where the $\bar{f}^0$ corresponds to a sampling at the coarsest resolution and each sequence e^k represents the details which are necessary to recover $\bar{f}^k$ from $\bar{f}^{k-1}$. Nevertheless, in the case of Harten's multiresolution the process which connects two scales can be nonlinear. The nonlinear process allows a better adapted treatment of singularities, in the sense that they do not generate so many large detail coefficients as in standard wavelet transform.

On the other hand, nonlinear process are more difficult to analyze. In particular the stability of the multiresolution is not clear. In practical applications, as we

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said before, a discrete sequence $\bar{f}^L$ is encoded to produce a multiscale representation of its information contents, $(\bar{f}^0, e^1, e^2, \dots, e^L)$; this representation is then processed and the end result of this step is a modified multiscale representation $(\hat{f}^0, \hat{e}^1, \hat{e}^2, \dots, \hat{e}^L)$ which is close to the original one, i.e. such that (in some norm)

$$||\hat{f}^0 - \bar{f}^0|| \leq \epsilon_0 \qquad ||\hat{e}^k - e^k|| \leq \epsilon_k \quad 1 \leq k \leq L,$$

where the truncation parameters $\epsilon_0, \epsilon_1, \dots, \epsilon_L$ are chosen according to some criteria specified by the user. After decoding the processed representation, we obtain a discrete set $\hat{f}^L$ which is expected to be close to the original discrete set $\bar{f}^L$. In order for this to be true, some form of stability is needed, i.e. we must require that

$$||\hat{f}^L - \bar{f}^L|| \leq \sigma(\epsilon_0, \epsilon_1, \dots, \epsilon_L)$$

where $\sigma(\cdot, \dots, \cdot)$ satisfies

$$\lim_{\epsilon_l \to 0,\ 0 \leq l \leq L} \sigma(\epsilon_0, \epsilon_1, \dots, \epsilon_L) = 0.$$

The stability analysis for linear prediction processes can be carried out using tools coming from wavelet theory, subdivision schemes and functional analysis (see [10], [4]), however none of these techniques is applicable in general when the prediction process is nonlinear.

In the nonlinear case, stability can be ensured by modifying the encoding algorithm. The idea of a modified-encoding to deal with nonlinear multiresolution schemes is due to Harten; one dimensional algorithms in several settings can be found in [9], [5], [3]. The goal of a modified-encoding procedure is to keep track of the accumulation error in processing the values in the multiscale representation. The aim of this paper is to present two-dimensional multiresolution algorithms that ensure stability in the case of nonlinear prediction processes. We will consider two frameworks: point-values and cell-average. In the case of linear reconstructions the first corresponds with the interpolatory wavelets and the second with some biorthogonal wavelets.

The paper is organized as follows: we recall in section 2 the discrete framework for multiresolution introduced by Harten. We present the point-values and the cell-average (2-D) multiresolutions in sections 3 and 4 respectively. The error-control algorithms are discussed in section 5. Finally we give stability results in section 6.

2. Harten's framework for multiresolution

Harten's framework is based on two fundamental tools: discretization and reconstruction. The discretization operator D_k obtains discrete information from a (non-discrete) signal ($f \in F$ functional space) at a particular resolution level k ($\bar{f}^k \in V^K$). The reconstruction operator R_k, on the other hand, produces an approximation to a signal from its discrete values. This reconstruction can be nonlinear. Then we can connect the different resolution levels by $D_k^{k-1} := D_{k-1}R_k$ (decimation operator) and $P_{k-1}^K := R_{k-1}D_k$ (prediction operator). The basic consistency property that these two operators have to satisfy is

$$D_k^{k-1} P_{k-1}^k = I_{V^{k-1}}$$

where $I_{V^{k-1}}$ is the identity operator in V^{k-1}. In this case, $D_k^{k-1} e^k = 0$ where e^k denotes the prediction error, $e^k = \bar{f}^k - P_{k-1}^k \bar{f}^{k-1}$.

