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A note on Chebyshev series solution of the Crank-Gupta equation

Dal, F.

Abstract

A moving boundary problem arising in biomechanical diffusion theory which has previously been investigated by Crank and Gupta (1972) is one of the simplest examples of a moving boundary problem in a partial differential equation; it has been used as a test problem by number of different authors.In this paper ; we show an accurate solution may be obtained by Chebyshev series solution.

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1 INTRODUCTION

The one-dimensional problem of the oxygen diffusion in an absorbing medium is one of the simplest examples of parabolic equation involving moving boundary.So, it has been used as a test problem in development of several new methods for such problems. A recent summary of the methods may be found in Furzeland (1977) , Rubinstein (1977) and comprehensive

Ege University, Department of Mathematics
E-mail: dal@fenfak.ege.edu.tr

book of Crank (1984) and related papers. The mathematical features of such problem can be stated as follows: The frontal velocity is explicitly related to the first spatial derivative of the solution value at the front through a mass or energy balance across the front. Thus, we wish to determine $C(x,t)$; the solution of

$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial x^2} - 1 \tag{1}$$

x in $0 \leq x \leq S(t)$, $t \geq 0$ with the initial conditions

$$C(x,0) = \frac{1}{2}(1-x)^2, \quad 0 < x < 1 \tag{2}$$

$$S(0) = 1 \tag{3}$$

and the boundary conditions

$$\frac{\partial C}{\partial x} = 0 \quad on \quad x = 0 \tag{4}$$

$$C = 0 \quad \text{and} \quad \frac{\partial \mathrm{C}}{\partial \mathrm{x}} = 0 \quad \text{on} \quad \mathrm{x} = \mathrm{S(t)} \tag{5}$$

2 SOLUTION IN CHEBYSHEV SERIES

If the moving boundary conditions were replaced by a fixed boundary at $x = 1$ with the condition $\frac{\partial C}{\partial t} = 0$ on it , we may write the Chebyshev series as a solution of equation (1) as

$$C(x,t) = \sum_{r=0}^{n} a_r(t) T_r^*(x) \tag{6}$$

In order to determine the coefficients , we insert the eq. (6) into eq. (1) to obtain ;

$$\sum_{r=0}^{n} a_r^{\bullet}(t) T_r^*(x) = \sum_{r=0}^{n} a_r^{(2)} T_r^*(x) - 1 \tag{7}$$

where

$$\frac{\partial C}{\partial t} = \sum_{r=0}^{n} a_r^{\bullet}(t) T_r^*(x) \tag{8}$$

$$a_r^{\bullet}(t) = \frac{da_r}{dt} \tag{9}$$

$$\frac{\partial^2 C}{\partial x^2} = \sum_{r=0}^{n} a_r^{(2)}(t) T_r^*(x) \tag{10}$$

$$a_r^{(2)}(t) = 16 \sum_{s=1}^{n} s(r+s)(r+2s) a_{(r+2s)}(t) r = 0,1,...,n \tag{11}$$

and $T_r^*(x)$ is the Chebyshev polinomial defined in the interval $0 \leq x \leq 1$. More clearly, the equation eq. (7) can be written as;

$$\frac{1}{2} a_0^\bullet(t) T_0^*(x) + a_1^\bullet(t) T_1^3 * (x) + \quad ... \quad + a_n^\bullet(t) T_n^*(x) = \tag{12}$$

$$\frac{1}{2} a_0^{(2)}(t) T_0^*(x) + a_1^{(2)}(t) T_1^*(x) + \quad ... \quad + a_n^{(2)}(t) T_n^*(x) - T_0^*(x) \tag{13}$$

and using undefined coefficients for $T_r^*(x)$ $r = 0, 1, ..., n$ Namely ;

$$\begin{aligned} a_0^\bullet(t) &= a_0^{(2)}(t) - 2 \\ a_1^\bullet(t) &= a_1^{(2)}(t) \\ &\vdots \\ a_n^\bullet(t) &= a_n^{(2)}(t) \end{aligned} \tag{14}$$

