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Vector spaces over function fields Vector spaces over analytic function fields being associated to ordinary differential equations

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Abstract

It is a well-known fact, that the general solution of a linear ordinary differential equation belongs to a function vector space over the real (complex) number field generated by a finite exponential function set. This circumstance allows us to handle the large-time behavior and stability problems easily. However there is a wide class of non-linear differential equations whose solutions belong to some function space being also generated by an arbitrary exponential function set; so that the stability problems can be handled as in the linear case. To discern whether a given autonomous system belongs to such an equation class, sufficient conditions have been stated in a previous article. Now, we generalize those conditions to a wider class of non-linear autonomous systems.

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These generalized conditions imply the existence of an analytic function field K(X) for each smooth vector X, and a vector space V over the function field K(X) for the corresponding equation dx(t))/dt=X(x(t)), such that each solution of this equation belongs to the vector space V. Thus, a vector space V over an analytic function field K(X) is associated to every member of a wide class of autonomous systems of ordinary differential equations. The algebraic and topological structures for V and K(X) are also investigated.

Finally, by means of a useful integral transform we build the general solution for those autonomous systems satisfying our generalized conditions.

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1. Introduction

The large time behavior and stability problems in ordinary differential equations are not difficult to handle in linear autonomous systems, because the solution of a linear equation belongs to a vector space over the real (complex) number field generated by an exponential function set, say the function set $\left\{e^{\lambda_j} \mid j \in A\right\}$, where A stands for any finite index set. However, there is a wide class of non-linear equations each solution of which belongs to a function vector space being also generated by an exponential function set, hence they can be handled as if they were linear. In this case, we will say the non-linear system to be equivalent to some linear equation. Sufficient conditions for equivalence have been stated in [6], but the involved vector spaces are built over the real number field. As we shall show, in $\mathcal{C}^k$-differentiable manifolds whose dimension is larger than one, the results obtained in [6] can be generalized to a wide family of non-linear autonomous systems. To this end, the structure of vector space over function fields is necessary. It is not difficult to see, that for every smooth manifold $(M_n, \mathcal{A}_k)$ and each chart $(U, \varphi) \in \mathcal{A}_k$ one can associate some function field $\Theta[\mathbf{X}, U]$ to each smooth vector $\mathbf{X}$ as follows.

Let $\mathcal{C}^k(U, \mathbb{C})$ stand for the set of all $\mathcal{C}^k$-maps from U into $\mathbb{C}$. It is easy to see, that the collection of all members of $\mathcal{C}^1(U, \mathbb{C})$ being solutions of the equation

$$\mathbf{X}\phi = 0 \tag{1.1}$$

is a commutative ring, because if ϕ_1 and ϕ_2 are solutions of (1.1), then so are both $\phi_1 + \phi_2$ and $\phi_1\phi_2$. Likewise, if for every p lying in some open subset A of $\operatorname{dom}(\phi)$ it is true that $\phi(p) \neq 0$, besides, ϕ satisfies (1.1), then $\dfrac{1}{\phi|_A}$ is again a solution of (1.1); therefore the solutions of (1.1) form a field. However, the condition $\phi(p) \neq 0$ reduces the domain of the inverse of ϕ to the set $A = \{p \in \operatorname{dom}(\phi) \mid \phi(p) \neq 0\}$, but the ordinary sum and product of two functions require to have the same domain. To avoid this inconvenient we state the following conditions.

- **a):** The topology $\mathcal{T}$ for M_n is separated.
- **b):** For every $p \in U$ the vector $\mathbf{X}$ satisfies the relation $\mathbf{X}(p) \neq 0$, besides, if $(X^1, X^2, \cdots X^n)$ are the coordinates of $\mathbf{X}$ with respect to the chart (U, φ), then for every $j \leq n$ the function $X^j : \varphi[U] \to \mathbb{R}$ is analytic.
- **c):** The domain of every member of $\Theta[\mathbf{X}, U]$ belongs to the subset $\mathcal{D}(U)$ of $\wp(U)$ defined by

$$\mathcal{D}(U) = \left\{V \subseteq U \mid \overline{\overset{\circ}{V}} = U\right\} \tag{1.2}$$

accordingly if $\phi \in \Theta[\mathbf{X}, U]$, then the interior of $\operatorname{dom}(\phi)$ is dense in U.

Notation 1.1. *From now on, denote by $\mathcal{C}^k(\mathcal{D}(U),\mathbb{C})$ the set consisting of every $\mathcal{C}^k$-map $\phi : V \to \mathbb{C}$ whose domain V belongs to $\mathcal{D}(U)$.*

Lemma 1.2. *The family $\mathcal{D}(U)$ satisfies the following properties.*

a): *$\mathcal{D}(U)$ contains with every finite collection of sets their intersection, and with every arbitrary collection their union.*
b): *U belongs to $\mathcal{D}(U)$.*
c): *$W \subseteq V \subseteq U$ and $W \subseteq \mathcal{D}(U)$ imply $V \subseteq \mathcal{D}(U)$.*

In section §2.1 we define two composition laws $\dot{+}$ and $\star$ in $\mathcal{C}^k(\mathcal{D}(U),\mathbb{C})$, closely related with the ordinary sum and product, such that in order to apply each of these laws between two maps ϕ_1 and ϕ_2 the domains of ϕ_1 and ϕ_2 need not be the same. To this end, denote by $()^{\smile} : \mathcal{C}^k(\mathcal{D}(U),\mathbb{C}) \to \mathcal{C}^k(\mathcal{D}(U),\mathbb{C})$ the map carrying each function f into its extension by continuity. Now, define the composition $f \dot{+} g$ by $(f|_{\mathrm{dom}(f)\cap\mathrm{dom}(g)} + g|_{\mathrm{dom}(f)\cap\mathrm{dom}(g)})^{\smile}$, and likewise define $f \star g$ by $(f|_{\mathrm{dom}(f)\cap\mathrm{dom}(g)} \cdot g|_{\mathrm{dom}(f)\cap\mathrm{dom}(g)})^{\smile}$. Obviously, to apply both composition laws $\dot{+}$ and $\star$ between f and g the sets $\mathrm{dom}\,(f)$ and $\mathrm{dom}\,(g)$ need not be the same, besides, by virtue of the preceding lemma the intersection $\mathrm{dom}\,(f) \cap \mathrm{dom}\,(g)$ is non-void and it is dense in U. Analogously, define the inverse $f^\flat$ of f by $f^\flat = \left(\dfrac{1}{f|A}\right)^{\smile}$ being $A = \{p \in \mathrm{dom}\,(f) \mid f(p) \neq 0\}$.

By means of $\dot{+}$, $\star$ and $^\flat$ one can build the algebraic field structure for some subset $\Theta[\mathbf{X}, U]$ of $\mathcal{C}^k(\mathcal{D}(U),\mathbb{C})$ the members of which are analytic solutions of the equation (1.1). In Theorem 2.12, we prove that if the coefficients of an algebraic equation belong to $\Theta[\mathbf{X}, U]$, besides, they are rational functions of a fixed $g \in \Theta[\mathbf{X}, U]$, then its solutions are also members of $\Theta[\mathbf{X}, U]$.

In section §2.2, we define compatible topological structures for function fields. To this end, the ordinary functional families cannot be handled in the usual way. This is why we build these compatible topologies by means of partial seminorms. Likewise, in section §3 we briefly describe both algebraic and topological structures for those vector spaces over analytic function fields used in the proofs for the results in the next section.

Section §4 begins with a generalization of the equivalence criterion stated in [6], and consisting of two conditions, namely, Condition 1 and Condition 2, by means of which one can discern whether a given non-linear autonomous system is equivalent to some linear one, that is to say, by means of these conditions one can discern whether the general solution of a given autonomous system belongs to a vector space over a function field being generated by an arbitrary exponential function set $\mathcal{B}$. Both conditions can be summarized as follows:

If $(X^1, X^2 \cdots X^n)$ are the coordinates of the vector $\mathbf{X}$ with respect to the chart (U, φ), then for each $k \leq n$ there are an integration path C and two functions F^k and Φ^k satisfying the relations

$$\sum_{m=1}^{n} \frac{\partial}{\partial x^m}\Phi^k = h^k \in \Theta[\mathbf{X}, U] \tag{1.3}$$

$$\forall p \in U : \qquad X^k(p) = F^k\big(p, \Phi^k(p)\big) \tag{1.4}$$

besides, there are two functions $\mathcal{F}^k, g^k \in \Theta[\mathbf{X}, U]$ such that for some positive number K and every positive integer r it is true that

$$F^k(p,s) = \int_C \mathcal{F}^k(p,z) \star e^{g^k(p,z)s} dz \tag{1.5}$$

$$\left| \int_C \Big(g^k(p,z) \star h^k(p)\Big)^{(r-1)} \star \mathcal{F}^k(p,z) \star e^{g^k(p,z)s} dz \right|^{\vee} \leq Kr! \tag{1.6}$$

In Theorem 4.3 we show that the general solution for each autonomous system

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{X}\left(\mathbf{x}(t)\right) \tag{1.7}$$

whose associated vector $\mathbf{X}$ satisfies the above conditions can be obtained by means of the integral transform

$$x^k_{p_0}(\gamma(t)) = x^k_0 \star 1_U \dotplus \int_C \mathcal{F}^k(p_0,z) \star \Big(g^k(p_0,z) \star h^k(p_0)\Big)^{\flat} \star \\ e^{g^k(p_0,z)\star\Phi^k(p_0)} \star \Big(e^{g^k(p_0,z)\star h^k(p_0)(t-t_0)} \dot{-} 1_U\Big) dz \tag{1.8}$$

where $(x^1_{p_0}, x^2_{p_0} \cdots x^k_{p_0}) = \mathbf{x}_{p_0}$ stands for the solution of (1.7) satisfying the initial condition $\mathbf{x}_{p_0}(t_0) = \varphi(p_0)$ and 1_U denotes the map with domain U being constantly equal to 1.

Obviously, the former transform implies that the solution coordinate $x^k_{p_0}(\gamma(t))$ belongs to a vector space over the analytic function field $\Theta[\mathbf{X}, U]$ being generated by the non-countable exponential function set

$$\mathcal{B} = \left\{ e^{g^k(p_0,z)\star\Phi^k(p_0) \dotplus g^k(p_0,z)\star h^k(p_0)(t-t_0)} \mid z \in C \right\} \tag{1.9}$$

Thus, one can associate a vector space $\mathcal{V}$ over the analytic function field $\Theta[\mathbf{X}, U]$ to the ordinary differential equation (1.7). This fact justifies the title of this article.

To handle stability problems, the integral transform (1.8) is very useful, because if every value of z satisfying $\Re\big(g^k(p_0,z) \star h^k(p_0)\big) > 0$ can be neglected, then each non-periodic part of the solution vanishes when $t \mapsto \infty$; for instance, if $\mathcal{F}^k(p_0,z) \star \big(g^k(p_0,z) \star h^k(p_0)\big)^{\flat} = 0$ whenever $\Re\big(g^k(p_0,z) \star h^k(p_0)\big) > 0$.

In Corollary 4.6 we state a certain relation between those autonomous systems satisfying the above conditions and an associated partial differential equation. In the same section, we introduce the concept of Θ-linear equation, that is to say, systems being linear over the associated function field, which need not be linear over the real (complex) number field. In Theorem 4.12 we show that Θ-linear equations satisfy our generalized equivalence criterion, accordingly they can be solved by means of the integral transform (4.9).

2. Analytic function fields

2.1. **The algebraic structure.** In the set $\mathcal{C}^0(\mathcal{D}(U), \mathbb{C})$ consisting of every continuous map $f : V \to \mathbb{C}$, whose domain V belongs to $\mathcal{D}(U)$, define an ordering $\preceq$ as follows.

Definition 2.1. *For every pair $\phi_1, \phi_2 \in \mathcal{C}^0(\mathcal{D}(U), \mathbb{C})$, $\phi_1 \preceq \phi_2$ if and only if the following conditions hold:*

c1): $\operatorname{dom}(\phi_1) \subseteq \operatorname{dom}(\phi_2)$ *and* $\forall p \in \operatorname{dom}(\phi_1)$: $\phi_1(p) = \phi_2(p)$.

c2): *The map ϕ_2 is the extension by continuity of ϕ_1; i.e., for each p_0 in $\operatorname{dom}(\phi_2)$ there is a sequence $\{p_n \mid n \in \mathbb{N}\}$ in $\operatorname{dom}(\phi_1)$, converging to p_0, such that* $\lim\limits_{p_n \to p_0} \phi_1(p_n) = \phi_2(p_0)$.

It is worth pointing out, that since both sets $\operatorname{dom}(\phi_1)$ and $\operatorname{dom}(\phi_2)$ belong to $\mathcal{D}(U)$, then for every $p_0 \in U$ there is a sequence in $\operatorname{dom}(\phi_1)$ converging to p_0, for the standard topology for $\mathbb{R}^n$ satisfies the first axiom of countability, and U is homeomorphic to some open subset of $\mathbb{R}^n$, besides, $\operatorname{dom}(\phi_1)$ is dense in U.