3. Interpolatory multiresolution analysis in 2-D

Let $X^k = \{x_i^k, y_j^k\}_{i,j=0}^{J_k}$, $J_k = 2^k J_0$, J_0 some integer, $x_{2i}^k = x_i^{k-1}$, $y_{2j}^k = y_j^{k-1}$ and $x_{2i-1}^k = (x_i^{k-1} + x_{i-1}^{k-1})/2$, $y_{2j-1}^k = (y_j^{k-1} + y_{j-1}^{k-1})/2$.

We consider

$$D_k : C([0,1] \times [0,1]) \to V^k \qquad \bar{f}_{i,j}^k = (D_k f)_{i,j} = f(x_i^k, y_j^k,), \quad 0 \leq i,j \leq J_k.$$

In this case, $\dim V^k = (J_k+1) \times (J_k+1)$ and the decimation operators D_k^{k-1} are

$$\bar{f}_{i,j}^{k-1} = (D_k^{k-1}\bar{f}^k)_{i,j} = \bar{f}_{2i,2j}^k, \quad i,j = 0,1,\dots,J_{k-1}.$$

Since $\bar{f}_{i,j}^{k-1} = \bar{f}_{2i,2j}^k$, we obtain $N(D_k^{k-1}) = \{v^k \in V^k | v_{2i,2j}^k = 0\}$. Thus, if we denote by e^k the prediction errors, we will need to keep $e_{2i-1,2j-1}^k, e_{2i-1,2j}^k, e_{2i,2j-1}^k$ only.

A reconstruction procedure for this discretization is given by any operator R_k such that

$$R_k : V^k \to C([0,1] \times [0,1]); \qquad D_k R_k \bar{f}^k = \bar{f}^k,$$

which means

$$(R_k \bar{f}^k)(x_i^k, y_j^k) = \bar{f}_{i,j}^k = f(x_i^k, y_j^k).$$

Therefore, R_k should be a continuous function that interpolates the data $\bar{f}^k$ on at the grid points of X^k. Finally the prediction operators are

$$P_{k-1}^k := D_k R_{k-1}$$

Many different approaches have been used to derive the prediction operators, see [2], [6], [7], [8], [11].

The encoding and decoding algorithms take the following form:

$\mu(\bar{f}^L) = M\bar{f}^L$ (Encoding)

$$
\begin{array}{l}
for\ k = L, \dots, 1 \\
\quad for\ i,j = 0, \dots, J_{k-1} \\
\quad\quad \bar{f}_{i,j}^{k-1} = \bar{f}_{2i,2j}^k \\
\quad end \\
\quad for\ i = 1, \dots, J_{k-1} \\
\quad for\ j = 0, \dots, J_{k-1} \\
\quad\quad f_{2i-1,2j}^P = (P_{k-1}^k \bar{f}^{k-1})_{2i-1,2j} \\
\quad\quad e_{2i-1,2j}^k = \bar{f}_{2i-1,2j}^k - f_{2i-1,2j}^P \\
\quad end \\
\quad end \\
\quad for\ i = 0, \dots, J_{k-1} \\
\quad for\ j = 1, \dots, J_{k-1} \\
\quad\quad f_{2i,2j-1}^P = (P_{k-1}^k \bar{f}^{k-1})_{2i,2j-1}
\end{array}
$$

$$
\begin{array}{l}
\qquad\qquad e^k_{2i,2j-1} = \bar{f}^k_{2i,2j-1} - f^P_{2i,2j-1} \\
\qquad end \\
\qquad end \\
\qquad for\ i,j = 1,\ldots, J_{k-1} \\
\qquad\qquad f^P_{2i-1,2j-1} = (P^k_{k-1}\bar{f}^{k-1})_{2i-1,2j-1} \\
\qquad\qquad e^k_{2i-1,2j-1} = \bar{f}^k_{2i-1,2j-1} - f^P_{2i-1,2j-1} \\
\qquad end \\
end \\
M^M \bar{f}^L = \{\bar{f}^0, e^1, \ldots, e^L\}
\end{array}
$$

