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About the Existence of the Thermodynamic Limit for some Deterministic Sequences of the Unit Circle

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Abstract We show that in the set $\Omega = \mathbb{R}_+ \times (1, +\infty) \subset \mathbb{R}_+^2$, endowed with the usual Lebesgue measure, for almost all $(h, \lambda) \in \Omega$ the limit $\lim_{n\to+\infty} \frac{1}{n} \ln \big| h(\lambda^n - \lambda^{-n}) \operatorname{mod} [-1/2, 1/2) \big|$ exists and is equal to zero. The result is related to a characterization of relaxation to equilibrium in mixing automorphisms of the two-torus. It is nothing but a curiosity, but maybe you will find it nice.

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1. Introduction.

1. Introduction.
In the analysis of relaxation to equilibrium of mixing automorphisms of the two-torus[1−2] one encounters the following problem. Suppose that the one-torus is parametrized by the unit interval $[-1/2, 1/2)$ and for appropriate constants $h > 0$ and $\lambda > 1$ consider the real sequence

$$x_n = h\big(\lambda^n - \lambda^{-n}\big) \operatorname{mod} [-1/2, 1/2) \qquad \forall\, n \in \mathbb{N}\,. \tag{1}$$

A significant definition of an exponential "relaxation rate" can be given if the so-called "thermodynamic" limit[3]

$$\lim_{n\to+\infty} -\frac{1}{n} \ln |x_n| \tag{2}$$

exists and is equal to zero. Existence of (2) is clearly not obvious, since the x_n's typically wander through the whole interval $[-1/2, 1/2)$ but every so often they visit a small neighborhood of zero, where the logarithm is singular. Actually, not even if one replaces the ordinary limit in (2) with a supremum limit the finiteness of the result is assured.

This short note is devoted to a measure theoretical discussion of the previous problem. One can show that existence to zero of limit (2) occurs almost surely, for almost any choice of the parameters h and λ, with respect to a measure suitably defined.

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2. Results.

Our goal is to prove the statement below.

Theorem 1 *Consider the set* $\Omega = \mathbb{R}_+ \times (1, +\infty) \subset \mathbb{R}_+^2$ *endowed with the usual Lebesgue measure* μ. *Then, for* μ*-almost all* $(h, \lambda) \in \Omega$ *there holds*

$$\lim_{n \to +\infty} \frac{1}{n} \ln \left| h\big(\lambda^n - \lambda^{-n}\big) \bmod [-1/2, 1/2) \right| = 0 \ . \tag{3}$$

This result can be easily deduced by means of standard arguments of measure theory once the following main theorem is proved.

Theorem 2 *Let* $h > 0$ *and* $Q \in \mathbb{N}$, $Q > 1$, *some fixed constants. Consider the set* G *of all* $\lambda \in [1, Q]$ *for which a (possibly* λ*-dependent) real sequence* $(a_n)_{n \in \mathbb{N}}$ *and an integer* $n' \in \mathbb{N}$ *exist such that*

$$\begin{aligned} &(a) \quad a_n > 0 \quad \forall\, n > n'; \\ &(b) \quad a_n \le \left| h\big(\lambda^n - \lambda^{-n}\big) \bmod [-1/2, 1/2) \right| \quad \forall\, n > n'; \\ &(c) \quad \lim_{n \to +\infty} \frac{1}{n} \ln a_n = 0 \end{aligned}$$

Then, if μ *denotes the Lebesgue measure on* $\mathbb{R}$:

(1) *the set* $G \subseteq [1, Q]$ *is actually nonempty;*

(2) G *is* μ*-measurable and its measure holds* $\mu(G) = Q - 1$.

As a consequence, the set $B = [1, Q] \backslash G$ *where conditions* (a), (b), (c) *are not simultaneously satisfied is also* μ*-measurable and of vanishing measure.*

We firstly prove the result by considering values of λ in the interval $[1 + \eta, Q]$, with η small positive number arbitrarily fixed ($\eta < 1/2$). We therefore look for the subset G_η of $\lambda \in [1 + \eta, Q]$ where hypotheses (a), (b), (c) are satisfied by a suitable choice of the sequence $(a_n)_{n \in \mathbb{N}}$ and of the integer $n' \in \mathbb{N}$. The basic idea of the proof is that the μ-measure of G_η turns out to be $Q - 1 - \eta$ even if we confine ourselves to choose the sequence $(a_n)_{n \in \mathbb{N}}$ in the form

$$a_n = \frac{1}{n^2} \quad \forall\, n \in \mathbb{N},$$

which certainly fulfills requirements (a) and (c), and enable us to deal with the *only* condition (b) on λ.