By means of the ordering $\preceq$, and taking into account the well-known Zorn lemma, one can define a mapping

$$(\;)^{\smile} : \mathcal{C}^0(\mathcal{D}(U), \mathbb{C}) \to \mathcal{C}^0(\mathcal{D}(U), \mathbb{C}) \tag{2.1}$$

carrying each $\phi \in \mathcal{C}^0(\mathcal{D}(U), \mathbb{C})$ into the maximal element $\phi^{\smile}$, such that $\phi \preceq \phi^{\smile}$. In other words, $^{\smile}$ carries each map ϕ into its extension by continuity.

Notation 2.2. *From now on, denote by 0_U the map with domain U being constantly equal to 0; and for every map $\phi \in \mathcal{C}^0(\mathcal{D}(U), \mathbb{C})$ denote by $\operatorname{zero}(\phi)$ the subset of U defined by $\operatorname{zero}(\phi) = \{p \in \operatorname{dom}(\phi) \mid \phi(p) = 0\}$.*

Definition 2.3. *Henceforth, we are concerned with those analytic manifolds whose underlying topology $\mathcal{T}$ is separated. Given such a manifold $(M_n, \mathcal{A})$ and a chart (U, φ), let $\mathcal{C}^\omega(\mathcal{D}(U), \mathbb{C}) \subset \mathcal{C}^0(\mathcal{D}(U), \mathbb{C})$ stand for the subset consisting of each map $\phi \in \mathcal{C}^0(\mathcal{D}(U), \mathbb{C})$ such that the function $\phi \circ \varphi^{-1}$ is analytic; i.e., letting $A = \varphi[\operatorname{dom}(\phi)] \subseteq \mathbb{R}^n$, then both real functions $\Re(\phi \circ \varphi^{-1}) : A \to \mathbb{R}$ and $\Im(\phi \circ \varphi^{-1}) : A \to \mathbb{R}$ are analytic. Given a $\mathcal{C}^\omega$-vector $\mathbf{X}$ satisfying the relation $\mathbf{X}(p) \neq 0$ for every $p \in U$, let $\Theta[\mathbf{X}, U]$ stand for the subset of $\mathcal{C}^\omega(\mathcal{D}(U), \mathbb{C})$ consisting of every function being maximal with respect to $\preceq$, and satisfying the following condition: ϕ is an analytic solution of the equation*

$$\mathbf{X}\phi = 0 \tag{2.2}$$

explicitly

$$\sum_{m=1}^{n} X^m \frac{\partial}{\partial x^m} \widehat{\phi}(x^1, x^2 \cdots x^n) = 0 \tag{2.3}$$

where $\widehat{\phi} : \varphi[\mathrm{dom}\,(\phi)] \to \mathbb{C}$ *is the mapping defined by* $\widehat{\phi} = \phi \circ \varphi^{-1}$*, and* $(x^1, x^2 \cdots x^n)$ *are the coordinates of the homeomorphism* φ *of the considered chart* (U, φ)*. It is worth pointing out, that because it is assumed* $\mathbf{X}$ *to be a* $\mathcal{C}^\omega$*-vector, then the existence of analytic solutions of (2.3) is guaranteed* [1].

Obviously, letting $\mathbf{X}\breve{\ }\phi = (\mathbf{X}\phi)\breve{\ }$ *the equation (2.2) can be written as follows,*

$$\mathbf{X}\breve{\ }\phi = 0_U \tag{2.4}$$

In the function set $\mathcal{C}^0(\mathcal{D}(U), \mathbb{C})$ define the composition laws $\dot{+}$ and $\star$ by means of the following statements: $\forall \phi_1, \phi_2 \in \mathcal{C}^0(\mathcal{D}(U), \mathbb{C})$,

1): $\phi_1 \dot{+} \phi_2 = (\phi_1|_W + \phi_2|_W)\breve{\ }$.
2): $\phi_1 \star \phi_2 = (\phi_1|_W \cdot \phi_2|_W)\breve{\ }$.

where $W = \mathrm{dom}\,(\phi_1) \cap \mathrm{dom}\,(\phi_2)$. Since for every $\mathcal{C}^\omega$-vector $\mathbf{X}$ the set $\Theta[\mathbf{X}, U]$ belongs to $\mathcal{C}^0(\mathcal{D}(U), \mathbb{C})$, then both laws $\dot{+}$ and $\star$ are also defined for $\Theta[\mathbf{X}, U]$.

Now, let $()^\flat : (\Theta[\mathbf{X}, U] - \{0_U\}) \to (\Theta[\mathbf{X}, U] - \{0_U\})$ stand for the map carrying each $\phi \in (\Theta[\mathbf{X}, U] - \{0_U\})$ into $\phi^\flat = \left(\dfrac{1}{\phi|_V}\right)\breve{\ }$, where

$$V = \{p \in \mathrm{dom}\,(\phi) \mid \phi(p) \neq 0\} = \mathrm{dom}\,(\phi) - \mathrm{zero}\,(\phi)$$

It is worth pointing out, that according to the definition of the mapping $^\flat$, for every $\phi \in \Theta[\mathbf{X}, U]$ the image under $^\flat$ belongs to $\Theta[\mathbf{X}, U]$, but Definition 2.3 requires that $\mathrm{dom}\left(\phi^\flat\right) \in \mathcal{D}(U)$, and this relation cannot be satisfied whenever the interior of zero (ϕ) does not vanish. As the Lemma 2.7 shows, the required analyticity for the members of $\Theta[\mathbf{X}, U]$ implies $(\mathrm{zero}\,(\phi))^\circ = \emptyset$.

Remark 2.4. *For one-dimensional differentiable manifolds the system 2.3 consists of the only equation* $X^1 \frac{\partial}{\partial x^1} \widehat{\phi}(x^1) = 0$*, hence* $\widehat{\phi}$ *is a constant map and* $\Theta[\mathbf{X}, U]$ *can be identified with the complex number field.*

Lemma 2.5. *For every couple* $\phi_1, \phi_2 \in \Theta[\mathbf{X}, U]$ *the following is true,*

a): $\mathrm{dom}\,(\phi_1) \cap \mathrm{dom}\,(\phi_2) \subseteq \mathrm{dom}\,(\phi \dot{+} \phi_2)$.
b): $\mathrm{dom}\,(\phi_1) \cap \mathrm{dom}\,(\phi_2) \subseteq \mathrm{dom}\,(\phi \star \phi_2)$.

besides, for each $\phi \in \Theta[\mathbf{X}, U]$ *the following relation holds:*

$$\mathrm{dom}\,(\phi) - \mathrm{zero}\,(\phi) \subseteq \mathrm{dom}\left(\phi^\flat\right)$$

Proof. It follows immediately from the definitions of the laws $\dot{+}$, $\star$ and $^\flat$. $\square$

Lemma 2.6. *For every vector* $\mathbf{X}$ *each constant map* $\phi : U \to \mathbb{C}$ *belongs to* $\Theta[\mathbf{X}, U]$*, besides, if a map* $\phi : V \subseteq U \to$ *lying in* $\Theta[\mathbf{X}, U]$ *is constantly equal to* $c \in \mathbb{C}$*, then* $V = U$*.*

Proof. On the one hand, every constant map satisfies equation (2.4), besides, if the domain is U, then it satisfies Definition 2.3. On the other hand, let $\phi : V \in \mathcal{D}(U) \to \mathbb{C}$ be a map being constantly equal to $c \in \mathbb{C}$. Since V is dense in U, then for each $p_0 \in U$ there is a sequence $(p_n)_{n\in\mathbb{N}}$ in V which converges to p_0; so that

$$\lim_{n\to\infty} \phi(p_n) = c \tag{2.5}$$

and because ϕ is maximal with respect to $\preceq$, then $\phi(p_0) = c$, therefore $p_0 \in \mathrm{dom}\,(\phi)$ and $\mathrm{dom}\,(\phi) = U$. □

Lemma 2.7. *For every member ϕ of $\Theta[\mathbf{X}, U]$ the following statements hold.*

a): *If there is an open subset V of U such that $\forall p \in V$, $\phi(p) = 0$, then $\phi = 0_U$; consequently, if $\phi \neq 0_U$, then $(\mathrm{zero}\,(\phi))^\circ = \emptyset$.*

b): *If $\phi \neq 0_U$, then $U - \mathrm{zero}\,(\phi) \in \mathcal{D}(U)$.*

c): *If $\phi \neq 0_U$, then $A = \mathrm{dom}\,(\phi) \cap (U - \mathrm{zero}\,(\phi)) \in \mathcal{D}(U)$.*

Proof. **a**) It follows immediately from the required analyticity of each member of $\Theta[\mathbf{X}, U]$.

b) On the one hand, because the topology $\mathcal{T}$ of the considered analytic manifold $(M_n, \mathcal{A})$ is assumed to be separable, then $\mathrm{zero}\,(\phi)$ is closed and $U - \mathrm{zero}\,(\phi)$ is open in the relative topology $\mathcal{T}\big|_U$. On the other hand, if $E = U - \mathrm{zero}\,(\phi)$ is not dense in U, then there is $p \in \mathrm{zero}\,(\phi)$ and an open neighborhood B_p of p such that $B_p \cap E = \emptyset$, therefore $B_p \subseteq \mathrm{zero}\,(\phi)$, and taking into account the preceding statement it follows that $\phi = 0_U$, which contradicts the assumption $\phi \neq 0_U$, hence E must be dense in U and because it is open, then $E \in \mathcal{D}(U)$.

c) By Definition 2.3 $\mathrm{dom}\,(\phi)$ belongs to $\mathcal{D}(U)$, and by the preceding statement we know that $U - \mathrm{zero}\,(\phi)$ is also a member of $\mathcal{D}(U)$, therefore taking into account Lemma 1.2, $A \in \mathcal{D}(U)$. □

Remark 2.8. *From now on, for every $c \in \mathbb{C}$ denote by c_U the map with domain U being constantly equal to c. Likewise, denote by $\mathbb{C}_U$ the subset $\{c_U \mid c \in \mathbb{C}\}$ of $\Theta[\mathbf{X}, U]$.*

Corollary 2.9. *If a map $\phi \in \Theta[\mathbf{X}, U]$ is constantly equal to $c \in \mathbb{C}$ over an open subset V of U, then ϕ is equal to c at every point of U, therefore $\mathrm{dom}\,(\phi) = U$.*

Proof. Let $-c_U$ stand for the map with domain U being constantly equal to $-c$. Thus, $\psi = \phi \dotplus (-c_U) = 0$ at every point of V. Now, since V is open and $V \subseteq \mathrm{zero}\,(\psi)$, then $U - \mathrm{zero}\,(\psi)$ is not dense in U, therefore from Lemma 2.7 it follows, that $\psi = 0_U$ and $\phi = c_U$. □

Lemma 2.10. *The set $\Theta[\mathbf{X}, U]$ together with the laws $\dotplus$, $\star$ and $^\flat$ form a field.*

Proof. Let ϕ_1 and ϕ_2 be two members of $\Theta[\mathbf{X}, U]$. On the one hand, by Lemma 1.2 one can obtain easily, that $W = \mathrm{dom}\,(\phi_1) \cap \mathrm{dom}\,(\phi_2)$ belongs to $\mathcal{D}(U)$, hence taking into account Lemma 2.5, we have that $\mathrm{dom}\,(\phi_1 \dotplus \phi_2) \supseteq$

W and $\operatorname{dom}(\phi_1 \star \phi_2) \supseteq W$ are also members of $\mathcal{D}(U)$. On the other hand, by Definition 2.3

$$(\mathbf{X}(\phi_1 \dotplus \phi_2))\check{} = \mathbf{X}\check{}\phi_1 \dotplus \mathbf{X}\check{}\phi_2 = 0_U \tag{2.6}$$

$$(\mathbf{X}(\phi_1 \star \phi_2))\check{} = (\mathbf{X}\check{}\phi_1) \star \phi_2 \dotplus \phi_1 \star \mathbf{X}\check{}\phi_2 = 0_U \tag{2.7}$$

therefore both maps $\phi_1 \dotplus \phi_2$ and $\phi_1 \star \phi_2$ satisfy Definition 2.3. Taking into account Lemma 2.6, for each $\phi \in \Theta[\mathbf{X}, U]$ it is true that $\phi \dotplus (-\phi) = 0_U$. The remaining properties defining the ring structure are immediate.