$\bar{f}^L = M^{-1}\mu(\bar{f}^L)$ (Decoding)

$$
\begin{array}{l}
for\ k = 1, \dots, L \\
\quad for\ i, j = J_{k-1}, \dots, 0 \\
\quad\quad \bar{f}^k_{2i,2j} = \bar{f}^{k-1}_{i,j} \\
\quad end \\
\quad for\ i = J_{k-1}, \dots, 1 \\
\quad for\ j = J_{k-1}, \dots, 0 \\
\quad\quad f^P_{2i-1,2j} = (P^k_{k-1}\bar{f}^{k-1})_{2i-1,2j} \\
\quad\quad \bar{f}^k_{2i-1,2j} = e^k_{2i-1,2j} + f^P_{2i-1,2j} \\
\quad end \\
\quad end \\
\quad for\ i = J_{k-1}, \dots, 0 \\
\quad for\ j = J_{k-1}, \dots, 1 \\
\quad\quad f^P_{2i,2j-1} = (P^k_{k-1}\bar{f}^{k-1})_{2i,2j-1} \\
\quad\quad \bar{f}^k_{2i,2j-1} = e^k_{2i,2j-1} + f^P_{2i,2j-1} \\
\quad end \\
\quad end \\
\quad for\ i, j = J_{k-1}, \dots, 1 \\
\quad\quad f^P_{2i-1,2j-1} = (P^k_{k-1}\bar{f}^{k-1})_{2i-1,2j-1} \\
\quad\quad \bar{f}^k_{2i-1,2j-1} = e^k_{2i-1,2j-1} + f^P_{2i-1,2j-1} \\
\quad end \\
end
\end{array}
$$

4. Cell-Averages multiresolution analysis in 2-D

In this case we have

$$D_k : L^1([0,1]\times[0,1]) \to V^k,$$

$$\bar{f}^k_{i,j} = (D_k f)_{i,j} = \frac{1}{h_k^2}\int_{x^k_{i-1}}^{x^k_i}\int_{y^k_{j-1}}^{y^k_j} f(x,y)dydx, \quad 1 \le i,j \le J_k;$$

where $L^1([0,1]\times[0,1])$ is the space of absolutely integrable functions in $[0,1]\times[0,1]$ and the set of nested grids X^k is as in above section. This analysis turns out to be appropriate for data compression of discontinuous, piecewise smooth signals.

It is sufficient to consider weighted averages $\bar{f}^k_{i,j}$ for $1 \le i,j \le J_k$ since these contain information on f over $[0,1]\times[0,1]$. Thus, V^k is the space of sequences with $J_k \times J_k$ components.

Moreover

$$\bar{f}^{k-1}_{i,j} = (D^{k-1}_k \bar{f}^k)_{i,j} = \frac{1}{4}(\bar{f}^k_{2i-1,2j-1} + \bar{f}^k_{2i-1,2j} + \bar{f}^k_{2i,2j-1} + \bar{f}^k_{2i,2j})$$

$i,j = 1,2,\dots,J_{k-1}$.

On the other hand, since

$$0 = (D^{k-1}_k e^k)_{i,j} = (e^k_{2i-1,2j-1} + e^k_{2i-1,2j} + e^k_{2i,2j-1} + e^k_{2i,2j})/4$$

we will keep $e^k_{2i-1,2j-1}, e^k_{2i-1,2j}, e^k_{2i,2j-1}$ only.

A reconstruction operator for this discretization is any operator R_k satisfying

$$R_k : V_k \to L^1([0,1]\times[0,1]),$$
$$(D_k R_k \bar{f}^k)_{i,j} = \frac{1}{h_k^2}\int_{x_{i-1}^k}^{x_i^k}\int_{y_{j-1}^k}^{y_j^k}(R_k\bar{f}^k)(x,y)dxdy = \bar{f}_{i,j}^k.$$

That is, $R_k\bar{f}^k(x,y)$ has to be a function in $L^1([0,1]\times[0,1])$ whose mean value on the (i,j)-th cell coincides with $\bar{f}^k_{i,j}$, $\forall i,j$, and $P_{k-1}^k := D_k R_{k-1}$.