Using the second terms of eq. (9) , eq. (10) and eq. (11) we get the system for $a_r^\bullet(t)$; $r = 0, 1, ..., n$ i.e:

$$\begin{aligned} \frac{da_0(t)}{dt} &= 16 \sum_{s=1}^{\frac{n-r}{2}} (s)(s)(2s) a_{2s-2} \\ \frac{da_1(t)}{dt} &= 16 \sum_{s=1}^{\frac{n-r-1}{2}} (s)(1+s)(1+2s) a_{1+2s} \\ &\vdots \\ \frac{da_r(t)}{dt} &= 16 \sum_{s=1}^{\frac{n-r}{2}} (s)(r+s)(r+2s) a_{r+2s} \end{aligned} \tag{15}$$

where $r = 0, 1, ..., n$; $r + 2s = n$ and $s = \frac{n-r-1}{2}$ if r is odd $s = \frac{n-r}{2}$ if r is even. This function clearly satisfies the boundary condition

$$\frac{\partial C}{\partial x} = 0 \quad \text{on} \quad \mathrm{x = S(t)} \quad \text{for} \quad \mathrm{t} \geq 0$$

Hence

$$\frac{\partial C}{\partial x}=\sum_{r=1}^{n}\frac{d}{dx}T_r^*(x)a_r(t) \tag{16}$$

$$\frac{d}{dx}T_r^*(x)=(-1)^{(r-1)}2r^2, x=0 \tag{17}$$

Inserting eq. (17) into eq. (16) and using the boundary condition ;

$$\sum_{r=1}^{n}(-1)^{(r-1)}2r^2a_r(t)=0 \tag{18}$$

is obtained and also

$$a_n(t)=\frac{2a_1(t)}{2n^2}-\frac{\sum_{r=2}^{(n-1)}(-1)^r2r^2a_r(t)}{2n^2} \tag{19}$$

is easily written which is the extra condition which we need. Solving system eq. (15) with eq. (19) one can easily find the coefficients $a_0(t), a_1(t), ..., a_n(t)$. Also the function will satisfy the initial condition eq. (2) on $t=0$ provided that

$$a_0(0)=\frac{3}{8}, a_1(0)=-\frac{1}{4}, a_2(0)=\frac{1}{16}, a_4(0)=0, ..., a_n(0)=0. \tag{20}$$

3 POSITION OF MOVING BOUNDARY

In order to determine the location of the moving boundary , $S(t)$, we use transformation $x=\frac{x}{S(t)}$ on to fix the boundary and hence $T_m^*(x)$ can be given as

$$T_m^*(x)=\frac{1}{2}\sum_{k=0}^{m}(-1)^kA2^{2m-2k}\left(\frac{x}{S(t)}\right)^{m-k} \tag{21}$$

where

$$A=\frac{2(2m-k)!}{(2m-2k)!k!}-\frac{(2m-(k+1))!}{(2m-2k-1)!k!} \tag{22}$$

(see, Fox and Parker,1968) Inserting eq. (21) into the series, we find the solution on the moving boundary by

$$C(x,t)=\sum_{r=0}^{n}{}'a_r(t)\frac{1}{2}\sum_{k=0}^{m}(-1)^kA2^{2m-2k}(\frac{x}{S(t)})^{(m-k)} \tag{23}$$

where the coefficients $a_r(t)$'s are the coefficients which are found previously in section 2. Now to find the position of moving boundary , we integrate both side of the differential equation eq. (1) with respect to x over the range for which it is valid.i.e. $0 \leq x \leq S(t)$. Thus we obtain

$$\frac{\partial}{\partial t} \int_0^{S(t)} C dx = -S(t) \tag{24}$$

Inserting equation (23) into eq. (24) we get an ordinary differential equation for the position of moving boundary $S(t)$, with t as the independent variable, i.e.