Now, for every $\phi \in \Theta[\mathbf{X}, U]$, $\operatorname{dom}\left(\phi^\flat\right)$ contains the set $\operatorname{dom}(\phi) - \operatorname{zero}(\phi)$, hence by virtue of Lemma 2.7 we have that $\operatorname{dom}\left(\phi^\flat\right)$ belongs to $\mathcal{D}(U)$, because $\operatorname{dom}\left(\phi^\flat\right) \supseteq \operatorname{dom}(\phi) - \operatorname{zero}(\phi) = \operatorname{dom}(\phi) \cap (U - \operatorname{zero}(\phi))$. Denoting by 1_U the map with domain U being constantly equal to 1, by definition, for each $p \in \operatorname{dom}(\phi) \cap \operatorname{dom}\left(\phi^\flat\right)$, $\phi(p)\phi^\flat(p) = 1$, therefore taking into account Lemma 2.6, $\phi \star \phi^\flat = 1_U$, and $\phi^\flat$ is the inverse of ϕ. Finally, for every $p \in (\operatorname{dom}(\phi) - \operatorname{zero}(\phi))$

$$\frac{-1}{\phi^2(p)} \mathbf{X}_p \phi(p) = \mathbf{X}_p \phi^\flat(p) = 0 \tag{2.8}$$

hence

$$\mathbf{X}\check{}\phi^\flat = 0_U \tag{2.9}$$

therefore $\phi^\flat$ satisfies Definition 2.3 and $\phi^\flat \in \Theta[\mathbf{X}, U]$. □

Definition 2.11. *Notice, that by virtue of Lemma 2.6, for each chart (U, φ) and every smooth vector $\mathbf{X}$, the set $\mathbb{C}_U = \{c_U : U \to \mathbb{C} |\ c \in \mathbb{C}\}$ consisting of every constant map belongs to $\Theta[\mathbf{X}, U]$, besides, $\mathbb{C}_U$ equipped with the relative laws $\dotplus$, $\star$ and $^\flat$, is a field being isomorphic to the complex number field. Thus, the analytic function field $\left(\Theta[\mathbf{X}, U], \dotplus, \star, ^\flat\right)$ can be regarded as a vector space over $\left(\mathbb{C}_U, \dotplus, \star, ^\flat\right)$, which we shall say to be the underlying vector space of $\left(\Theta[\mathbf{X}, U], \dotplus, \star, ^\flat\right)$.*

Theorem 2.12. *If each of the coefficients $a_0, a_1 \cdots a_m$ of the algebraic equation*

$$P\big(z\big) = a_0 \dotplus a_1 \star z \cdots \dotplus a_m \star (z)^{\smile m} = 0_U \tag{2.10}$$

belongs to the function field $\Theta[\mathbf{X}, U]$, besides, there are a fixed $g \in \Theta[\mathbf{X}, U]$ and for each $j \leq m$ a rational function $h_j : \mathbb{C} \to \mathbb{C}$ such that $a_j = h_j \circ g$, then each solution of (2.10) belongs to $\Theta[\mathbf{X}, U]$; where

$$(z)^{\smile m} = \underset{1}{z} \star \underset{2}{z} \cdots \star \underset{m}{z}$$

Proof. First, assume that $P(z)$ has no multiple root. In this case, taking into account equation (2.3), from (2.10) we obtain that

$$\begin{aligned} \mathbf{X}\check{}(a_0 \dotplus a_1 \star z \cdots \dotplus a_m \star (z)^{\smile m}) = \\ \left(a_1 \dotplus 2a_2 \star z \cdots \dotplus m a_m \star (z)^{\smile (m-1)}\right) \mathbf{X}\check{} z = 0_U \end{aligned} \tag{2.11}$$

and because we are assuming that $P(z)$ has no multiple root, then it is true that $\left(a_1 \dotplus 2a_2 \star z \cdots \dotplus ma_m \star (z)^{\smile(m-1)}\right) \neq 0_U$, hence from the equation above one can obtain that $\mathbf{X}\breve{} z = 0_U$, accordingly z satisfies equation (2.3).

Conversely, if $P(z)$ has some multiple root, then denoting by $Q(z)$ the greatest common factor of $P(z)$ and its derivative, the equation $\dfrac{P(z)}{Q(z)} = 0_U$ has the same solutions as (2.10), however it has not multiple root and the proof goes as in the preceding case.

It remains to be shown that $\operatorname{dom}(z)$ belongs to $\mathcal{D}(U)$. The equation (2.10) is defined at every point p lying in the set

$$E = \left(\bigcap_{0\le k\le m} \operatorname{dom}(a_k(p))\right) \bigcap \left(\bigcap_{1\le k\le m} \left(U - \operatorname{zero}\left(a_k(p)\right)\right)\right) \in \mathcal{D}(U) \tag{2.12}$$

therefore, each solution $z(p)$ of (2.10) is defined for every $p \in E$. Thus, by continuity, we have that $(P(z^{\smile}))^{\smile} = 0_U$, so that each solution z of 2.10 is maximal, besides, $\operatorname{dom}(z)$ contains E, and taking into account Lemma 1.2, it follows that $\operatorname{dom}(z)$ belongs to $\mathcal{D}(U)$.

Finally, because each of the h_j is assumed to be rational, then it is a known fact that each solution $f(g)$ of the equation

$$\begin{aligned} h_0(g) \dotplus h_1(g) \star f(g) \cdots \dotplus h_m(g) \star (f(g))^{\smile m} &= \\ a_0 \dotplus a_1 \star z \cdots \dotplus a_m \star (z)^{\smile m} &= 0_U \end{aligned} \tag{2.13}$$

is an analytic function of g, and because g is assumed to be a member of $\Theta[\mathbf{X}, U]$, then the composition $z = f \circ g$ is also an analytic function. $\square$

2.2. **Topological function fields.** Since for each $\mathcal{C}^\omega$-vector $\mathbf{X}$ the field $(\mathbb{C}, \dotplus, \star, ^\flat)$ belongs to each analytic function field $\left(\Theta[\mathbf{X}, U], \dotplus, \star, ^\flat\right)$, then every topology $\mathfrak{T}$ being compatible with the algebraic field structure of $\left(\Theta[\mathbf{X}, U], \dotplus, \star, ^\flat\right)$ is also compatible with its underlying vector space structure described in Definition 2.11. However, because the domains of two members of $\Theta[\mathbf{X}, U]$ need not be the same, then such a topology $\mathfrak{T}$ cannot be built from the ordinary functionals or seminorms in the usual way.

Notation 2.13. *Given any member A of $\mathcal{D}(U)$ denote by $\mathcal{R}(\mathbf{X}, A)$ the subset of $\Theta[\mathbf{X}, U]$ consisting each map $\phi \in \Theta[\mathbf{X}, U]$ whose domain contains A. By Lemma 2.6 we know that every constant map in $\Theta[\mathbf{X}, U]$ belongs to $\mathcal{R}(\mathbf{X}, A)$, for the domain of each constant map is $U \supseteq A$. As a immediate consequence of Definition 2.3 we have that the set $\mathcal{R}(\mathbf{X}, A)$ together with both relative laws $\dotplus$ and $\star$ form a function ring $(\mathcal{R}(\mathbf{X}, A), \dotplus, \star)$. Since $\mathcal{R}(\mathbf{X}, A)$ contains each constant map from U into $\mathbb{C}$, it is not difficult to see that $(\mathcal{R}(\mathbf{X}, A), \dotplus, \star)$ can be regarded as a vector space over the field $(\mathbb{C}, \dotplus, \star, ^\flat)$, that we shall say to be the underlying vector space of $(\mathcal{R}(\mathbf{X}, A), \dotplus, \star)$; i.e. the subspace of the vector space described in Definition 2.11 whose underlying set is $\mathcal{R}(\mathbf{X}, A)$.*

Definition 2.14. *For each $p \in A$ the mapping $q_p : \mathcal{R}(\mathbf{X}, A) \to [0, \infty)$ defined by*

$$\forall \phi \in \mathcal{R}(\mathbf{X}, A) : \qquad q_p(\phi) = |\phi(p)| \tag{2.14}$$

is an ordinary seminorm with respect to the underlying vector space structure of $\mathcal{R}(\mathbf{X}, A)$. Thus, the family of seminorms $\mathcal{S}(A) = \{q_p \mid p \in A\}$ induces a topology $\mathfrak{T}_A$ for $\mathcal{R}(\mathbf{X}, A)$ such that $(\mathcal{R}(\mathbf{X}, A), \mathfrak{T}_A)$ is a locally convex space. If the mapping $m_{\mathcal{R}(\mathbf{X},A)} : \mathcal{R}(\mathbf{X}, A) \to \Theta[\mathbf{X}, U]$ is the canonical inclusion, then the pair $(m_{\mathcal{R}(\mathbf{X}A)}, q_p)$, explicitly

$$\Theta[\mathbf{X}, U] \xleftarrow{m_{\mathcal{R}(\mathbf{X},A)}} \mathcal{R}(\mathbf{X}, A) \xrightarrow{q_p} [0, \infty)$$

can be denoted by partial seminorm in $\Theta[\mathbf{X}, U]$. *Given a partial seminorm family $\mathfrak{M} = \{(m_{R(\mathbf{X},A)}, q_p) \mid A \in \mathcal{D}(U), p \in A\}$, let $\mathfrak{T}_{\mathfrak{M}}$ denote the topology for $\Theta[\mathbf{X}, U]$ a subbase of which is the set collection*

$$\mathfrak{S} = \{V_{p,\varepsilon}(\phi_0) \mid p \in U, \varepsilon > 0, \phi_0 \in \Theta[\mathbf{X}, U]\}$$

where each of the $V_{p,\varepsilon}(\phi_0)$ stands for the subset of $\Theta[\mathbf{X}, U]$ defined by

$$\begin{aligned} V_{p,\varepsilon}(\phi_0) = \Big\{\phi \in \Theta[\mathbf{X}, U] \Big| \\ \forall A \in \mathcal{D}(U), \exists p_0 \in \operatorname{dom}(\phi_0) \cap \operatorname{dom}(\phi) \cap A : \\ \|\varphi(p_0) - \varphi(p)\| < \varepsilon \text{ and } q_{p_0}\left(\phi \dot{-} \phi_0\right) < \varepsilon \Big\} \end{aligned} \tag{2.15}$$

where $\varphi : U \to \mathbb{R}^n$ is the homeomorphism of the considered chart (U, φ), and $\phi \dot{-} \phi_0$ denotes the composition $\phi \dot{+} (-\phi_0)$. Obviously, the seminorm q_{p_0} occurring in (2.15) is defined in the ring $\mathcal{R}(\mathbf{X}, B)$ being B the intersection $\operatorname{dom}(\phi_0) \cap \operatorname{dom}(\phi) \cap A$.

Remark 2.15. *From the preceding definition it follows immediately, that a net $S = \{\phi_\delta \mid \delta \in (D, \leq)\}$ in $\Theta[\mathbf{X}, U]$ $\mathfrak{T}_{\mathfrak{M}}$-converges to ϕ_0 if and only if the following condition holds: For each $p \in U$ and every $\varepsilon > 0$ there exists $\delta_0 \in D$ such, that for every $\delta \geq \delta_0$ and each $A \in \mathcal{D}(U)$ there is $p_\delta \in A \cap \operatorname{dom}(\phi_d) \cap \operatorname{dom}(\phi_0)$ satisfying the following relations*

$$\begin{cases} \|\varphi(p_\delta) - \varphi(p)\| < \varepsilon \\ q_{p_\delta}(\phi_\delta \dot{-} \phi_0) = \left|\phi_\delta(p_\delta) \dot{-} \phi_0(p_\delta)\right| < \varepsilon \end{cases} \tag{2.16}$$

Theorem 2.16. *The topology $\mathfrak{T}_{\mathfrak{M}}$ is compatible with the algebraic structure of $\left(\Theta[\mathbf{X}, U], \dot{+}, \star, ^\flat\right)$.*

Proof. To see the continuity of the map

$$(\,) \dot{+} (\,) : \Theta[\mathbf{X}, U] \times \Theta[\mathbf{X}, U] \to \Theta[\mathbf{X}, U]$$

it must be shown that for every pair $\phi_1, \phi_2 \in \Theta[\mathbf{X}, U]$ and each neighborhood $V_{p,\varepsilon}(\phi_1 \dot{+} \phi_2)$ of $\phi_1 \dot{+} \phi_2$ there are a neighborhood $V_{p,\varepsilon_1}(\phi_1)$ of ϕ_1 and a neighborhood $V_{p,\varepsilon_1}(\phi_2)$ of ϕ_2 such that

$$V_{p,\varepsilon_1}(\phi_1) \dot{+} V_{p,\varepsilon_1}(\phi_2) \subset V_{p,\varepsilon}(\phi_1 \dot{+} \phi_2) \tag{2.17}$$

On the one hand, $\psi_1 \in V_{p,\varepsilon_1}(\phi_1)$ implies that for each positive number $\varepsilon_1 < \varepsilon/6$ and every $A_1 \in \mathcal{D}(U)$ there is $p_1 \in A_1 \cap \operatorname{dom}(\phi_1) \cap \operatorname{dom}(\psi_1)$ such that

$$\left\{\begin{array}{l} \|\varphi(p_1) - \varphi(p)\| < \varepsilon_1 < \varepsilon/6 \\ \left|\psi_1(p_1) \dot{-} \phi_1(p_1)\right| < \varepsilon_1 < \varepsilon/6 \end{array}\right. \tag{2.18}$$

Analogously, the relation $\psi_2 \in V_{p,\varepsilon_1}(\phi_2)$ implies that for every A_2 in $\mathcal{D}(U)$ there is $p_2 \in A_2 \cap \operatorname{dom}(\phi_2) \cap \operatorname{dom}(\psi_2)$ such that

$$\left\{\begin{array}{l} \|\varphi(p_2) - \varphi(p)\| < \varepsilon_1 < \varepsilon/6 \\ \left|\psi_2(p_2) \dot{-} \phi_2(p_2)\right| < \varepsilon_1 < \varepsilon/6 \end{array}\right. \tag{2.19}$$