The multiresolution transform and its inverse are now

$\mu(\bar{f}^L) = M\bar{f}^L$ (Encoding)

$$\begin{array}{l}
for\ k = L,\dots,1\\
\quad for\ i,j = 1,\dots,J_{k-1}\\
\quad\quad \bar{f}^{k-1}_{i,j} = \frac{1}{4}(\bar{f}^k_{2i-1,2j-1} + \bar{f}^k_{2i-1,2j} + \bar{f}^k_{2i,2j-1} + \bar{f}^k_{2i,2j})\\
\quad end\\
\quad for i,j = 1,\dots,J_{k-1}\\
\quad\quad f^P_{2i-1,2j-1} = (P^k_{k-1}\bar{f}^{k-1})_{2i-1,2j-1}\\
\quad\quad e^k_{2i-1,2j-1} = \bar{f}^k_{2i-1,2j-1} - f^P_{2i-1,2j-1}\\
\quad\quad f^P_{2i-1,2j} = (P^k_{k-1}\bar{f}^{k-1})_{2i-1,2j}\\
\quad\quad e^k_{2i-1,2j} = \bar{f}^k_{2i-1,2j} - f^P_{2i-1,2j}\\
\quad\quad f^P_{2i,2j-1} = (P^k_{k-1}\bar{f}^{k-1})_{2i,2j-1}\\
\quad\quad e^k_{2i,2j-1} = \bar{f}^k_{2i,2j-1} - f^P_{2i,2j-1}\\
\quad end\\
end
\end{array}$$

$M^M\bar{f}^L = \{\bar{f}^0, e^1,\dots,e^L\}$

$\bar{f}^L = M^{-1}\mu(\bar{f}^L)$ (Decoding)

$$\begin{array}{l}
for\ k = 1,\dots,L\\
\quad for\ i,j = J_{k-1},\dots,1\\
\quad\quad f^P_{2i-1,2j-1} = (P^k_{k-1}\bar{f}^{k-1})_{2i-1,2j-1}\\
\quad\quad \bar{f}^k_{2i-1,2j-1} = f^P_{2i-1,2j-1} + e^k_{2i-1,2j-1}\\
\quad\quad f^P_{2i-1,2j} = (P^k_{k-1}\bar{f}^{k-1})_{2i-1,2j}\\
\quad\quad \bar{f}^k_{2i-1,2j} = f^P_{2i-1,2j} + e^k_{2i-1,2j}\\
\quad\quad f^P_{2i,2j-1} = (P^k_{k-1}\bar{f}^{k-1})_{2i,2j-1}\\
\quad\quad \bar{f}^k_{2i,2j-1} = f^P_{2i,2j-1} + e^k_{2i,2j-1}\\
\quad\quad \bar{f}^k_{2i,2j} = 4\bar{f}^{k-1}_{i,j} - \bar{f}^k_{2i-1,2j-1} - \bar{f}^k_{2i-1,2j} - \bar{f}^k_{2i,2j-1}\\
\quad end\\
end
\end{array}$$

5. Multiresolution-based compression schemes with error control

Multiresolution representations lead naturally to data-compression algorithms. The simplest data compression procedure is obtained by setting to zero all scale coefficients which fall below a prescribed tolerance. Let us denote

$$(\hat{e}^k)_{i,j} = tr(e^k_{i,j};\epsilon_k)$$

and refer to this operation as truncation. This type of data compression is used primarily to reduce the "dimensionality" of the data. A different strategy, which is used to reduce the digital representation of the data is "quantization", which can be modeled by

$$(\hat{e}^k)_{i,j} = qu(e^k_{i,j}; \epsilon_k) = 2\epsilon_k round\left[\frac{e^k_{i,j}}{2\epsilon_k}\right],$$

where $round[\cdot]$ denotes the integer obtained by rounding. For example, if $|e^k_{i,j}| \leq 256$ and $\epsilon_k = 4$ then we can represent $e^k_{i,j}$ by an integer which is not larger than 32 and commit a maximal error of 4. Observe that if $|e^k_{i,j}| < \epsilon_k \Rightarrow qu(e^k_{i,j}; \epsilon_k) = 0$ and that in both cases

$$|e^k_{i,j} - \hat{e}^k_{i,j}| \leq \epsilon_k.$$

By applying the inverse multiresolution transform to the compressed representation, we obtain $\hat{f}^L = M^{-1}\{\bar{f}^0, \hat{e}^1, \ldots, \hat{e}^L\}$, an approximation to the original signal $\bar{f}^L$. We expect the information contents of $\hat{f}^L$ to be very close to those of the original signal $\bar{f}^L$, and in order for this to be true, the stability of the multiresolution scheme with respect to perturbations is essential. Studying the effect of using $\hat{e}^k_{i,j}$ instead of $e^k_{i,j}$ in the input of M^{-1} is equivalent to studying the effect of a perturbation in the scale coefficients in the outcome of the inverse multiresolution transform.