Let us take then $a_n = 1/n^2 \ \forall\, n \in \mathbb{N}$ and an arbitrarily given value of $n \in \mathbb{N}$. Before tackling the real proof, we need some definitions.

Definition 1. We introduce the set $B_n \subseteq [1 + \eta, Q]$

$$B_n = \Big\{ \lambda \in [1 + \eta, Q] \, : \, a_n > \left| h\big(\lambda^n - \lambda^{-n}\big) \bmod [-1/2, 1/2) \right| \Big\}$$

i.e. the set of $\lambda \in [1 + \eta, Q]$ where the condition $a_n \le \left| h\big(\lambda^n - \lambda^{-n}\big) \bmod [-1/2, 1/2) \right|$ is not satisfied for the assigned $n \in \mathbb{N}$.

Notice that B_n is a finite union of intervals because the function $\Phi_n(\lambda) = \lambda^n - \lambda^{-n}$ is strictly increasing in $[+1, +\infty)$ at fixed n. In fact

$$\Phi_n'(\lambda) = n\big(\lambda^{n-1} + \lambda^{-(n+1)}\big) > 0 \quad \forall\, \lambda \in [1, +\infty) .$$

Consequently, B_n is μ-measurable as a finite union of bounded intervals.

Definition 2. We further introduce the set $\hat{B}_{n'} \subseteq [1+\eta, Q]$, $n' \in \mathbb{N}$, given by

$$\hat{B}_{n'} = \Big\{ \lambda \in [1+\eta, Q] \,:\, a_n > \big|h\big(\lambda^n - \lambda^{-n}\big) \bmod [-1/2, 1/2)\big| \,,\ n > n' \Big\} = \bigcup_{n>n'} B_n$$

which is obviously μ-measurable as a countable union of μ-measurable sets.

Definition 3. We finally introduce the "bad" set $B_\eta \subseteq [1+\eta, Q]$

$$B_\eta = \bigcap_{n'=1}^{\infty} \hat{B}_{n'} \ ,$$

where condition (b) is not satisfied — with this particular choice of the sequence $(a_n)_{n\in\mathbb{N}}$.

B_η is also a μ-measurable set, as a countable intersection of μ-measurable sets.

An immediate consequence of the previous definitions is that $[1+\eta, Q] \setminus G_\eta = B_\eta$. Our goal is to prove that $\mu(B_\eta) = 0$. To this end, since $\forall\, n' \in \mathbb{N} B_\eta \subseteq \hat{B}_{n'}$ by definition, it is enough to show that

$$\lim_{n'\to+\infty} \mu(\hat{B}_{n'}) = 0 \ .$$

Therefore, we can confine ourselves to consider values of $n' \in \mathbb{N}$ large enough, and owing to Definition 2 we can also assume values of $n \in \mathbb{N}$ greater that n'. More precisely, we impose the following technical requirements on the size of n' and n. We take $n > n' \in \mathbb{N}$ such that:

$$\begin{array}{ll}
(i) & a_{n'} = \dfrac{1}{n'^2} < \eta \quad \Longrightarrow \quad a_n < \eta \quad \forall\, n > n' \\
(ii) & a_n - \dfrac{h}{(1+\eta)^n} = \dfrac{1}{n^2} - \dfrac{h}{(1+\eta)^n} > 0 \quad \forall\, n > n' \\
(iii) & h\big[(1+\eta)^n - (1+\eta)^{-n}\big] > 3/2 \quad \mathit{and} \quad h\big[Q^n - Q^{-n}\big] > 5/2 \quad \forall\, n > n' \ .
\end{array}$$

Under the previous conditions we can state the lemmas below.

Lemma 1 *The μ-measure of B_n, n as above, admits the upper bound*

$$\mu(B_n) \leq 2\varepsilon_n \left(\frac{1}{h}\right)^{\frac{1}{n}} [h(Q^n - Q^{-n})]^{\frac{1}{n}} \ .$$

where $\varepsilon_n = a_n + \frac{h}{(1+\eta)^n} > 0$.

Proof

We firstly notice that $1+a_n<1+\eta$ by (i); on the other hand, since $\eta<1/2$ by hypothesis, (i) implies $a_n<1/2$, so that all the intervals $(p-a_n, p+a_n)$, $p=2,\dots\lfloor h(Q^n-Q^{-n})\rfloor+1$ are disjoint.