On the other hand, setting $A_1 = A \cap \operatorname{dom}(\phi_2) \cap \operatorname{dom}(\psi_2)$ and $A_2 = A \cap \operatorname{dom}(\phi_1) \cap \operatorname{dom}(\psi_1)$ for any $A \in \mathcal{D}(U)$, then, by continuity and taking into account that φ is a homeomorphism, for every $\varepsilon/6 > 0$ there is ε_1 such that the relations

$$\left\{\begin{array}{l} \|\varphi(p_1) - \varphi(p)\|\, \varepsilon_1 < \varepsilon/6 \\ \|\varphi(p_2) - \varphi(p)\|\, \varepsilon_1 < \varepsilon/6 \\ \|\varphi(p_1) - \varphi(p_2)\| \leq \|\varphi(p_1) - \varphi(p)\| + \|\varphi(p_2) - \varphi(p)\| < 2\varepsilon_1 \end{array}\right. \tag{2.20}$$

imply the following

$$\left\{\begin{array}{l} \left|\phi_2(p_1) \dot{-} \phi_2(p_2)\right| < \varepsilon/6 \\ \left|\psi_2(p_1) \dot{-} \psi_2(p_2)\right| < \varepsilon/6 \end{array}\right. \tag{2.21}$$

hence

$$\begin{aligned} \left|\psi_2(p_1) \dot{-} \phi_2(p_1)\right| \leq \left|\psi_2(p_1) \dot{-} \psi_2(p_2)\right| + \\ \left|\psi_2(p_2) \dot{-} \phi_2(p_2)\right| + \left|\phi_2(p_2) \dot{-} \phi_2(p_1)\right| < \varepsilon/2 \end{aligned} \tag{2.22}$$

therefore from (2.18) and (2.22) together with Lemma 2.5 it follows, that for each $A \in \mathcal{D}(U)$ there is p_1 lying in

$$\begin{aligned} A \cap \operatorname{dom}(\phi_1) \cap \operatorname{dom}(\phi_2) \cap \operatorname{dom}(\psi_1) \cap \operatorname{dom}(\psi_2) \subseteq \\ A \cap \operatorname{dom}(\phi_1 \dot{+} \phi_2) \cap \operatorname{dom}(\psi_1 \dot{+} \psi_2) \end{aligned}$$

satisfying the relations,

$$\left\{\begin{array}{l} \|\varphi(p_1) - \varphi(p)\| < \varepsilon_1 < \varepsilon/6 < \varepsilon \\ \left|(\psi_1(p_1) \dot{+} \psi_2(p_1)) \dot{-} (\phi_1(p_1) \dot{+} \phi_2(p_1))\right| < \varepsilon \end{array}\right. \tag{2.23}$$

accordingly $(\psi_1 \dot{+} \psi_2)$ belongs to $V_{p,\varepsilon}(\phi_1 \dot{+} \phi_2)$ and because both maps ψ_1 and ψ_2 are arbitrary, then the relation (2.17) follows.

Likewise, setting $\varepsilon = \varepsilon_0$ in (2.18) and (2.22) we obtain, that

$$\left\{\begin{array}{l} \|\varphi(p_1) - \varphi(p)\| < \varepsilon_1 < \varepsilon_0/6 \\ \left|\psi_1(p_1) \dot{-} \phi_1(p_1)\right| < \varepsilon_1 < \varepsilon_0/6 \\ \left|\psi_2(p_1) \dot{-} \phi_2(p_1)\right| < \varepsilon_0/2 \end{array}\right. \tag{2.24}$$

therefore

$$\left|(\psi_1(p_1) \star \psi_2(p_1)) \dot{-} (\phi_1(p_1) \star \phi_2(p_1))\right| < (|\psi_2(p_1)| + |\phi_1(p_1)|)\, \varepsilon_0 \tag{2.25}$$

and from (2.24) one can obtain $|\psi_2(p_1)| < \varepsilon_0/2 + |\phi_2(p_1)|$. Using this relation in (2.25) it follows, that

$$\left|(\psi_1(p_1) \star \psi_2(p_1)) \dot{-} (\phi_1(p_1) \star \phi_2(p_1))\right| < (|\phi_2(p_1)| + |\phi_1(p_1)| + \varepsilon_0/2)\, \varepsilon_0/6 \tag{2.26}$$

therefore, setting $(|\phi_2(p_1)| + |\phi_1(p_1)| + \varepsilon_0/2)\, \varepsilon_0/6 = \varepsilon$ it follows that $\psi_1 \star \psi_2$ belongs to $V_{p,\varepsilon}(\phi_1 \star \phi_2)$, hence one can deduce that for each neighborhood $V_{p,\varepsilon}(\phi_1 \star \phi_2)$ of $\phi_1 \star \phi_2$ there are $V_{p,\varepsilon_1}(\phi_1)$ and $V_{p,\varepsilon_1}(\phi_2)$ satisfying the relations

$$V_{p,\varepsilon_1}(\phi_1) \star V_{p,\varepsilon_1}(\phi_2) \subset V_{p,\varepsilon}(\phi_1 \star \phi_2) \tag{2.27}$$

and the mapping $(\,) \star (\,) : \Theta[\mathbf{X}, U] \times \Theta[\mathbf{X}, U] \to \Theta[\mathbf{X}, U]$ is continuous.

Finally, consider a net $S = \{\phi_\delta \mid (D, \leq)\}$ converging to ϕ_0 and assume that $\phi_0 \neq 0_U$, accordingly for every $p \in U$ and each $\varepsilon > 0$ there is $\delta_0 \in D$ such that $\forall \delta \geq \delta_0$ and $\forall A \in \mathcal{D}(U)$ there is $p_\delta \in A \cap \operatorname{dom}(\phi_\delta) \cap \operatorname{dom}(\phi_0)$ satisfying the following relations

$$\begin{cases} \|\varphi(p_\delta) - \varphi(p)\| < \varepsilon \\ \left|\phi_\delta(p_\delta) \dot{-} \phi_0(p_\delta)\right| < \varepsilon \end{cases} \tag{2.28}$$

Letting $A = U - \operatorname{zero}(\phi_0)$ by Lemma 2.7 we know that $A \in \mathcal{D}(U)$, besides $p_\delta \in A$ implies that

$$|\phi_0(p_\delta)| > 0 \tag{2.29}$$

and from (2.28) one can obtain, that

$$\begin{aligned} \left|\phi_\delta^\flat(p_\delta) \dot{-} \phi_0^\flat(p_\delta)\right| &= \\ &\left|\left((\phi_\delta(p_\delta) \dot{-} \phi_0(p_\delta)\right) \star \phi_\delta^\flat(p_\delta) \star \phi_0^\flat(p_\delta)\right| \leq \\ &\qquad \left|\phi_\delta(p_\delta) \dot{-} \phi_0(p_\delta)\right| |\phi_\delta(p_\delta)|^{-1} |\phi_0(p_\delta)|^{-1} \end{aligned} \tag{2.30}$$

therefore setting $\varepsilon < |\phi_0(p_\delta)|$, from the relations (2.28) and (2.29) we obtain that $|\phi_\delta(p_\delta)| > |\phi_0(p_\delta)| - \varepsilon > 0$. Using this relation in (2.30) it follows that

$$\forall \delta \geq \delta_0 : \qquad \left|\phi_\delta^\flat(p_\delta) \dot{-} \phi_0^\flat(p_\delta)\right| < \frac{\varepsilon}{|\phi_0(p_\delta)|\,(|\phi_0(p_\delta)| - \varepsilon)} \tag{2.31}$$

and now it is easy to see, that the net $\{\phi_\delta^\flat \mid \delta \in D \leq\}$ converges to $\phi_0^\flat$. $\square$

Lemma 2.17. *If $S = \{\phi_\delta \mid \delta \in (D, \leq)\}$ is a net in $\Theta[\mathbf{X}, U]$ such that there are a function ring $(\mathcal{R}(\mathbf{X}, A), \dot{+}, \star)$ and a $\delta_0 \in D$ satisfying the relation*

$$\forall \delta \geq \delta_0 : \qquad \phi_\delta \in \mathcal{R}(\mathbf{X}, A) \tag{2.32}$$

then the net S $\mathfrak{T}_{\mathfrak{M}}$-converges to some function ϕ_0 in $\Theta[\mathbf{X}, U]$ if and only if S $\mathfrak{T}_A$-converges to ϕ_0.

Proof. First, suppose that S $\mathfrak{T}_A$-converges to ϕ_0. On the one hand, $\forall p \in U$, $\forall \varepsilon > 0$, $\forall B \in \mathcal{D}(U)$, $\forall \delta \in D$, by virtue of Lemma 1.2 together with Definition 2.3 there is $p_0 \in B \cap \operatorname{dom}(\phi_\delta) \cap \operatorname{dom}(\phi_0) \cap A \subseteq B \cap \operatorname{dom}(\phi_\delta) \cap \operatorname{dom}(\phi_0)$ satisfying the relation

$$\|\varphi(p_0) - \varphi(p)\| < \varepsilon \tag{2.33}$$

On the other hand, because it is assumed that the net S $\mathfrak{T}_A$-converges to ϕ_0, for every $p_0 \in A$ there is $\delta_1 \geq \delta_0$, such that

$$\forall \delta \geq \delta_1 : \qquad \left|\phi_\delta(p_0) \dot{-} \phi_0(p_0)\right| < \varepsilon \tag{2.34}$$

accordingly setting $p_\delta = p_0$ we have that $\forall B \in \mathcal{D}(U)$, $\forall \delta \geq \delta_1$ there is $p_\delta \in B \cap \operatorname{dom}(\phi_\delta) \cap \operatorname{dom}(\phi_0)$ satisfying the relations

$$\begin{cases} \|\varphi(p_\delta) - \varphi(p)\| < \varepsilon \\ \left|\phi_\delta(p_\delta) \dot{-} \phi_0(p_\delta)\right| < \varepsilon \end{cases} \tag{2.35}$$

so that S $\mathfrak{T}_{\mathfrak{M}}$-converges to ϕ_0.

Conversely, suppose that the net S $\mathfrak{T}_{\mathfrak{M}}$-converges to ϕ_0; accordingly, because $A \in \mathcal{D}(U)$, then $\forall p \in A \subseteq U$, $\forall \varepsilon_1 > 0$, there is $\delta_1 \geq \delta_0$ such that for each $\delta \geq \delta_1$, there is p_δ lying in $A \cap \operatorname{dom}(\phi_\delta) \cap \operatorname{dom}(\phi_0)$ satisfying the relations

$$\begin{cases} \|\varphi(p_\delta) - \varphi(p)\| < \varepsilon_1 \\ \left|\phi_\delta(p_\delta) \dot{-} \phi_0(p_\delta)\right| < \varepsilon_1 \end{cases} \tag{2.36}$$

By continuity for every ε_2 there is $\varepsilon_1 < \varepsilon_2$ such that the relation

$$\|\varphi(p_\delta) - \varphi(p)\| < \varepsilon_1 < \varepsilon_2$$

implies

$$\begin{cases} \left|\phi_\delta(p_\delta) \dot{-} \phi_\delta(p)\right| < \varepsilon_2 \\ \left|\phi_0(p_\delta) \dot{-} \phi_0(p)\right| < \varepsilon_2 \end{cases} \tag{2.37}$$

Finally, setting $\varepsilon_2 = \varepsilon/3 > 0$ from (2.36) and (2.37) one can deduce that

$$\begin{aligned} \forall \delta \geq \delta_1 : \qquad & \left|\phi_\delta(p) \dot{-} \phi_0(p)\right| \leq \\ & \left|\phi_\delta(p) \dot{-} \phi_\delta(p_\delta)\right| + \left|\phi_\delta(p_\delta) \dot{-} \phi_0(p_\delta)\right| + \left|\phi_0(p_\delta) \dot{-} \phi_0(p)\right| < \varepsilon \end{aligned} \tag{2.38}$$

accordingly, the net S $\mathfrak{T}_A$-converges to ϕ_0. □

Corollary 2.18. *For every $A \in \mathcal{D}(U)$ the inclusion*

$$m_{\mathcal{R}(\mathbf{X},A)} : (\mathcal{R}(\mathbf{X}, A), \mathfrak{T}_A) \to (\Theta[\mathbf{X}, U], \mathfrak{T}_{\mathfrak{M}})$$

is a continuous map.