Given a discrete sequence $\bar{f}^L$ and a tolerance level ϵ for accuracy, our task is to come up with a compressed representation

$$\{\bar{f}^0, \hat{e}^1, \ldots, \hat{e}^L\} \tag{10}$$

such that if $\hat{f}^L = M^{-1}\{\bar{f}^0, \hat{e}^1, \ldots, \hat{e}^L\}$, we have

$$||\bar{f}^L - \hat{f}^L|| \leq C\epsilon \tag{11}$$

for an appropriate norm.

As observed by Harten [9], one possible way to accomplish this goal is to modify the encoding procedure in such a way that the modification allows us to keep track of the cumulative error and truncate accordingly.

In what follows we present a two-dimensional extension of the one dimensional algorithms in [9], [3] and the two dimensional tensor product in [1]. Given a tolerance level ϵ, the outcome of the modified encoding procedure is a compressed representation (10) satisfying (11). This enables us to specify the desired level of accuracy in the decompressed signal. A modified encoding procedure is designed keeping in mind the particular decoding procedure to be used.

We will consider truncation, but the algorithms are identical for another compression process.

Both point-values and cell-averages have been analyzed.

In next section, we will obtain explicit error bound corresponding to these algorithms, and in particular we will obtain their stability.

(Encoding for point-values)

$$
\begin{array}{l}
for\ k = L, \dots, 1 \\
\quad for\ i, j = 0, \dots, J_{k-1} \\
\quad\quad \bar{f}^{k-1}_{i,j} = \bar{f}^k_{2i,2j} \\
\quad end \\
end \\
\\
Set\ \hat{f}^0 = \bar{f}^0 \\
\\
for\ k = 1, \dots, L \\
\\
\quad for\ i = 1, \dots, J_{k-1} \\
\quad for\ j = 0, \dots, J_{k-1} \\
\quad\quad \hat{e}^k_{2i-1,2j} = tr(\bar{f}^k_{2i-1,2j} - (P^k_{k-1}\hat{f}^{k-1})_{2i-1,2j}, \epsilon_k) \\
\quad\quad \hat{f}^k_{2i-1,2j} = (P^k_{k-1}\hat{f}^{k-1})_{2i-1,2j}, +\hat{e}^k_{2i-1,2j} \\
\quad end \\
\quad end \\
\\
\quad for\ i = 0, \dots, J_{k-1} \\
\quad for\ j = 1, \dots, J_{k-1} \\
\quad\quad \hat{e}^k_{2i,2j-1} = tr(\bar{f}^k_{2i,2j-1} - (P^k_{k-1}\hat{f}^{k-1})_{2i,2j-1}, \epsilon_k) \\
\quad\quad \hat{f}^k_{2i,2j-1} = (P^k_{k-1}\hat{f}^{k-1})_{2i,2j-1} + \hat{e}^k_{2i,2j-1} \\
\quad end \\
\quad end \\
\\
\quad for\ i = 1, \dots, J_{k-1} \\
\quad for\ j = 1, \dots, J_{k-1} \\
\quad\quad \hat{e}^k_{2i-1,2j-1} = tr(\bar{f}^k_{2i-1,2j-1} - (P^k_{k-1}\hat{f}^{k-1})_{2i-1,2j-1}, \epsilon_k) \\
\quad\quad \hat{f}^k_{2i-1,2j-1} = (P^k_{k-1}\hat{f}^{k-1})_{2i-1,2j-1} + \hat{e}^k_{2i-1,2j-1} \\
\quad end \\
\quad end \\
\\
\quad for\ i = 0, \dots, J_{k-1} \\
\quad for\ j = 0, \dots, J_{k-1} \\
\quad\quad \hat{f}^k_{2i,2j} = \hat{f}^{k-1}_{i,j} \\
\quad end \\
\quad end \\
\\
end
\end{array}
$$