By using (iii) and denoted with I_n the integer set $\{2,3,\dots,\lfloor h(Q^n-Q^{-n})\rfloor+1\}$, we deduce

$$\begin{aligned} B_n \subseteq \Big\{\lambda\in[1+\eta,Q]\,:\,h\big(\lambda^n-\lambda^{-n}\big)\in \textstyle\bigcup_{p=2}^{\lfloor h(Q^n-Q^{-n})\rfloor+1}(p-a_n,p+a_n)\Big\} &= \\ =\Big\{\lambda\in[1+\eta,Q]\,:\,p-a_n<h\big(\lambda^n-\lambda^{-n}\big)<p+a_n\ ,\ p\in I_n\Big\} &= \qquad (4) \\ =\Big\{\lambda\in[1+\eta,Q]\,:\,p-\big(a_n-h\lambda^{-n}\big)<h\lambda^n<p+a_n+h\lambda^{-n}\ ,\ p\in I_n\Big\}&\ . \end{aligned}$$

Now it is clear that $\forall\,\lambda\in[1+\eta,Q]$

$$a_n-\frac{h}{(1+\eta)^n}\le a_n-h\lambda^{-n}<a_n+h\lambda^{-n}\le a_n+\frac{h}{(1+\eta)^n}$$

and by (ii)

$$a_n-\frac{h}{(1+\eta)^n}>0$$

from which we obtain

$$0<a_n-\frac{h}{(1+\eta)^n}\le a_n-h\lambda^{-n}<a_n+h\lambda^{-n}\le a_n+\frac{h}{(1+\eta)^n}\quad\forall\,\lambda\in[1+\eta,Q]\ .$$

By enlarging each covering interval in (4), we are led then to the inclusion

$$B_n\subseteq\left\{\lambda\in[1+\eta,Q]\,:\,p-\left(a_n+\frac{h}{(1+\eta)^n}\right)<h\lambda^n<p+a_n+\frac{h}{(1+\eta)^n}\ ,\ p\in I_n\right\}$$

and recalling the definition of ε_n

$$B_n\subseteq\left\{\lambda\in[1+\eta,Q]\,:\,\left(\frac{p-\varepsilon_n}{h}\right)^{\frac1n}<\lambda<\left(\frac{p+\varepsilon_n}{h}\right)^{\frac1n}\ ,\ p\in I_n\right\}\subseteq$$

$$\subseteq\bigcup_{p=2}^{\lfloor h(Q^n-Q^{-n})\rfloor+1}\left(\left(\frac{p-\varepsilon_n}{h}\right)^{\frac1n},\left(\frac{p+\varepsilon_n}{h}\right)^{\frac1n}\right)\ ,$$

the final set being μ-measurable as a finite union of intervals. Whence

$$\mu(B_n)\le\sum_{p=2}^{\lfloor h(Q^n-Q^{-n})\rfloor+1}\left(\frac1h\right)^{\frac1n}\left[(p+\varepsilon_n)^{\frac1n}-(p-\varepsilon_n)^{\frac1n}\right]\ .$$

Moreover, $\forall\,p=2,3,\dots\lfloor h(Q^n-Q^{-n})\rfloor+1$ Lagrange Mean Value Theorem implies the equalities below

$$(p+\varepsilon_n)^{\frac1n}-(p-\varepsilon_n)^{\frac1n}\ =\ \frac1n\,(p+\xi_p)^{\frac1n-1}\ 2\varepsilon_n$$

for some $\xi_p \in (-\varepsilon_n, \varepsilon_n)$, and since

$$(p+\xi_p)^{\frac{1}{n}-1} = \frac{1}{(p+\xi_p)^{1-\frac{1}{n}}} \leq \frac{1}{(p-1)^{1-\frac{1}{n}}}$$

we conclude that

$$\mu(B_n) \leq \sum_{p=2}^{\lfloor h(Q^n-Q^{-n})\rfloor+1} \left(\frac{1}{h}\right)^{\frac{1}{n}} \frac{1}{n} \frac{1}{(p-1)^{1-\frac{1}{n}}} 2\varepsilon_n = \frac{2\varepsilon_n}{n} \left(\frac{1}{h}\right)^{\frac{1}{n}} \sum_{p=1}^{\lfloor h(Q^n-Q^{-n})\rfloor} \frac{1}{p^{1-\frac{1}{n}}} .$$

