Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISBN 1083-0464
Issue 2 December 2001, pp. 1 – 6
Submitted: April 4, 2001. Published: December 31, 2001

On σ-Chebyshev Subspaces of Banach Spaces

H. Mazaheri and A. Dehghan Nezhad

Abstract

σ-Chebyshev subspaces of a Banach space X are defined as those in which the set of best approximants of any vector x in X is non-empty and σ-compact. Moreover, it is shown that there exists a proximinal subspace G of X which is σ-Chebyshev, but is not quasi-Chebyshev in X. Also, some other related results are presented.

(1991) A.M.S Subject Classification Codes. 46B25, 46A20, 46A25, 41A65.
Keywords and Phrases. σ-Chebyshev subspace, Quasi-Chebyshev subspace, Reflexive Banach space.

1 Introduction

Let X be a (complex or real) Banach space and let G be a linear subspace of X. A point $y_0 \in G$ is said to be a best approximation for $x \in X$ if

$$\|x - y_0\| = d(x, G) = \inf\{\|x - y\| : y \in G\}.$$

Departments of Mathematics, Yazd University, Yazd, Iran
E-mail : hmazaheri@yazduni.net, anezhad@yazduni.net

If each $x \in X$ has at least one best approximation in G, then G is called a proximinal subspace of X. If each $x \in X$ has a unique best approximation in G, then G is called a Chebyshev subspace of X (see[7,8]). For $x \in X$, put

$$P_G(x) = \{y \in G : \|x - y\| = d(x, G)\},$$

and

$$\hat{G} = P_G^{-1}(0) = \{x \in X : \|x\| = d(x, G)\}.$$

It is clear that $P_G(x)$ is a closed, bounded and convex subset of X. We say that G is a quasi-Chebyshev subspace of X if $P_G(x)$ is a non-empty and compact set in X for every $x \in X$.

In the following we give a definition which extends the definition of quasi-Chebyshev subspace.

1.1. Definition. Let X be a Banach space. A linear subspace G of X is called σ-Chebyshev if $P_G(x)$ is non-empty and σ-compact set in X for every $x \in X$ (see[3,4,5,6]).

It is clear that every quasi-Chebyshev subspace is σ-Chebyshev. In the following we shall give an example in which the converse is not true.

Let X^* be the dual space of the Banach space X. For $0 \neq f \in X^*$, put

$$M_f = \{x \in X : f(x) = \|f\|, \|x\| = 1\}.$$

We say that a Banach space X is quasi-strictly convex if for every $0 \neq f \in X^*$ the set M_f is compact (see [7]). Every k-strictly convex Banach space is quasi-strictly convex ($k = 1, 2, \ldots$).

1.2. Lemma([1]). Let $C_R([0, 1])$ be class of all continuous real functions on $[0, 1]$ with the supremum norm. Let $c_0, c_1 > 0$ and let S consist of all differentiable functions $f : [0, 1] \longrightarrow R$ such that $\|f\| \leq c_0$ and $\|f'\| \leq c_1$ then S is a compact subset of $C_R([0, 1])$.

2 Introduction

Now, we are ready to state and prove our main results.

2.1. lemma. *Let X be a Banach space and let G is a proximinal subspace of X. If $\hat{G}$ is a σ-compact set, then G is a σ-Chebyshev subspace.*

Proof. Let $x \in X \backslash G$. We have

$$P_G(x) = \{y \in G : \|x - y\| = d(x, G)\}$$
$$= \{y \in G : x - y \in \hat{G}\} = G \bigcap (x - \hat{G})$$

.

Since $\hat{G}$ is a σ-Chebyshev set, then $x - \hat{G}$ and hence $P_G(x)$ is σ-compact set.□

2.2.Example. Let $X = G^1_R([0,1])$ be the Banach space of all real-valued differentiable continuously functions on $[0,1]$, and equipped with the supremum norm. Suppose $G = \{g \in G^1_R([0,1]) : g(0) = 0\}$. Then for every $f \in X$, we have $g = f - f(0) \in G$ is a best approximation for f in X. In fact

$\|f - g\| = |f(0)| = |f(0) - h(0)| \leq \|f - h\|$, for all $h \in G$.

Thus G is proximinal subspace of X. Now it is sufficent that prove $\hat{G}$ is a σ-Chebyshev set.

We have

$$\hat{G} = \{f \in C^1([0,1]) : \|f\| = d(f, G)\}$$
$$= \{f \in C^1([0,1]) : \|f\| = |f(0)|\}$$
$$= \bigcup_{n=1}^{\infty} \{f \in C^1([0,1]) : \|f\| = |f(0)|, \|f^{'}\| \leq n\}$$

.

By lemma 1.3, the set $\{f \in C^1([0,1]) : \|f\| = |f(0)|, \|f^{'}\| \leq n\}$ is a compact subset of $C^1([0,1])$. Therefore $\hat{G}$ is σ-Chebyshev subspace of X. Now, we show that G is not quasi-Chebyshev subspace of X.