(Encoding for cell-average)

$$
\begin{array}{l}
for\ k = L, \dots, 1 \\
\quad for\ i, j = 1, \dots, J_{k-1} \\
\quad\quad \bar{f}^{k-1}_{i,j} = \frac{1}{4}(\bar{f}^k_{2i-1,2j-1} + \bar{f}^k_{2i-1,2j} + \bar{f}^k_{2i,2j-1} + \bar{f}^k_{2i,2j}) \\
\quad end \\
end
\end{array}
$$

$Set\ \hat{f}^0 = \bar{f}^0$

$$
\begin{array}{l}
for\ k = 1, \dots, L \\
\\
\quad for\ i = 1, \dots, J_{k-1} \\
\quad for\ j = 1, \dots, J_{k-1} \\
\quad\quad f^P_{2i-1,2j-1} = (P^k_{k-1}\hat{f}^{k-1})_{2i-1,2j-1} \\
\quad\quad \hat{e}^k_{2i-1,2j-1} = tr([\bar{f}^k_{2i-1,2j-1} - f^P_{2i-1,2j-1}] - [\bar{f}^{k-1}_{i,j} - \hat{f}^{k-1}_{i,j}], \epsilon_k) \\
\quad\quad \hat{f}^k_{2i-1,2j-1} = f^P_{2i-1,2j-1} + \hat{e}^k_{2i-1,2j-1} \\
\quad\quad f^P_{2i-1,2j} = (P^k_{k-1} hat f^{k-1})_{2i-1,2j} \\
\quad\quad \hat{e}^k_{2i-1,2j} = tr([\bar{f}^k_{2i-1,2j} - f^P_{2i-1,2j}] - [\bar{f}^{k-1}_{i,j} - \hat{f}^{k-1}_{i,j}], \epsilon_k) \\
\quad\quad \hat{f}^k_{2i-1,2j} = f^P_{2i-1,2j} + \hat{e}^k_{2i-1,2j} \\
\quad\quad f^P_{2i,2j-1} = (P^k_{k-1}\hat{f}^{k-1})_{2i,2j-1} \\
\quad\quad \hat{e}^k_{2i,2j-1} = tr([\bar{f}^k_{2i,2j-1} - f^P_{2i,2j-1}] - [\bar{f}^{k-1}_{i,j} - \hat{f}^{k-1}_{i,j}], \epsilon_k) \\
\quad\quad \hat{f}^k_{2i,2j-1} = f^P_{2i,2j-1} + \hat{e}^k_{2i,2j-1} \\
\quad\quad \hat{f}^k_{2i,2j} = 4\hat{f}^{k-1}_{i,j} - \hat{f}^k_{2i-1,2j-1} - \hat{f}^k_{2i-1,2j} - \hat{f}^k_{2i,2j-1} \\
\quad end \\
\quad end \\
\\
end
\end{array}
$$

6. Stability Analysis

We use the following matrix norms:

$$
\begin{aligned}
||A||_1 &= max_{||x||_1}||Ax||_1 \\
||A||_\infty &= max_{||x||_\infty}||Ax||_\infty \\
||A||_2 &= max_{||x||_2}||Ax||_2
\end{aligned}
$$

The proofs are similar with norm 1 and ∞. Moreover, to derive the l^2 bound, we simply note that for a vector, x, one has $||x||_2 \le ||x||_1$. Thus we are proving the propositions with $p = \infty$.

Proposition 6.1. *Given a discrete sequence $\bar{f}^L$ and a tolerance level ϵ, if the truncation parameters ϵ_k in the modified encoding algorithm (point values) are chosen so that*

$$\epsilon_k = \epsilon$$

then the sequence $\hat{f}^L = M^{-1}\{\bar{f}^0, \hat{e}^1, \dots, \hat{e}^L\}$ satisfies

$$||\bar{f}^L - \hat{f}^L||_p \le \epsilon \tag{12}$$

for $p = \infty, 1$ *and* 2.