As $\left(p^{1-\frac{1}{n}}\right)^{-1}$ is a decreasing function of p, the following upper estimate holds

$$\sum_{p=1}^{\lfloor h(Q^n-Q^{-n})\rfloor} \frac{1}{p^{1-\frac{1}{n}}} \leq \int_0^{\lfloor h(Q^n-Q^{-n})\rfloor} p^{\frac{1}{n}-1}\, dp = \left[np^{\frac{1}{n}}\right]_0^{\lfloor h(Q^n-Q^{-n})\rfloor} = n \lfloor h(Q^n - Q^{-n})\rfloor^{\frac{1}{n}}$$

and finally $\mu(B_n) \leq 2\varepsilon_n \left(\frac{1}{h}\right)^{\frac{1}{n}} [h(Q^n - Q^{-n})]^{\frac{1}{n}}$, which completes the proof. $\square$

Lemma 2 *If $n' > 0$ (satisfying (i), (ii) and (iii)) is sufficienlty large, the μ-measure of $\hat{B}_{n'}$ is bounded by*

$$\mu(\hat{B}_{n'}) \leq 2(Q+\varepsilon) \sum_{n=n'+1}^{\infty} \varepsilon_n$$

for some small $\varepsilon > 0$.

Proof
Because of the identity $\hat{B}_{n'} = \bigcup_{n>n'} B_n$ and using Lemma 1, we have the following estimate

$$\mu(\hat{B}_{n'}) \leq \sum_{n=n'+1}^{\infty} \mu(B_n) \leq 2 \sum_{n=n'+1}^{\infty} \varepsilon_n \left(\frac{1}{h}\right)^{\frac{1}{n}} [h(Q^n - Q^{-n})]^{\frac{1}{n}} .$$

Notice that $\forall\, h > 0, \forall\, Q \in \mathbb{N}, Q > 1$

$$\lim_{n\to+\infty} \left(\frac{1}{h}\right)^{\frac{1}{n}} [h(Q^n - Q^{-n})]^{\frac{1}{n}} = Q .$$

so that for some $\varepsilon > 0$, $\varepsilon \ll Q$, and n' sufficienlty large there holds

$$Q - \varepsilon < \left(\frac{1}{h}\right)^{\frac{1}{n}} [h(Q^n - Q^{-n})]^{\frac{1}{n}} < Q + \varepsilon \quad \forall\, n \in \mathbb{N} , \quad n > n' .$$

Whence for n' as above

$$\mu(\hat{B}_{n'}) \leq 2 \sum_{n=n'+1}^{\infty} \varepsilon_n (Q+\varepsilon) = 2(Q+\varepsilon) \sum_{n=n'+1}^{\infty} \varepsilon_n$$

which is finite, owing to

$$\sum_{n=1}^{\infty} \varepsilon_n \;=\; \sum_{n=1}^{\infty} \left(\frac{1}{n^2} + \frac{h}{(1+\eta)^n} \right) \;=\; \frac{\pi^2}{6} \;+\; \frac{h}{\eta} \; . \qquad \square$$

Lemma 3 *The measure of B_η is zero*

$$\mu(B_\eta) \;=\; 0 \; .$$

Proof
Since $\forall\, n' \in \mathbb{N}$ we have that $B_\eta \subseteq \hat{B}_{n'}$, in particular this will be true for all $n' \in \mathbb{N}$ large enough to satisfy the requirements of the previous lemmas. Thus

$$\mu(B_\eta) \leq \mu(\hat{B}_{n'}) \leq 2(Q+\varepsilon) \sum_{n=n'+1}^{\infty} \varepsilon_n$$

and therefore

$$\mu(B_\eta) \leq \lim_{n' \to +\infty} 2(Q+\varepsilon) \sum_{n=n'+1}^{\infty} \varepsilon_n$$

where the limit is obviously zero, because of $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$. By the nonnegativity of measure we have the result. $\square$

Proof of theorem 2
As a consequence of Lemma 3, the "good" set $G_\eta = [1+\eta, Q] \backslash B_\eta$ of λ-values in $[1+\eta, Q]$ satisfying condition (b) for the particular choice of $(a_n)_{n\in\mathbb{N}}$, $a_n = 1/n^2$, is of course μ-measurable and with Lebesgue measure

$$\mu(G_\eta) \;=\; Q - 1 - \eta - \mu(B_\eta) \;=\; Q - 1 - \eta \; .$$

If we now consider an *arbitrary* choice of the sequence $(a_n)_{n\in\mathbb{N}}$, compatibile again with conditions (a) and (c), the previous set G_η will maybe "grow" by a subset $\tilde{G}_\eta \subseteq [1+\eta, Q] \backslash G_\eta$:

$$G_\eta \quad \longrightarrow \quad G_\eta^o \;=\; G_\eta \cup \tilde{G}_\eta \; .$$