For that, let $f = 1$ and $g_n(x) = 1 - cosnx(n = 1, 2, ..., x \in [0,1])$. It is clear that $g_n \in P_G(f)$ and $g_n(1/n) = 1 - cos1$ for every $n = 1, 2, ...,$. Since $g_n(0) = 1(n = 1, 2, ...)$ and $g(0) = 1$ for all $g \in G$, it follows that $\{g_n\}_{n \geq 1}$ has not any uniformly convergent subsequence in $G_G(f)$. Hence G is not a quasi-Chebyshev subspace of $C^1_R([0,1])$.

2.3. Theorem. *Let X be a Banach space and let G is a proximinal subspace of X. If M_f is σ-compact for every $0 \neq f \in G^{\perp}$, then G is σ-Chebyshev in X.*

Proof. Let M_f is σ-compact for every $0 \neq f \in G^{\perp}$. It is clear that for each $0 \neq f \in G^{\perp}$ and each $x_0 \in M_f$

$$x_0 - P_G(x_0) = M_f \bigcap (x_0 + G)$$

.

Now, suppose that $x_0 \in X - G$, put $y_0 = x_0/\|x_0\|$. Then by an consequence of the Hahn-Banach theorem, there exists $0 \neq f \in G^{\perp}$ such that $\|f_0\| = f_0(y_0)$, therefore $y_0 \in M_f$. Since M_{f_0} is σ-compact, then $y_0 - P_G(y_0)$ is σ-compact and hence $P_G(x_0) = \|x_0\| P_G(y_0)$ is σ-compact. $\square$

The following example show that the inverse of Theorem 2.3 does not hold in general.

Example 2.4. Let ℓ^1 with the basis $\{e_n\}_{n\geq 1}$. Put $X = \ell^1 \bigoplus_1 \Re$and $G = \{0\} \times \Re$. First, we show that G is a σ-Chebyshev subspace of X. Let $z = (x,y) \in X$ be arbitrary. let $g = (0,y)$. Then $g \in G$and we have

$$\|z - g\|_1 = \|(x,y) - (0,y)\|_1$$

$$= \|(x,0)\|_1 = \|x\|_1 \leq \|x\|_1 + |y - u|$$

$$\leq \|(x,y) - (0,u)\|_1$$

for all $(o,u) \in G$. It follows that $g \in P_G(z)$ and hence $P_G(z)$ is non-empty for every $z \in X$. Therefore, G is a proximinal subspace of X.

It is not difficult to show that for every $z = (x,y) \in X$ we have $P_G(z) = \{(o,y)\}$, hence $P_G(z)$ is σ-compact for each $(x,y) \in X$.

Finally, define $f_0 : X \longrightarrow \mathcal{C}$ by $f_0(x, y) = \sum_{n=1}^{\infty} \mathcal{C}_n$ for all $(x, y) \in X$, where $x = \sum_{n=1}^{\infty} e_n$ and $\mathcal{C}$ is the field of complex numbers. Then we have

$$|f_0(x, y)| = |\sum_{n=1}^{\infty} \mathcal{C}_n| \leq \sum_{n=1}^{\infty} |\mathcal{C}_n| = \|x\|_1 \leq \|x\|_1 + |y| = \|(x, y)\|_1$$

for all $(x, y) \in X$. It follows that $f_0 \in X^*$, $f_0|_G = 0$ and $\|f_0\| = 1$. But, we have

$$\begin{aligned} M_{f_0} &= \{(x, y) \in X : f_0(x, y) = \|f_0\|, \|(x, y)\|_1 = 1\} \\ &= \{(x, y) \in X : \sum_{n=1}^{\infty} \mathcal{C}_n = 1, \|x\|_1 + |y| = 1\} \\ &= \{(x, y) \in X : \|x\|_1 = 1, y = 0\} = \{(x, 0) \in X : \|x\|_1 = 1\} \\ &= S_{\ell^1} \times \{0\} \end{aligned}$$

where $S_{\ell^1} = \{x \in \ell^1 : \|x\|_1 = 1\}$.

Hence M_{f_0} is not σ-compact.

References

1. Bridges, D.S., Foundations of Real and Abstract Analysis, Springer-Verlag, 1997.

2. Conway, J.B., A Coures in Functional Analysis, Springer-Verlag, 1985.

3. Mohebi, H., On Quasi-Chebyshev Subspaces of Banach Spaces, to appear in Journal of Approximation Theory (2000).

4. Mohebi, H., Pseudo-Chebyshev Subspaces in L^1, The Korean Journal of Computational and Applied Mathematics, 7(2000), 465-475.

5. Mohebi, H., Pseudo-Chebyshev Subspaces in Dual Spaces, Journal of Natural Geometry, 17(2000), 93-104.

6. Mohebi, H., Radjavi, H., On Compactness of the Best Approximant Set, Submitted.

7. Singer, I., The Theory of Best Approximation and Functional Analysis, Regional conference series in Applied Mathematics, 13(1974).

8. Singer, I., Best Approximation in Normal Linear Spaces by Elements of Linear Subspaces, Springer-Verlag, 1970.