Proof

Let us define the cumulative compression error at the k-th level by $E^k_{i,j}$,

$$E^k_{i,j} = \bar{f}^k_{i,j} - \hat{f}^k_{i,j}$$

Notice that

$$|E^k_{2i,2j}| = |\bar{f}^k_{2i,2j} - \hat{f}^k_{2i,2j}| = |E^{k-1}_{i,j}| \leq ||E^{k-1}||_\infty$$

$$\begin{aligned}|E^k_{2i,2j-1}| &= |\bar{f}^k_{2i,2j-1} - \hat{f}^k_{2i,2j-1}| \\ &= |\bar{f}^k_{2i,2j-1} - f^P_{2i,2j-1} - tr(\bar{f}^k_{2i,2j-1} - f^P_{2i,2j-1}, \epsilon_k)| \leq \epsilon_k \\ |E^k_{2i-1,2j}| &= |\bar{f}^k_{2i-1,2j} - \hat{f}^k_{2i-1,2j}| \\ &= |\bar{f}^k_{2i-1,2j} - f^P_{2i-1,2j} - tr(\bar{f}^k_{2i-1,2j} - f^P_{2i-1,2j}, \epsilon_k)| \leq \epsilon_k\end{aligned}$$

$$\begin{aligned}|E^k_{2i-1,2j-1}| &= |\bar{f}^k_{2i-1,2j-1} - \hat{f}^k_{2i-1,2j-1}| \\ &= |\bar{f}^k_{2i-1,2j-1} - f^P_{2i-1,2j-1} - tr(\bar{f}^k_{2i-1,2j-1} - f^P_{2i-1,2j-1}, \epsilon_k)| \leq \epsilon_k\end{aligned}$$

then

$$||E^k||_\infty \leq \max\left(\frac{1}{J_k+1}\sum_{i=0}^{J_k}|\epsilon_k|, \frac{1}{J_k+1}(||E^{k-1}||_\infty + \sum_{i=1}^{J_{k-1}}|\epsilon_k|)\right) \qquad k = 1, \ldots, l$$

$||E^0||_\infty = 0$ which implies

$$||E^L||_\infty \leq \max(\epsilon_L, \frac{1}{J_L+1}||E^{L-1}||_\infty + \frac{J_{L-1}}{J_L+1}\epsilon_L)$$

taking $\epsilon_k = \epsilon$ we obtain $||E^L||_\infty \leq \epsilon$, and hence (12).

□

Proposition 6.2. *Given a discrete sequence* $\bar{f}^L$ *and a tolerance level* ϵ*, if the truncation parameters* ϵ_k *in the modified encoding algorithm (cell-average) are chosen so that*

$$\epsilon_k = \frac{\epsilon}{2} \cdot q^{L-k} \qquad 0 < q < 1$$

then the sequence $\hat{f}^L = M^{-1}\{\bar{f}^0, \hat{d}^1, \ldots, \hat{d}^L\}$ *satisfies*

$$||\bar{f}^L - \hat{f}^L||_p \leq (1-q)^{-1}\epsilon \tag{13}$$

for $p = \infty, 1$ *and* 2.

Proof Let us define the cumulative compression error at the k-th level by $E^k_{i,j}$,

$$E^k_{i,j} = \bar{f}^k_{i,j} - \hat{f}^k_{i,j}$$

and the modified prediction error at the k-th level by $\tilde{e}^k_{i,j}$,

$$\tilde{e}^k_{i,j} = \bar{f}^k_{i,j} - (P^k_{k-1}\hat{f}^{k-1})_{i,j}$$

With this notation we get

$$\begin{aligned} E^k_{2i-1,2j-1} &= \bar{f}^k_{2i-1,2j-1} - \hat{f}^k_{2i-1,2j-1} \\ &= \bar{f}^k_{2i-1,2j-1} - f^P_{2i-1,2j-1} - \hat{e}^k_{2i-1,2j-1} \\ &= \tilde{e}^k_{2i-1,2j-1} - tr(\tilde{e}^k_{2i-1,2j-1} - E^{k-1}_{i,j}, \epsilon_k), \end{aligned} \tag{14}$$

$$\begin{aligned} E^k_{2i-1,2j} &= \bar{f}^k_{2i-1,2j} - \hat{f}^k_{2i-1,2j} \\ &= \bar{f}^k_{2i-1,2j} - f^P_{2i-1,2j} - \hat{e}^k_{2i-1,2j} \\ &= \tilde{e}^k_{2i-1,2j} - tr(\tilde{e}^k_{2i-1,2j} - E^{k-1}_{i,j}, \epsilon_k), \end{aligned} \tag{15}$$