But as $\mu([1+\eta, Q] \backslash G_\eta) = 0$ it follows that $\tilde{G}_\eta$ is also μ-measurable and of vanishing μ-measure. Hence we finally conclude that the *full* set G_η^o, corresponding to arbitrary (a)-(c)-conditioned sequences $(a_n)_{n\in\mathbb{N}}$, is μ-measurable with measure

$$\mu(G_\eta^o) \;=\; Q - 1 - \eta$$

and that the corresponding *full* set $B_\eta^o [1+\eta, Q] \backslash G_\eta^o$ of λ values where condition (b) is not fulfilled for *any* (a)-(c)-conditioned sequence $(a_n)_{n\in\mathbb{N}}$ is in turn μ-measurable with vanishing μ-measure:

$$\mu(B_\eta^o) \;=\; Q - 1 - \eta - \mu(G_\eta^o) \;=\; 0 \; .$$

So far we have proved that B_η is a set of vanishing measure in any closed interval $[1+\eta, Q]$ with $\eta > 0$. Consider now B, G in $[1, Q]$, that is, according to the previous notation

$$B \;=\; B_0 \qquad \textit{and} \qquad G \;=\; G_0 \; .$$

We firstly notice that B and G are both μ-measurable because

$$G = \bigcup_{n'=1}^{\infty} \bigcap_{n>n'} S_n$$

where S_n is the finite union of *subintervals* in $[1, Q]$ (dependent on n), and $B = [1, Q]\backslash G$. Take then

$$B = \Big(B \cap [1, 1+\eta)\Big) \cup \Big(B \cap [1+\eta, Q]\Big)$$

union of disjoint sets, for some fixed $\eta \in (0, 1/2)$. The μ-measurable set $B \cap [1+\eta, Q]$ is the "bad" set in $[1+\eta, Q]$, so that by Lemma 3

$$\mu\big(B \cap [1+\eta, Q]\big) \; = \; 0 \; .$$

As for the μ-measurable set $B \cap [1, 1+\eta)$, we have the identity

$$B \cap [1, 1+\eta) \; = \; B \cap \left(\{1\} \cup \bigcup_{k=1}^{\infty} \left[1 + \frac{\eta}{k+1}, 1 + \frac{\eta}{k}\right)\right) \; = \; B \cap \{1\} \cup \bigcup_{k=1}^{\infty} B \cap \left[1 + \frac{\eta}{k+1}, 1 + \frac{\eta}{k}\right)$$

union of disjoint sets. But the μ-measurable set

$$B \cap \left[1 + \frac{\eta}{k+1}, 1 + \frac{\eta}{k}\right) \; , \quad k \in \mathbb{N}\backslash\{0\} \; ,$$

satisfies

$$B \cap \left[1 + \frac{\eta}{k+1}, 1 + \frac{\eta}{k}\right) \subseteq B \cap \left[1 + \frac{\eta}{k+1}, Q\right]$$

and since $1/2 > \dfrac{\eta}{k+1} > 0$ for any given $k \in \mathbb{N}\backslash\{0\}$, we obtain

$$\mu\left(B \cap \left[1 + \frac{\eta}{k+1}, 1 + \frac{\eta}{k}\right)\right) \; = \; 0 \; .$$

On the other hand there trivially holds $\mu\big(B \cap \{1\}\big) \; = \; 0$, so that

$$\mu\big(B \cap [1, 1+\eta)\big) \; = \; \mu(B \cap \{1\}) \; + \; \sum_{k=1}^{\infty} \mu\left(B \cap \left[1 + \frac{\eta}{k+1}, 1 + \frac{\eta}{k}\right)\right) \; = \; 0 \; .$$

Whence, finally,

$$\mu(B) \; = \; \mu\big(B \cap [1, 1+\eta)\big) \; + \; \mu\big(B \cap [1+\eta, Q)\big) \; = \; 0$$

that is $\mu(B) = 0$ and $\mu(G) = Q - 1$. The proof is complete. $\square$

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