Proof. By Lemma 2.17, if a net S $\mathfrak{T}_A$-converges to $\phi \in \mathcal{R}(\mathbf{X}, A)$, then the net $m_{\mathcal{R}(A)} \circ S$ $\mathfrak{T}_{\mathfrak{M}}$-converges to $m_{\mathcal{R}(A)}(\phi) = \phi$. □

Theorem 2.19. $(\Theta[\mathbf{X}, U], \mathfrak{T}_{\mathfrak{M}})$ *is Hausdorff space.*

Proof. Let $S = \{\phi_\delta \mid \delta \in (D, \leq)\}$ be a net which converges to both maps ϕ_1 and ϕ_2, and suppose that there is $p \in \operatorname{dom}(\phi_1) \cap \operatorname{dom}(\phi_2)$ such that

$$\left|\phi_1(p) \dot{-} \phi_2(p)\right| = a > 0 \tag{2.39}$$

By convergence, for each ε_1 there is $\delta_0 \in D$ such that for every $\delta \geq \delta_0$ there exist $p_1, p_2 \in \operatorname{dom}(\phi_\delta) \cap \operatorname{dom}(\phi_1) \cap \operatorname{dom}(\phi_2)$ satisfying the relations

$$\begin{cases} \|\varphi(p_1) - \varphi(p)\| < \varepsilon_1/2 \\ \|\varphi(p_2) - \varphi(p)\| < \varepsilon_1/2 \\ \left|\phi_\delta(p_1) \dot{-} \phi_1(p_1)\right| < \varepsilon_1/2 \\ \left|\phi_\delta(p_2) \dot{-} \phi_2(p_2)\right| < \varepsilon_1/2 \end{cases} \tag{2.40}$$

hence

$$\|\varphi(p_1) - \varphi(p_2)\| < \varepsilon_1 \tag{2.41}$$

Now, by continuity, for each $\varepsilon_0 > 0$ there is $\varepsilon_1 < \varepsilon_0$ such that the relations (2.40) and (2.41) imply the following

$$\begin{cases} \left|\phi_1(p_1) \dot{-} \phi_1(p)\right| < \varepsilon_0 \\ \left|\phi_2(p_1) \dot{-} \phi_2(p)\right| < \varepsilon_0 \\ \left|\phi_2(p_1) \dot{-} \phi_2(p_2)\right| < \varepsilon_0 \\ \left|\phi_\delta(p_1) \dot{-} \phi_\delta(p_2)\right| < \varepsilon_0 \end{cases} \tag{2.42}$$

and from (2.40) and (2.42) it follows that, $\forall \varepsilon_0 > 0$:

$$\begin{aligned} \left|\phi_\delta(p_1) \dot{-} \phi_2(p_1)\right| \leq& \left|\phi_\delta(p_1) \dot{-} \phi_\delta(p_2)\right| \\ &+ \left|\phi_\delta(p_2) \dot{-} \phi_2(p_2)\right| + \left|\phi_2(p_2) \dot{-} \phi_2(p_1)\right| < 3\varepsilon_0 \end{aligned} \tag{2.43}$$

hence $\forall \varepsilon_0 > 0$:

$$\begin{aligned} \left|\phi_1(p_1) \dot{-} \phi_2(p_2)\right| \leq& \left|\phi_1(p_1) \dot{-} \phi_\delta(p_1)\right| + \\ &\left|\phi_\delta(p_1) \dot{-} \phi_2(p_1)\right| + \left|\phi_2(p_1) \dot{-} \phi_2(p_2)\right| < \varepsilon_1 + 4\varepsilon_0 < 5\varepsilon_0 \end{aligned} \tag{2.44}$$

and analogously, $\forall \varepsilon_0 = \varepsilon/8 > 0$:

$$\begin{aligned} \left|\phi_1(p) \dot{-} \phi_2(p\right| \leq& \left|\phi_1(p) \dot{-} \phi_1(p_1)\right| + \\ &\left|\phi_1(p_1) \dot{-} \phi_2(p_2)\right| + \left|\phi_2(p_2) \dot{-} \phi_2(p_1)\right| + \left|\phi_2(p_1) \dot{-} \phi_2(p)\right| < 8\varepsilon_0 = \varepsilon \end{aligned} \tag{2.45}$$

which contradicts (2.39), consequently for every $p \in A = \operatorname{dom}(\phi_1) \cap \operatorname{dom}(\phi_2)$ the equation $\left|\phi_1(p) \dot{-} \phi_2(p)\right| = 0$ holds, accordingly

$$\phi_1\big|_A = \phi_2\big|_A = 0 \tag{2.46}$$

Finally, because A is dense in U it follows that $\left(\phi_1\big|_A\right)^{\smile} = \phi_1 = \left(\phi_2\big|_A\right)^{\smile} = \phi_2$, therefore $(\Theta[\mathbf{X}, U], \mathfrak{T}_{\mathfrak{M}})$ is a Hausdorff space. $\square$

3. Topological vector spaces over $\Theta[\mathbf{X}, U]$

3.1. **Function vector spaces.** The subset $\mathcal{V}$ of $\left(\mathcal{C}^0(\mathcal{D}(U), \mathbb{C}), \preceq\right)$ consisting of every maximal function lying in $\mathcal{C}^0(\mathcal{D}(U), \mathbb{C})$ equipped with the law $\dot{+}$ is a commutative group containing every constant map in $\mathcal{C}^0(\mathcal{D}(U), \mathbb{C})$, because by definition, for every couple $f_1, f_2 \in \mathcal{V}$ the composition $f_1 \dot{+} f_2$ is also a maximal member of $\mathcal{C}^0(\mathcal{D}(U), \mathbb{C})$. Analogously, from the definition of the law $\star$ it follows immediately, that for every analytic function field $\Theta[\mathbf{X}, U] \subseteq \mathcal{C}^0(\mathcal{D}(U), \mathbb{C})$ and each pair $(\phi, f) \in \Theta[\mathbf{X}, U] \times \mathcal{V}$ the composition $\phi \star f$ is also maximal, therefore $\phi \star f \in \mathcal{V}$. Now, it is not difficult to see, that if $(M_n, \mathcal{A})$ is an analytic manifold, then for every $\mathcal{C}^\omega$-vector $\mathbf{X}$ and each chart $(U, \varphi) \in \mathcal{A}$ the subset $\mathcal{V} \subseteq \mathcal{C}^0(\mathcal{D}(U), \mathbb{C})$ consisting of every maximal element is a vector space over the analytic function field $\Theta[\mathbf{X}, U]$.

Enriching the vector space $\mathcal{V}$ with a topology $\mathfrak{T}$, being compatible with its algebraic vector space structure, we can calculate the sums of function series, for instance, the Taylor expansion of an analytic function. To prove some results described in the following section, we are only concerned with subspaces of $\mathcal{V}$, besides, for our purposes we only consider the coarser topology $\mathfrak{T}$ for $\mathcal{V}$ such that every member of its algebraic dual $\mathcal{V}^*$ is a continuous map.

Definition 3.1. *Let $\mathfrak{T}$ stand for the coarser topology for $\mathcal{V}$ under which every map $h : (\mathcal{V}, \mathfrak{T}) \to (\Theta[\mathbf{X}, U], \mathfrak{T}_{\mathfrak{M}})$ lying in the algebraic dual $\mathcal{V}^*$ of $\mathcal{V}$ is a continuous map. It is not difficult to see, that a subbase for $\mathfrak{T}$ is the set collection $\mathfrak{F} = \{W_{p,\varepsilon,h}(f_0) \mid f_0 \in \mathcal{V}, p \in U, \varepsilon > 0, h \in \mathcal{V}^*\}$ defined as follows:*

$$W_{p,\varepsilon,h}(f_0) = \left\{ f \in \mathcal{V} \middle| \; h(f) \in V_{p,\varepsilon}\left(h(f_0)\right) \right\} \tag{3.1}$$

where $V_{p,\varepsilon}\left(h(f_0)\right) \in \mathfrak{T}_{\mathfrak{M}}$ is a member of the subbase for $\mathfrak{T}_{\mathfrak{M}}$ defined in (2.15).

Taking into account (2.15) from the preceding relation it follows immediately, that

$$\begin{aligned} W_{p,\varepsilon,h}(f_0) = \Big\{ f \in \mathcal{V} \Big| \; & \\ & \forall A \in \mathcal{D}(U), \exists p_0 \in \operatorname{dom}\left(h(f_0)\right) \cap \operatorname{dom}\left(h(f)\right) \cap A : \\ & \qquad \|\varphi(p_0) - \varphi(p)\| < \varepsilon \text{ and } q_{p_0}\left(h(f) \dot{-} h(f_0)\right) < \varepsilon \Big\} \end{aligned} \tag{3.2}$$

In locally convex spaces the coarser topology for which the algebraic dual coincides with the topological one is compatible with the vector space structure. By virtue of relation (3.1), the compatibility of $\mathfrak{T}$ with the algebraic structure of $\mathcal{V}$ does not depend on the nature of the considered field $\Theta[\mathbf{X}, U]$, as one can see proving that for every pair $f_1, f_2 \in \mathcal{V}$ and every neighborhood $W_{p,\varepsilon,h}(f_1 \dot{+} f_2)$ of $f_1 \dot{+} f_2$ there are a neighborhood $W_{p,\varepsilon_1,h}(f_1)$ of f_1 and a neighborhood $W_{p,\varepsilon_1,h}(f_2)$ of f_2 such that

$$W_{p,\varepsilon_1,h}(f_1) \dot{+} W_{p,\varepsilon_1,h}(f_2) \subset W_{p,\varepsilon,h}(f_1 \dot{+} f_2)$$

Of course, from (3.1) we have that

$$W_{p,\varepsilon,h}(f_1 \dot{+} f_2) = \{f \in \mathcal{V} \mid h(f) \in V_{p,\varepsilon}(h(f_1 \dot{+} f_2))\} \tag{3.3}$$

$$W_{p,\varepsilon_1,h}(f_1) = \{f \in \mathcal{V} \mid h(f) \in V_{p,\varepsilon_1}(h(f_1))\} \tag{3.4}$$

$$W_{p,\varepsilon_1,h}(f_2) = \{f \in \mathcal{V} \mid h(f) \in V_{p,\varepsilon_1}(h(f_2))\} \tag{3.5}$$

therefore,

$$\begin{aligned} W_{p,\varepsilon_1,h}(f_1) \dot{+} W_{p,\varepsilon_1,h}(f_2) = \\ \{f \dot{+} g \mid h(f) \in V_{p,\varepsilon_1}(h(f_1)), h(g) \in V_{p,\varepsilon_1}(h(f_2))\} \subseteq \\ \{f \in \mathcal{V} \mid h(f) \in V_{p,\varepsilon_1}(h(f_1)) \dot{+} V_{p,\varepsilon_1}(h(f_2))\} \end{aligned} \tag{3.6}$$

Now, by Theorem 2.16 there is $\varepsilon_1 > 0$ such that

$$V_{p,\varepsilon_1}(h(f_1)) \dot{+} V_{p,\varepsilon_1}(h(f_2)) \subset V_{p,\varepsilon}(h(f_1) \dot{+} h(f_2))$$

Using this relation in (3.6) we obtain that

$$\begin{aligned} W_{p,\varepsilon_1,h}(f_1) \dot{+} W_{p,\varepsilon_1,h}(f_2) \subset \{f \in \mathcal{V} \mid h(f) \in V_{p,\varepsilon}(h(f_1) \dot{+} h(f_2))\} = \\ \{f \in \mathcal{V} \mid h(f) \in V_{p,\varepsilon}(h(f_1 \dot{+} f_2))\} = W_{p,\varepsilon,h}(f_1 \dot{+} f_2) \end{aligned} \tag{3.7}$$

therefore the mapping $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$ carrying each pair (f_1, f_2) into $f_1 \dot{+} f_2$ is continuous. The proof for the continuity of the mapping $\Theta[\mathbf{X}, U] \times \mathcal{V} \to \mathcal{V}$ carrying each pair $(\phi, f) \in \Theta[\mathbf{X}, U] \times \mathcal{V}$ into $\phi \star f$ is similar; so that $\mathfrak{T}$ is compatible with the algebraic vector space structure of $\mathcal{V}$.

4. Application to differential equation theory

4.1. **General equivalence conditions.** The general solution of a linear system of ordinary differential equations belongs to some vector space over the real (complex) number field generated by an exponential function set of the form

$$\mathcal{B} = \left\{e^{\lambda_\alpha (t-t_0)} \mid (t, t_0) \in \mathbb{R}^2, \lambda_\alpha \in \mathbb{C}, \alpha \in A\right\}$$

where A is any finite index set. A system of ordinary differential equations we shall say to be equivalent to some linear equation, provided that its general solution belongs to a vector space generated by a function set like $\mathcal{B}$, but now A can be also a non-countable index set [6]. We generalize such an equivalence concept to vector spaces over function fields. Thus, an equation system we shall say to be equivalent to some linear system, provided that its solution belongs to a vector space over a function field $\Theta[\mathbf{X}, U]$ being generated by a function set of the form

$$\mathcal{B} = \left\{e^{\phi_1 \star \Phi \dot{+} \phi_2 (t-t_0)} \mid (t, t_0) \in \mathbb{R}^2, \phi_1, \phi_2 \in \Theta[\mathbf{X}, U], \mathbf{X}\Phi \in \Theta[\mathbf{X}, U]\right\} \tag{4.1}$$

In the finite dimensional case, each solution satisfies an equation lying in an n-parameter linear system family. To state the equivalence between an autonomous system

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{X}(\mathbf{x}(t)) \tag{4.2}$$

and some linear equation system, i.e., to determine whether the general solution of (4.2) belongs to a vector space over the function field $\Theta[\mathbf{X}, U]$ generated by the function set (4.1), the following conditions are sufficient.