$$\begin{aligned} E^k_{2i,2j-1} &= \bar{f}^k_{2i,2j-1} - \hat{f}^k_{2i,2j-1} \\ &= \bar{f}^k_{2i,2j-1} - f^P_{2i,2j-1} - \hat{e}^k_{2i,2j-1} \\ &= \tilde{e}^k_{2i,2j-1} - tr(\tilde{e}^k_{2i,2j-1} - E^{k-1}_{i,j}, \epsilon_k), \end{aligned} \tag{16}$$

$$\frac{1}{4}(E^k_{2i-1,2j-1} + E^k_{2i-1,2j} + E^k_{2i,2j-1} + E^k_{2i,2j}) = E^{k-1}_{i,j}$$

Let us now examine the two possibilities in (14), (15) and (16),

$$|\tilde{e}^k_{2i-1,2j-1} - E^{k-1}_{i,j}| > \epsilon_k \Rightarrow E^k_{2i-1,2j-1} = E^{k-1}_{i,j},$$

$$|\tilde{e}^k_{2i-1,2j-1} - E^{k-1}_{i,j}| \leq \epsilon_k \Rightarrow E^k_{2i-1,2j-1} = \tilde{e}^k_{2i-1,2j-1}$$

$$|\tilde{e}^k_{2i-1,2j} - E^{k-1}_{i,j}| > \epsilon_k \Rightarrow E^k_{2i-1,2j} = E^{k-1}_{i,j},$$

$$|\tilde{e}^k_{2i-1,2j} - E^{k-1}_{i,j}| \leq \epsilon_k \Rightarrow E^k_{2i-1,2j} = \tilde{e}^k_{2i-1,2j}$$

$$|\tilde{e}^k_{2i,2j-1} - E^{k-1}_{i,j}| > \epsilon_k \Rightarrow E^k_{2i,2j-1} = E^{k-1}_{i,j},$$

$$|\tilde{e}^k_{2i,2j-1} - E^{k-1}_{i,j}| \leq \epsilon_k \Rightarrow E^k_{2i,2j-1} = \tilde{e}^k_{2i,2j-1}$$

We have:

$$\begin{aligned} ||E^k||_\infty = \max_i(\max(\frac{1}{J_k}(\sum_{j=1}^{\frac{J_k}{2}}(|E^k_{2i-1,2j-1}| + |E^k_{2i-1,2j}|), \\ \frac{1}{J_k}\sum_{j=1}^{\frac{J_k}{2}}(|E^k_{2i,2j-1}| + |E^k_{2i,2j}|))) \end{aligned}$$

Moreover:

$$|E^k_{2i-1,2j-1}| + |E^k_{2i-1,2j}| = \max(|E^k_{2i-1,2j-1} + E^k_{2i-1,2j}|, |E^k_{2i-1,2j-1} - E^k_{2i-1,2j}|)$$

$$|E^k_{2i,2j-1}| + |E^k_{2i,2j}| = \max(|E^k_{2i,2j-1} + E^k_{2i,2j}|, |E^k_{2i,2j-1} - E^k_{2i,2j}|)$$

And after straightforward algebraic manipulations we obtain:

$$
\begin{aligned}
||E^k||_\infty &\leq \max(\frac{1}{J_k}(\sum_{j=1}^{\frac{J_k}{2}}(2|E^{k-1}_{i,j}| + 4\epsilon_k))) \\
&\leq ||E^{k-1}||_\infty + 2\epsilon_k
\end{aligned}
$$

then

$$
||E^L||_\infty \leq ||E^{L-1}||_\infty + 2\epsilon_L \leq \ldots \leq 2\sum_{l=1}^{L}\epsilon_l
$$

taking $\epsilon_k = \frac{\epsilon}{2}q^{L-k}$ where $0 < q < 1$ we obtain $\sum_{l=1}^{L}\epsilon_l \leq \epsilon\frac{1}{1-q}$ and hence (13) for $p = \infty$.

□

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