$$\frac{dS(t)}{dt} = \frac{-S(t)}{f} \tag{25}$$

where $f = \int_0^{S(t)} C dx$

4 NUMERICAL RESULTS AND DISCUSSION

In order to evaluted the numerical solution, in some sense of "good approximation" one requires a means of selecting number of terms to be used in the solution series. To do this we made a algorithm in seek a way that the difference in n th step solution and $n-1$ th step should be less than the required tolerance value i.e. $|C_n(x,t) - C_{n-1}(x,t)| < \varepsilon$ then we stop the calculating higher steps of the solution in series and last stop is taken as "good approximation" solution to the problem. In calculation we have used only eight step of our series to get "good approximation" with in the tolerance of 10^{-4}. The results are shown in the Table 1 and Table 2.There is comparison of concentrations at different times in table 1. As shown in the table 2 there is a small discrepancy in the position of the boundary which shows increment as the critical point is reached. Our calculation give the critical value as $t_c = 0.1998$ and is belived to be correct to the of figures quoted. However the Fourier solution given by Dahmardah and Mayers (1983) obtains useful accuracy with n about 20 and they quotes that the Fourier series convergences rather slowly, for small values of t and is overcomed by using the numerical integration from the positive to, using as a starting condition the approximation given Crank and Gupta (1972) and this manipulation gives very good accuracy for to about 0.01 where the Fourier series now converges rapidly. Also they quot as t increases the boundary moves left so that s decreases the boundary reaches $s = 0$ at a time t_c shortly before $t = 0.2$ and at this pint their solution becomes singular and they investigate the behavior of the solution near this point with a special change of variables. This also effects the accuracy and the effeciency of the Fourier method besides economy. Also the calculation of Fourier coefficients is not possible analytically , therefore the choice of numerical method is very important. Moreover their system of solution gives stiff equations which needs special attention. In our case, no such problem were encountered. We , for example avoided the singularity of the moving boundary at time t_c shortly before $t = 0.2$ with the Chebyshev form of Fourier series with the advange that the terminal discontinuities are eliminated. Besides in the theories of minimax, if the given function is continous in given interval, the minimax polynomial $P_n(x)$ is that polynomial for which the error $e_n(x) = f(x) - p_n(x)$ has equal and successively opposite values at $n+2$ points in the interval and the error gets successively smaller in absulate value with increasing n. Also known that none of the finite approximations to infinite series have had exactly the minimax property that minimum error of the resulting polynomial is smaller than of any other polynomial of the same degree. The truncation of the Chebyshev series at a point where the coefficients are decreasing rapidly will , of course , give a very close approximation to the minimax polynomial of corresponding degree and in practice the effort to improve on this is well rewarded in our case , as it can be be seen from the results. But yet, it is not clear how satisfactory the technique may be when applied to more general moving boundary problems, which is , of course the interest of further research.

References

[1] Crank,J. and R. S. Gupta, 1972, Jour Inst. Math. and Its Appl. 10, 19.

[2] Crank,J. 1984, Free and Moving Boundary problems, Clarendon Press, Oxford.

[3] Dahmardah, H. O. and Mayers, D.F. ,1983, IMA J. Numer. Analysis, 3, 81.

[4] Fox , L. and Parker, I.B. 1968, Chebyshev Polynomials in Numerical Analysis, Tec. Officer, Oxford Universty Press, London.

[5] Furzeland, R. M. 1977, Brunel Universty Math. Report TR/76.

[6] Hansen, E. and Hougaard,P. 1974, Inst. Math. and Its Appl., 13, 385.

[7] Rubinstein, L. I. 1971 , Trans. Math. Monographs 27. Am. Math. Soc., Providence , R.I..

Table 1: Comparison of consentrations at different times

x	Crank and Gupta (1972)	Hansen and Hougaard (1974)	Present
t=0.001			
0.1	0.40500	0.40460	0.40461
0.2	0.32000	0.32000	0.32002
0.3	0.24500	0.24500	0.24500
0.4	0.18000	0.18000	0.17999
0.5	0.12500	0.12500	0.12500
0.6	0.08000	0.08000	0.07999
0.7	0.04500	0.04500	0.04500
0.8	0.02000	0.02000	0.02002
0.9	0.00500	0.00500	0.00499
t=0.01			
0.1	0.36567	0.36507	0.36507
0.2	0.31072	0.30995	0.30995
0.3	0.24372	0.24327	0.24327
0.4	0.17999	0.17980	0.17980
0.5	0.12500	0.12498	0.12497
0.6	0.08000	0.07999	0.07980
0.7	0.04500	0.04500	0.04327
0.8	0.02000	0.02000	0.01990
0.9	0.00500	0.00500	0.00450
t=0.02			
0.1	0.32622	0.32588	0.32588
0.2	0.28718	0.28667	0.28667
0.3	0.23379	0.23327	0.23325
0.4	0.17696	0.17660	0.17645
0.5	0.12437	0.12419	0.12330
0.6	0.07990	0.07984	0.07960
0.7	0.04499	0.04497	0.04317
0.8	0.01999	0.01999	0.01800
0.9	0.00500	0.00500	0.00410

Table 2: Comparison of position of moving boundary

Time	Crank and Gupta (1972)	Dahmardah and Mayers (1983)	Present
0.02	1.00000	0.99999	1.00000
0.1	0.93518	0.93501	0.93020
0.12	0.87885	0.87917	0.84199
0.14	0.79756	0.79894	0.74705
0.16	0.68502	0.68337	0.58839
0.19	0.33873	0.34537	0.31107
0.195	0.16128	0.20652	0.19897
0.196	–	0.16312	0.16687
0.198	–	–	0.00050
0.1989	–	–	0.000003
0.1990	–	–	0.0000028
0.1998	–	–	0.0000004