Condition 1. *Let $(M_n, \mathcal{A})$ an analytic manifold and $\mathbf{X}$ a $\mathcal{C}^\omega$-vector such that if its coordinates in a given chart $(U, \varphi) \in \mathcal{A}$ are $(X^1, X^2 \cdots X^n)$, then they satisfy the following properties: For each index $k \leq n$ there exist a subset $K \subseteq \mathbb{C}$, a continuous mapping $F^k : U \times K \to \mathbb{C}$, a member h^k of $\Theta[\mathbf{X}, U]$, and a maximal function $\Phi \in \mathcal{C}^1(\mathcal{D}(U), \mathbb{C})$ such that*

$$\mathbf{X}\breve{}\Phi^k = h^k \in \Theta[\mathbf{X}, U] \tag{4.3}$$

$$\forall p \in U : \quad X^k(p) = F^k\left(p, \Phi^k(p)\right) \neq 0 \tag{4.4}$$

Condition 2. *For each positive integer $k \leq n$ there are a subset $D \subseteq \mathbb{C}$, a member A of $\mathcal{D}(U)$ satisfying the relation $A \subseteq \operatorname{dom}\left(h^k\right)$, and two maps $\mathcal{F}^k : A \times D \to \mathbb{C}$ and $g^k : A \times D \to \mathbb{C}$ such that for some integration path $(\sigma : I \subseteq \mathbb{R} \to D)$, this being continuous and differentiable except, at most, in a finite point set, the following is true,*

1): *For every fixed $z \in D : \mathcal{F}^k(p, z) \in \Theta[\mathbf{X}, U]$ and $g^k(p, z) \in \Theta[\mathbf{X}, U]$.*

2): $F^k(p, s) = \displaystyle\int_C \mathcal{F}^k(p, z) \star e^{g^k(p,z)s} dz.$

3): $\forall r \in \mathbb{N}, \forall p \in A$ *there is a real number $K(p, s, k) > 0$ such that:*

$$\left| \int_C \left(g^k(p, z) \star h^k(p)\right)^{\smile(r-1)} \star \ \mathcal{F}^k(p, z) \ \star \ e^{g^k(p,z)s} dz \right|^{\smile} \leq K(p, s, k) r!.$$

Notice, that by virtue of Lemma 2.5, for each $r \in \mathbb{N}$ the following relation holds: $A \times D \subseteq \operatorname{dom}\left(\left(g^k \star h^k\right)^{\smile r} \star \mathcal{F}^k\right)$.

Remark 4.1. *It is worth pointing out, that in general, it is not difficult to obtain the function $\mathcal{F}^k(p, z)$ by knowing $F^k(p, s)$; for instance, if the integration path C is a Bromwich one and $g(p, z) = z$, then $\mathcal{F}^k(p, z)$ is nothing but the Laplace transform for $F^k(p, s)$. Conversely, if $C = [0, \infty)$, then $F^k(p, s)$ is the Laplace transform for $\mathcal{F}^k(p, z)$ etc.*

Lemma 4.2. *If a vector $\mathbf{X}$ satisfies both Condition 1 and Condition 2, then for each positive integer r the associated linear operator $\mathbf{X}\breve{}$ can be iterated r times over each of its coordinates.*

Proof. By virtue of Condition 1 and Condition 2 we obtain that

$$\forall p \in A : \qquad X^k(p) = F^k\left(p, \Phi^k(p)\right) = \int_C \mathcal{F}^k(p, z) \star e^{g^k(p,z) \star \Phi^k(p)} \, dz \tag{4.5}$$

therefore,

$$\mathbf{X}\breve{}X^k(p) = \int_C \mathbf{X}\breve{}\mathcal{F}^k(p, z) \star e^{g^k(p,z) \star \Phi^k(p)} \, dz \tag{4.6}$$

Since by assumption, regarding z as a parameter, both map families $\mathcal{F}^k(p,z)$ and $g^k(p,z)$ belong to the function field $\Theta[\mathbf{X},U]$ and $\Phi^k(p)$ satisfies equation 4.3, then from the equation above it follows immediately, that $\forall p \in A$,

$$\mathbf{X}\check{}X^k(p) = \int_C \mathcal{F}^k(p,z) \star g^k(p,z) \star h^k(\mathbf{x}) \star e^{g^k(p,z)\star\Phi^k(p)}\, dz \tag{4.7}$$

Now, by induction and taking into account Condition 2, it follows that $\forall r \in \mathbb{N}$:

$$\begin{aligned}(4.8)\quad & \left|(\mathbf{X}\check{})^r\, X^k(p)\right|^{\smile} = \\ & \left|\int_C \mathcal{F}^k(p,z) \star \left(g^k(p,z) \star h^k(p)\right)^{\smile r} \star e^{g^k(p,z)\star\Phi^k(p)} dz\right|^{\smile} \leq \\ & \qquad K(p,\Phi^k(p))(r+1)!\end{aligned}$$

□

Theorem 4.3. *If a vector field $\mathbf{X}$ satisfies both Condition 1 and Condition 2, then for each $p_0 \in U$ there is a positive number H_{p_0} and an open interval $I = (t_0 - H_{p_0},\ t_0 + H_{p_0}) \subseteq \mathbb{R}$ such that the map $\gamma : I \to M_n$ satisfying $\gamma(t_0) = p_0$ whose k-th coordinate in the considered chart (U,φ) is defined by the integral transform*

$$\begin{aligned}(4.9)\quad x^k_{p_0}(\gamma(t)) = x^k_0 \star 1_U \dot{+} \int_C \mathcal{F}^k(p_0,z) \star \left(g^k(p_0,z) \star h^k(p_0)\right)^{\flat} \star \\ e^{g^k(p_0,z)\star\Phi^k(p_0)} \star \left(e^{g^k(p_0,z)\star h^k(p_0)(t-t_0)} \dot{-} 1_U\right) dz\end{aligned}$$

is a solution for the autonomous system (4.2) satisfying the initial condition

$$\left(x^1_{p_0}(\gamma(t_0)), x^2_{p_0}(\gamma(t_0)) \cdots x^n_{p_0}(\gamma(t_0))\right) = (x^1_0, x^2_0 \cdots x^n_0) = \varphi(p_0)$$

Remark 4.4. *From the equation (4.9) it is easy to see, that letting $\phi_1 = g^k$ and $\phi_2 = g^k \star h^k$, then $x^k_{p_0}(\gamma(t))$ belongs to the subspace of the vector space $\mathcal{V}$ over the field $\Theta[\mathbf{X},U]$ generated by the function set $\mathcal{B}$ defined in (4.1). Thus, both Condition 1 and Condition 2 form a criterion for equivalence between a non-linear autonomous system and some linear equation.*

Proof. Since $\mathbf{X}$ is assumed to be a differentiable vector, then it satisfies the standard conditions for the existence of a solution $\gamma : I \subseteq \mathbb{R} \to M_n$ of the system (4.2), where I is some open subset of $\mathbb{R}$; accordingly if the coordinates of $\varphi(\gamma(t))$ in the considered chart (U,φ) are

$$\left(x^1_{p_0}(\gamma(t)), x^2_{p_0}(\gamma(t)), \cdots, x^n_{p_0}(\gamma(t))\right) = \mathbf{x}_{p_0}(\gamma(t))$$

then for each $k \leq n$,

$$\frac{d}{dt}\, x^k_{p_0}(\gamma(t)) = X^k(x^1_{p_0}(\gamma(t)), x^2_{p_0}(\gamma(t)) \cdots) \tag{4.10}$$

Now, by virtue of the preceding lemma we obtain that $\forall m \leq n$,

$$(4.11)\quad \frac{d^{m+1}}{dt^{m+1}}\, x^k_{p_0}(\gamma(t))\Big|_{t=t_0} = (\mathbf{X}^{\smile})^m\, X^k(p)\Big|_{p=p_0} =$$

$$\int_C \mathcal{F}^k(p_0,z) \star \left(g^k(p_0,z)\star h^k(p_0)\right)^{\smile m} \star e^{g^k(p_0,z)\star\Phi^k(p_0)}\, dz$$

therefore, the Taylor's expansion for $x^k_{p_0}(\gamma(t))$ is

$$(4.12)\quad x^k_{p_0}(\gamma(t)) = x^k_0 \star 1_U \dot{+}$$
$$\breve{\sum}_{m=1}^{\infty} \int_C \mathcal{F}^k(p_0,z)\star\left(\frac{(t-t_0)^{m+1}}{(m+1)!}\left(g^k(p_0,z)\star h^k(p_0)\right)^{\smile m}\right)\star e^{g^k(p_0,z)\star\Phi^k(p_0)} dz$$

where $\breve{\sum}_{m=1}^{\infty} = \left(\lim\limits_{r\to\infty}\sum\limits_{m=1}^{r}\right)^{\smile}$ is assumed to be the limit with respect to the topology $\mathfrak{T}$ in Definition 3.1. If $g^k(p_0,z)\star h^k(p_0)=0$, then the right-hand of the former equation converges; otherwise, as a consequence of Lemma 2.7, the set $A = \operatorname{dom}\left(g^k(p_0,z)\star h^k(p_0)\right) - \operatorname{zero}\left(g^k(p_0,z)\star h^k(p_0)\right)$ belongs to $\mathcal{D}(U)$. Obviously, for each $p_0 \in A$ the former equation can be written as follows,

$$(4.13)\quad x^k_{p_0}(\gamma(t)) =$$
$$x^k_0\star 1_U \dot{+} \breve{\sum}_{m=0}^{\infty}\int_C \left(\mathcal{F}^k(p_0,z)\star\frac{\left(g^k(p_0,z)\star h^k(p_0)(t-t_0)\right)^{\smile(m+1)}}{(m+1)!}\star\right.$$
$$\left.\left(g^k(p_0,z)\star h^k(p_0)\right)^{\flat}\star e^{g^k(p_0,z)\star\Phi^k(p_0)}\right)dz$$

Finally, using (4.8) it follows, that

$$(4.14)\quad \Bigg|\breve{\sum}_{m=0}^{\infty}\int_C \mathcal{F}^k(p_0,z)\star\frac{\left(g^k(p_0,z)\star((t-t_0)h^k(p_0))\right)^{\smile(m+1)}}{(m+1)!}\star$$
$$\left(g^k(p_0,z)\star h^k(p_0)\right)^{\flat}\star e^{g^k(p_0,z)\star\Phi^k(p_0)}\, dz\Bigg|^{\smile}$$
$$\leq \sum_{m=0}^{\infty}\left|\left(K\left(p_0,\phi^k(p_0)\right)(t-t_0)\right)^{m+1}\right|$$

Now, taking into account Lemma 2.5, for each $r\in\mathbb{N}$ it is true that the set $\operatorname{dom}\left(g^k(p_0,z)\star h^k(p_0)\right)$ belongs to $\operatorname{dom}\left(\left(g^k(p_0,z)\star h^k(p_0)\right)^{\smile r}\right)$, accordingly the map $\left(g^k(p_0,z)\star h^k(p_0)\right)^{\smile r}$ belongs to the ring $\mathcal{R}(\mathbf{X},A)$ for each $r\in\mathbb{N}$, therefore taking into account Lemma (2.17), we have that the convergence with respect to the topology $\mathfrak{T}_A$ is sufficient; i.e., one can calculate the limits with respect to the weak topology for the underlying vector space of $\mathcal{R}(\mathbf{X},A)$. Thus, letting $H_{p_0}=\min\limits_{k\leq n}\left(K\left(p_0,\phi^k(p_0),k\right)\right)^{-1}$, for every $|t-t_0|<H_{p_0}$ the right-hand of (4.12) converges and from (4.13) one can obtain (4.9) easily, for the sequence

$$\left(\breve{\sum}_{m=0}^{n}\frac{\left(g^k(p_0,z)\star h^k(p_0)(t-t_0)\right)^{m+1}}{(m+1)!}\right)_{n\in\mathbb{N}}$$

converges to $e^{g^k(p_0,z)\star h^k(p_0)(t-t_0)}\dot{-}1_U$. □

Remark 4.5. *The former theorem is nothing but a generalization to vector spaces over analytic function fields of a similar result proved in* [6] *for function spaces over the real (complex) number field.*

Corollary 4.6. *If the vector* $\mathbf{X}$ *satisfies both Condition 1 and Condition 2, being* $\mathbf{x}(x_0^1,x_0^2\cdots x_0^n,t)$ *the general solution of (4.2) for the initial values* $x_0^1,x_0^2\cdots x_0^n$*, then each coordinate* $x^k(x_0^1,x_0^2\cdots,t)$ *of* $\mathbf{x}(x_0^1,x_0^2\cdots x_0^n,t)$*, regarded as a function from* $\mathbb{R}^n\times\mathbb{R}$ *into* $\mathbb{C}$*, satisfies the linear partial differential equation*

$$\sum_{m=1}^{n}X^m\frac{\partial}{\partial x_0^m}x^k(x_0^1,x_0^2\cdots,t)-\frac{\partial}{\partial t}x^k(x_0^1,x_0^2\cdots,t)=0 \tag{4.15}$$

Proof. By straightforward computations and taking into account both Condition 1 and Condition 2, from (4.9) and (4.5) we obtain that

$$
(4.16)\quad \sum_{m=1}^{n} X^m \frac{\partial}{\partial x_0^m}\, x^k(x_0^1, x_0^2 \cdots, t) =
$$

$$
X^k + \int_C \widehat{\mathcal{F}}^k(x_0^1, x_0^2 \cdots, z) \star \left(\widehat{g}^k(x_0^1, x_0^2 \cdots, z) \star \widehat{h}^k(x_0^1, x_0^2 \cdots)\right)^{\flat} \star
$$

$$
e^{\widehat{g}^k(x_0^1,x_0^2\cdots,z)\star\,\widehat{\Phi}^k(x_0^1,x_0^2\cdots)} \star \left(\widehat{g}^k(x_0^1, x_0^2 \cdots, z) \star \widehat{h}^k(x_0^1, x_0^2 \cdots)\right) \star
$$

$$
\left(e^{\widehat{g}^k(x_0^1,x_0^2\cdots,z)\star\,\widehat{h}^k(x_0^1,x_0^2\cdots)(t-t_0)} \dot{-} 1_U\right)\, dz =
$$

$$
X^k + \int_C \widehat{\mathcal{F}}^k(x_0^1, x_0^2 \cdots, z) \star e^{\widehat{g}^k(x_0^1,x_0^2\cdots,z)\star\,\widehat{\Phi}^k(x_0^1,x_0^2\cdots)} \star
$$

$$
e^{\widehat{g}^k(x_0^1,x_0^2\cdots,z)\star\,\widehat{h}^k(x_0^1,x_0^2\cdots)(t-t_0)}\, dz -
$$

$$
\int_C \widehat{\mathcal{F}}^k(x_0^1, x_0^2 \cdots, z) \star e^{\widehat{g}^k(x_0^1,x_0^2\cdots,z)\star\,\widehat{\Phi}^k(x_0^1,x_0^2\cdots)}\, dz
$$

therefore, using equation (4.5) in the above one, we obtain that

$$
(4.17)\quad \sum_{m=1}^{n} X^m \frac{\partial}{\partial x_0^m}\, x^k(x_0^1, x_0^2 \cdots, t) =
$$

$$
\int_C \widehat{\mathcal{F}}^k(x_0^1, x_0^2 \cdots, z) \star e^{\widehat{g}^k(x_0^1,x_0^2\cdots,z)\star\,\widehat{\Phi}^k(x_0^1,x_0^2\cdots)} \star
$$

$$
e^{\widehat{g}^k(x_0^1,x_0^2\cdots,z)\star\,\widehat{h}^k(x_0^1,x_0^2\cdots)(t-t_0)}\, dz = \frac{\partial}{\partial t}\, x^k(x_0^1, x_0^2 \cdots, t)
$$

where we denote by $\widehat{\mathcal{F}} : \mathrm{img}\,(\varphi) \to \mathbb{C}$, $\widehat{g} : \mathrm{img}\,(\varphi) \to \mathbb{C}$, $\widehat{h} : \mathrm{img}\,(\varphi) \to \mathbb{C}$ and $\widehat{\Phi} : \mathrm{img}\,(\varphi) \to \mathbb{C}$ the maps defined by

$$
(4.18)\qquad \widehat{\mathcal{F}} = \mathcal{F} \circ \varphi^{-1}
$$

$$
(4.19)\qquad \widehat{g} = g \circ \varphi^{-1}
$$

$$
(4.20)\qquad \widehat{h} = h \circ \varphi^{-1}
$$

$$
(4.21)\qquad \widehat{\Phi} = \Phi \circ \varphi^{-1}
$$

□

Remark 4.7. *It worth noticing, that the solutions of (4.15) contains a function field* $\Theta[\mathbf{X}^*, U \times I]$*; where* $\mathbf{X}^* = \left(X^1 \frac{\partial}{\partial x^1}\big|_p, \cdots X^n \frac{\partial}{\partial x^n}\big|_p, -\frac{\partial}{\partial t}\big|_{t_0}\right)$.

4.2. **The finite dimensional case.** In this section we are concerned with the equivalence problem between non-linear and linear systems whose general solution belongs to a finite dimensional vector space. The following results form a criterion for the existence of such an equivalence.

Theorem 4.8. *Let $(M_n, \mathcal{A})$ be an analytic manifold and $\mathbf{X}$ a $\mathcal{C}^\omega$-vector whose coordinates in a chart $(U, \varphi) \in \mathcal{A}$ are $(X^1, X^2 \cdots X^n)$. If for each $k \leq n$ the linear operator $\mathbf{X}^{\smile}$ can be iterated r times over the k-th coordinate X^k of $\mathbf{X}$, and there is a linear dependence relation*

$$(4.22) \qquad (\mathbf{X}^{\smile})^r \, X^k = \breve{\sum}_{j=0}^{r-1} c_j^k \star (\mathbf{X}^{\smile})^j \, X^k$$

whose coefficients $c_0^k, c_1^k \cdots c_{r-1}^k$ belong to the function field $\Theta[\mathbf{X}, U]$, being r the smallest positive integer satisfying (4.22), besides, the polynomial

$$(4.23) \qquad P_k(z) = \big(z)\big)^{\smile r} \dot{-} c_{r-1}^k \big(z\big)^{\smile (r-1)} \dot{-} \cdots \dot{-} c_0^k$$

has no multiple root, and each of its roots belongs to $\Theta[\mathbf{X}, U]$, then X^k lies in the finite dimensional vector space over the associated function field $\Theta[\mathbf{X}, U]$ generated by the function set $\mathcal{B} = \left\{ e^{\Phi_m^k} \mid m \leq r \right\}$, where the Φ_m^k are maximal members of $\mathcal{C}^\omega(\mathcal{D}(U), \mathbb{C})$ satisfying the relation

$$(4.24) \qquad \mathbf{X}^{\smile} \Phi_m^k = \lambda_m^k \in \Theta[\mathbf{X}, U]$$

and the $\lambda_1^k \cdots \lambda_r^k$ are the solutions of

$$(4.25) \qquad P\big(z\big) = 0$$

moreover, the set $\mathcal{B}$ is a basis for the generated space.

Proof. On the one hand, by virtue of Theorem 2.12 every solution of 4.23 lies in $\Theta[\mathbf{X}, U]$. On the other hand, letting

$$(4.26) \qquad \forall n < r : \quad f_n^k = \Big(\mathbf{X}^{\smile}\Big)^n X^k$$

define the functions $\psi_1^k, \psi_2^k, \cdots \psi_r^k$ by
(4.27)

$$\forall p \in U : \begin{cases} \psi_1^k(p) &= \alpha_1^1(p) \star X^k(p) \dot{+} \alpha_2^1(p) \star f_1^k(p) \cdots \dot{+} \alpha_r^1(p) \star f_{r-1}^k(p) \\ \psi_2^k(p) &= \alpha_1^2(p) \star X^k(p) \dot{+} \alpha_2^2(p) \star f_1^k(p) \cdots \dot{+} \alpha_r^2(p) \star f_{r-1}^k(p) \\ \cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \psi_r^k(p) &= \alpha_1^r(p) \star X^k(p) \dot{+} \alpha_2^r(p) \star f_1^k(p) \cdots \dot{+} \alpha_r^r(p) \star f_{r-1}^k(p) \end{cases}$$

where $\alpha_j^i \in \Theta[\mathbf{X}, U]$ for each $i, j \leq r$; hence, using (4.26) and (4.22) we obtain, that $\forall i \leq r$:

$$(4.28) \quad \mathbf{X}^{\smile} \psi_i^k = \alpha_1^i \star f_1^k \dot{+} \cdots \dot{+} \alpha_{r-1}^i \star f_{r-1}^k$$
$$\alpha_r^i \star c_0^k \star X^k \dot{+} \alpha_r^i \star c_1^k \star f_1^k \dot{+} \cdots \alpha_r^i \star c_{r-1}^k \star f_{r-1}^k$$

therefore, if for some $A \in \mathcal{D}(U))$ we require the α^i_j to satisfy the conditions:

$$\forall i \leq r, \forall p \in A : \quad \begin{cases} c^k_0(p) \star \alpha^i_r(p) = \lambda^k_i(p) \star \alpha^i_1(p) \\ \alpha^i_1(p) \dotplus c^k_1(p) \star \alpha^i_r(p) = \lambda^k_i(p) \star \alpha^i_2(p) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha^i_{r-1}(p) \dotplus c^k_{r-1}(p) \star \alpha^i_r(p) = \lambda^k_i(p) \star \alpha^i_r(p) \end{cases} \tag{4.29}$$

then the maps ψ^k_i satisfy the equation

$$\mathbf{X}\check{}\psi^k_i = \lambda^k_i \star \psi^k_i \tag{4.30}$$

The equation (4.29) possesses non-trivial solution provided that

$$\begin{vmatrix} -\lambda^k_i & 0 & 0 & \cdots & 0 & c^k_0 \\ 1 & -\lambda^k_i & 0 & \cdots & 0 & c^k_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & 1 & c^k_{r-1} \dot{-} \lambda^k_i \end{vmatrix} = P(\lambda^k_i) = 0 \tag{4.31}$$

accordingly, the λ^k_i are solutions of (4.23), therefore they are also eigenvalues of the endomorphism associated to the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & c^k_0 \\ 1 & 0 & 0 & \cdots & 0 & c^k_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & 1 & c^k_{r-1} \end{pmatrix} \tag{4.32}$$

Setting $A = \bigcap_{0 \leq n \leq r} \operatorname{dom}(c_n)$, because by hypothesis the solutions of (4.31) belong to $\Theta[\mathbf{X}, U]$, then the corresponding eigenvectors

$$(\alpha^1_1, \alpha^1_2 \cdots), (\alpha^2_1, \alpha^2_2 \cdots) \cdots$$

belong to the vector space $(\Theta[\mathbf{X}, U])^r$, besides, since we are assuming that $P(z)$ has no multiple root, then the eigenvectors are linearly independent; consequently, taking into account (4.27), X^k can be written as a unique linear combination of the $\psi^k_1, \psi^k_2, \cdots \psi^k_r$ with coefficients in $\Theta[\mathbf{X}, U]$. Finally, setting

$$\forall j \leq r : \quad \Phi^k_j = \ln(\psi^k_j) \tag{4.33}$$

the equation (4.24) are satisfied and X^k can be written as a linear combination of $e^{\Phi^k_1}, e^{\Phi^k_2} \cdots e^{\Phi^k_r}$ with coefficients in $\Theta[\mathbf{X}, U]$. Since the $\psi^k_1, \psi^k_2, \cdots \psi^k_r$ are linearly independent, then $\mathcal{B}$ is a basis. $\square$

Corollary 4.9. *Let*

$$
\begin{cases}
\dfrac{d}{dt}\, x^1(t) = X^1\left(x^1(t), x^2(t)\cdots\right) \\
\dfrac{d}{dt}\, x^2(t) = X^2\left(x^1(t), x^2(t)\cdots\right) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\dfrac{d}{dt}\, x^n(t) = X^n\left(x^1(t), x^2(t)\cdots\right)
\end{cases} \tag{4.34}
$$

be an autonomous system. If $\mathbf{X}$ *satisfies the hypotheses of the preceding theorem, then it satisfies both Condition 1 and Condition 2, therefore the corresponding equation is equivalent to some linear system, besides, its solutions can be obtained by means of the integral transform (4.9).*

Proof. By the former theorem we know that the roots of $P\big(z\big)$ belong to $\Theta[\mathbf{X}, U]$, besides, from (4.24) we obtain that

$$\forall i \le r: \quad \mathbf{X}^{\breve{}}\Big((\lambda_i^k)^\flat \,\star\, \Phi_i^k\Big) = 1_U \tag{4.35}$$

consequently

$$\forall i \le r, \forall m \le r: \qquad \mathbf{X}^{\breve{}}\Big((\lambda_i^k)^\flat \star \Phi_i^k \dot{-} (\lambda_m^k)^\flat \star \Phi_m^k\Big) = 0_U \tag{4.36}$$

Since each of the coordinates of $\mathbf{X}$ is supposed to be an analytic function, and taking into account Definition 2.3, this equation implies that

$$(\lambda_i^k)^\flat \star \Phi_i^k \dot{-} (\lambda_m^k)^\flat \star \Phi_m^k \in \Theta[\mathbf{X}, U] \tag{4.37}$$

accordingly, setting $(\lambda_1^k)^\flat \star \Phi_1^k = \Phi^k$ there are

$$\vartheta_1^k, \vartheta_2^k \cdots \vartheta_r^k \in \Theta[\mathbf{X}, U]$$

such that

$$\forall m \le r: \quad \Phi_m^k = \lambda_m^k \star \Phi^k \dot{+} \vartheta_m^k \tag{4.38}$$

besides,

$$\mathbf{X}^{\breve{}}\Phi^k = 1_U \tag{4.39}$$

consequently, X^k can by written as a linear combination of the exponential function set $e^{\lambda_1^k \Phi^k}, e^{\lambda_1^k \Phi^k} \cdots e^{\lambda_r^k \Phi^k}$, for the maps $e^{\vartheta_1^k}, e^{\vartheta_2^k} \cdots$ are also members of $\Theta[\mathbf{X}, U]$. If $a_0^k, a_1^k \cdots a_r^k$ are the corresponding coefficients, we have that

$$X^k = \breve{\sum}_{m=1}^{r} a_m^k \star e^{\lambda_m^k \star\, \Phi^k} \tag{4.40}$$

Now, letting $\forall k \le n$, $\forall p \in U$:

$$F^k(p, s) = \breve{\sum}_{m=1}^{r} a_m^k(p) e^{\lambda_m^k(p) s} \tag{4.41}$$

$$g(p, z) = z \tag{4.42}$$

$$h^k(p) = 1_U \tag{4.43}$$

and

$$\mathcal{F}^k(p,z) = \frac{1}{2\pi i}\left(\sum_{m=1}^{r} \frac{a_m^k(p)}{z-\lambda_m^k(p)}\right)^{\smile} \tag{4.44}$$

it is not difficult to see, that these functions satisfy both Condition 1 and Condition 2 for a Bromwich integration path. Thus by virtue of theorem 4.3, each coordinate of the general solution of the corresponding autonomous system (4.34) can be written as a linear combination of the functions

$$1_U, e^{\lambda_1^k \star \left(\Phi^k \hat{+} (t-t_0)\right)}, \cdots e^{\lambda_r^k \star \left(\Phi^k \hat{+} (t-t_0)\right)}$$

with coefficients in the function field $\Theta[\mathbf{X}, U]$. □

Example 4.10. *Consider the space $\mathbb{R}^2$ with the standard smooth structure. Consider the vector $\mathbf{X} = \dfrac{x^1}{x^2}\mathbf{e}_1 + \dfrac{x^2}{x^1}\mathbf{e}_2$ and let (φ, U) be the chart whose domain is $U = \left\{(x^1,x^2) \in \mathbb{R}^2 \mid \left|x^1\right| > a, \left|x^2\right| > b\right\}$ and φ is the identity, where a and b are two fixed positive numbers. In this case, $\Theta[\mathbf{X}, U]$ is nothing but a function field $(\mathcal{K}, +, \cdot)$ whose underlying set $\mathcal{K}$ consists of functions of the form $f(\frac{1}{x^1} - \frac{1}{x^2})$, with domain in $\mathcal{D}(U)$, and $f : \mathbb{R} \to \mathbb{C}$ is a map such that both functions $\Re(f)$ and $\Im(f)$ are analytic.*

Now, consider the corresponding non linear autonomous system

$$\begin{cases} \dfrac{d}{dt}\, x^1(t) = \dfrac{x^1(t)}{x^2(t)} \\ \dfrac{d}{dt}\, x^2(t) = \dfrac{x^2(t)}{x^1(t)} \end{cases} \tag{4.45}$$

It is not difficult to see, that the field $\mathbf{X}$ satisfies the conditions of the preceding theorem, for there are the dependence relations

$$\begin{cases} \mathbf{X}\breve{\ } X^1 = \left(\dfrac{x^1}{x^2}\dfrac{\partial}{\partial x^1} + \dfrac{x^1}{x^2}\dfrac{\partial}{\partial x^1}\right)\dfrac{x^1}{x^2} = -\left(\dfrac{1}{x^1} - \dfrac{1}{x^2}\right)X^1 \\ \mathbf{X}\breve{\ } X^2 = \left(\dfrac{x^1}{x^2}\dfrac{\partial}{\partial x^1} + \dfrac{x^1}{x^2}\dfrac{\partial}{\partial x^1}\right)\dfrac{x^2}{x^1} = \left(\dfrac{1}{x^1} - \dfrac{1}{x^2}\right)X^2 \end{cases} \tag{4.46}$$

In this case, letting $\Phi = \widehat{\Phi} \circ \varphi$ we have that,

$$\widehat{\Phi}^m(x^1,x^2) = (-1)^{m+1}\left(\frac{1}{x^2} - \frac{1}{x^1}\right)^{-1} \ln\left(\frac{x^1}{x^2}\right) \quad (m = 1,2)$$

satisfies the equation

$$\sum_{j=1}^{2} X^j \frac{\partial}{\partial x^j}\, \widehat{\Phi}^m(x^1,x^2) = (-1)^{m+1} \tag{4.47}$$

hence the function $F^m\left(x^1,x^2,\widehat{\Phi}^m(x^1,x^2)\right) = X^m(x^1,x^2)$ defined in Condition 1, with respect to the chart (U,φ), for this example is:

$$F^m\left(x^1,x^2,s\right) = e^{(-1)^m\left(\frac{1}{x^1}-\frac{1}{x^2}\right)s} \quad (m = 1,2) \tag{4.48}$$

accordingly, setting $\mathcal{F}^m(z) = \dfrac{1}{2\pi i(z-(-1)^m)}$, $(m = 1, 2)$ *with the obvious integration path, we have that*

$$F^m\left(x^1, x^2, s\right) = \oint \mathcal{F}^m(z) e^{s\left(\frac{1}{x^1} - \frac{1}{x^2}\right)z} dz \tag{4.49}$$

which satisfies the Condition 2, accordingly (4.45) is equivalent to some 2-parameter linear differential equation family, as it can be deduced from the solution for (4.45),

$$\begin{cases} x^1(t) = -\dfrac{1}{\alpha(x_0^1, x_0^2)} + \dfrac{C(x_0^1, x_0^2)}{\alpha(x_0^1, x_0^2)}\, e^{\alpha(x_0^1, x_0^2)(t-t_0)} \\ x^2(t) = \dfrac{1}{\alpha(x_0^1, x_0^2)} - \dfrac{1}{C(x_0^1, x_0^2)\alpha(x_0^1, x_0^2)}\, e^{-\alpha(x_0^1, x_0^2)(t-t_0)} \end{cases} \tag{4.50}$$

where $\alpha(x_0^1, x_0^2) = \left(\dfrac{1}{x_0^2} - \dfrac{1}{x_0^1}\right)$ *and* $C(x_0^1, x_0^2) = \dfrac{x_0^1}{x_0^2}$, *for instance, the coordinate* $x^1(t)$ *is solution of* $y'(t) - \alpha(c_1, c_2)y(t) - 1 = 0$ *for* $c_1 = x_0^1$ *and* $c_2 = x_0^2$. *It is easy to see, that these functions* $\alpha(x^1, x^2)$ *and* $C(x^1, x^2)$ *satisfy the following equations,*

$$\sum_{j=1}^{2} X^j \frac{\partial}{\partial x_0^j}\, \alpha(x_0^1, x_0^2) = \sum_{j=1}^{2} X^j \frac{\partial}{\partial x_0^j}\left(\frac{1}{x_0^2} - \frac{1}{x_0^1}\right) = 0 \tag{4.51}$$

$$C(x_0^1, x_0^2) = e^{\alpha(x_0^1, x_0^2)\widehat{\Phi}(x_0^1, x_0^2)} = \frac{x_0^1}{x_0^2} \tag{4.52}$$

therefore $\mathbf{X}\check{\ }\Phi^m(p) = 1_U \in \Theta[\mathbf{X}, U]$, *besides, the coefficient* $\alpha(x_0^1, x_0^2)$ *lies in* $\Theta[\mathbf{X}, U]$. *Indeed, every coordinate in (4.50) belongs to a vector space over the function field* $\Theta[\mathbf{X}, U] = \Theta\left[\dfrac{x^1}{x^2}\mathbf{e}_1 + \dfrac{x^2}{x^1}\mathbf{e}_2, U\right]$.

Definition 4.11. *By means of the function field* $\Theta[\mathbf{X}, U]$ *associated to each smooth vector* $\mathbf{X}$, *one can generalize the concept of linear equation system as follows. Assume* $\left(a_j^i\right)_{i,j\le n}$ *to be a matrix such that for each couple* $i, j \le n$ *it is true that*

$$a_j^i \in \Theta\left[\sum_{m=1}^{n}\sum_{k=1}^{n} a_k^m \star x^k \mathbf{e}_m, U\right] \tag{4.53}$$

where $\mathbf{x} = (x^1, x^2 \cdots x^n)$ *are the coordinates of the homeomorphism* φ *of the considered chart* (U, φ). *In this case, the autonomous system*

$$
(4.54) \qquad \begin{cases} \dfrac{d}{dt}x^1(t) = \sum_{m=1}^n a_m^1(x^1(t), x^2(t)\cdots) \star x^m(t) \\ \dfrac{d}{dt}x^2(t) = \sum_{m=1}^n a_m^2(x^1(t), x^2(t)\cdots) \star x^m(t) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{d}{dt}x^n(t) = \sum_{m=1}^n a_m^n(x^1(t), x^2(t)\cdots) \star x^m(t) \end{cases}
$$

that, in general, is not linear over the real (complex) number field, can be regarded as a linear system over the associated function field

$$
\Theta[\mathbf{X}, U] = \Theta\left[\sum_{m=1}^n \sum_{k=1}^n a_k^m \star x^k \mathbf{e}_m, U\right]
$$

Henceforth, an equation system satisfying this property we shall say to be Θ*-linear.*

Theorem 4.12. *If the characteristic polynomial* $P(z) = c_0 \dot{+} c_1 \star z \dot{+} \cdots c_n z^{\smile n}$ *of an* $n \times n$*-matrix* $\left(a_j^i\right)_{i,j\leq n}$ *defining a* Θ*-linear equation system has no multiple root, and every solution of* $P(z) = 0$ *belongs to* $\Theta[\mathbf{X}, U]$*, then the corresponding vector field whose* i*-th coordinate is*

$$
X^i = \sum_{j=1}^n a_j^i \star x^j
$$

satisfies the conditions of theorem 4.8, therefore by virtue of Corollary 4.9 the equation system (4.54) satisfies both Condition 1 and Condition 2, accordingly the corresponding solution can be obtained from the integral transform (4.9).

Proof. Letting $\mathbf{A} = \left(a_j^i\right)_{i,j\leq n}$ and $\mathbf{X} = \mathbf{A} \star \mathbf{x}$, then the f_n^k defined in equation (4.26) satisfy the following relation,

$$
(4.55) \qquad f_r^k = \mathbf{A}^{r+1} \star \mathbf{x}
$$

where $\mathbf{x} = (x^1, x^2 \cdots x^n)$ stand for the coordinates of φ. Now, setting

$$
(4.56) \qquad P(\lambda) = \begin{vmatrix} a_1^1 \dot{-} \lambda & a_2^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2 \dot{-} \lambda & \cdots & a_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ a_1^n & a_2^n & \cdots & a_n^n \dot{-} \lambda \end{vmatrix}^{\smile}
$$

it is a well-known fact that each matrix satisfies its characteristic equation $P(\mathbf{A}) = 0$; therefore, if $c_0, c_1 \cdots c_n$ are the corresponding coefficients, we have that

$$
(4.57) \qquad P(\mathbf{A}) \star \mathbf{A} = c_n \star \mathbf{A}^{n+1} \dot{+} c_{n-1} \star \mathbf{A}^n \cdots \dot{+} c_0 \star \mathbf{A} = 0_U
$$

and by assumption the corresponding eigenvalues belong to the associated function field

$$\Theta\left[\sum_{m=1}^{n}\sum_{k=1}^{n} a_k^m \star x^k \mathbf{e}_m, U\right]$$

Finally, from the former equation it follows immediately, that

$$c_n \star \mathbf{A}^{n+1} \star \mathbf{x} \dotplus c_{n-1} \star \mathbf{A}^n \star \mathbf{x} \cdots \dotplus c_0 \star \mathbf{A} \star \mathbf{x} = 0_U \tag{4.58}$$

and taking into account (4.55), there is a linear dependence relation like (4.22), and the theorem follows. □

Remark 4.13. *Since every function field $\Theta[\mathbf{X}, U]$ contains $(\mathbb{C}_U, \dotplus, \star, ^\flat)$ as a subfield, this being isomorphic to the complex number field, then every linear equation system is also a Θ-linear one; consequently linear equations satisfy the conditions of the preceding theorem.*

References

[1] Dieudonné, J.: Éléments d' analyse, Cahiers scientifiques, fascicule XXXIV, Gauthier-Villars, Paris, 1971.
[2] Kelley, J.L.: General Topology, Springer-Verlag, New York, 1955.
[3] Kobayashi, S. and Nomizu, K.: Foundations of Differential Geometry, Interscience Publish, 1963.
[4] Köthe, G.: Topological Vector Spaces, Springer-Verlag, New York, 1969.
[5] Lang, S.: Algebra, Addison-Wesley, MA, 1965.
[6] Palomar, J. E.: Sufficient Conditions for Equivalence between non Linear and Linear Autonomous Systems, Dynamics of Cont., Discr. and Imp. Sys. (DCDIS), **6**,1999, pp. 237-249.
[7] K. Yosida, K.: Équations Différentielles et Intégrales, Dounod, Paris, 1971.