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On Solutions of Linear Difference Equations with Infinite Delay

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Abstract

The difference equation with infinite delay $(DE)_\varepsilon$ $x_{n+1} = Ax_n + \varepsilon \sum_{k=-\infty}^{n} R_{n-k}x_k$ *is considered. Sufficient conditions for the existence of a unique solution are proved. An* $O(\varepsilon)$*-approximation of this solution for* $-N \leq n \leq N$, $N > 0$, *by the solution of the equation* $(DE)_0$ *is obtained.*

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In [1], Ryabov considers a problem concerning the existence of so-called two-sided solutions of linear integrodifferential equations of the form

$$\frac{dx}{dt} = Ax + \varepsilon \int_{-\infty}^{t} R(t-s)x(s)ds, \tag{1}$$

where ε is a positive small parameter, $t \in \mathbb{R}$, $x \in \mathbb{R}^m$, A is a constant $m \times m$ matrix and R is a matrix of continuous functions satisfying the inequality

$$\|R(t)\| \leq c\frac{e^{-\gamma t}}{t^{1-\alpha}},$$

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where c, γ, α are positive constants and $0 < \alpha < 1$. It is proven there that if $\min\{\mathcal{R}e(\lambda_i) : 1 \leq i \leq m\} > -\gamma$, where $\lambda_1, \ldots, \lambda_m$ are eigenvalues of A, then there is an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and any $x_0 \in \mathbb{R}^m$ the equation (1) has a unique solution x_ε (so-called two-sided solution) defined on the entire axis t for every given initial condition $x_\varepsilon(0) = x_0$ and satisfying $\lim_{\varepsilon\to 0} \|x_\varepsilon - x\|_L = 0$ for any $L > 0$, where $\|x_\varepsilon - x\|_L = \sup\{\|x_\varepsilon(t) - x(t)\| : -L \leq t \leq L\}$, $x(t) = e^{At}x_0$.

The equation (1) has the discretization

$$x_{n+1} = Ax_n + \varepsilon(R_0x_n + R_1x_{n-1} + \cdots + R_kx_{n-k} + \cdots), \tag{2}$$

which is a difference equation with infinite delay. The purpose of this paper is to study (2) in a Banach space E, with $A, R_k \in \mathcal{L}(E)$, the space of continuous linear mappings from E into E $(k = 0, 1, \ldots)$, A is invertible and $A^{-1} \in \mathcal{L}(E)$. Under sufficient conditions, we shall prove the existence of solutions of (2) determined uniquely by the initial value and defined for all integers.

Theorem. *Suppose that the following conditions are satisfied*

$$\|A^{-1}\| < 1, \qquad \|R_0\| \leq 1, \qquad \|R_n\| \leq \frac{\gamma^{-n}}{n^{1-\alpha}}, \quad n = 1, 2, \ldots, \tag{3}$$

where γ, α are constants, $e \leq \gamma$, $0 < \alpha < 1$.

Then there exist a positive constant ε_0 and a unique operator $B \in \mathcal{L}(E)$, such that, for any $\varepsilon \in (0, \varepsilon_0]$ and any $x_0 \in E$, the sequence $x = \{x_n(\varepsilon)\}_{n=-\infty}^{+\infty} = \{(A + \varepsilon B)^n x_0\}_{n=-\infty}^{+\infty}$ is the unique solution of (2) satisfying the condition $x_0(\varepsilon) = x_0$ with $x \in F := \{y = \{y_n\}_{n=-\infty}^{\infty} : y_n \in E, \ \sup\{|y_n| : -\infty < n \leq 0\} < \infty\}$, where $|.|$ is the norm on E. Moreover, for any positive integer N, there exists a positive constant C (depending on ε_0, x_0 and N) such that $\sup\{|x_n(\varepsilon) - A^n x_0| : -N \leq n \leq N\} \leq C\varepsilon$.

Proof. Let us seek the solution of (2) in the form $x = \{x_n(\varepsilon)\}_{n=-\infty}^{+\infty} = \{(A + \varepsilon B)^n x_0\}_{n=-\infty}^{+\infty}$, where $B \in \mathcal{L}(E)$ is such that $AB = BA$ and $A + \varepsilon B$ is invertible. The matrix B satisfy the equation

$$B = P_\varepsilon(B) := R_0 + R_1(A + \varepsilon B)^{-1} + \cdots + R_k(A + \varepsilon B)^{-k} + \cdots. \tag{4}$$

Consider the mapping $P_\varepsilon : \mathcal{D} \to \mathcal{L}(E)$, defined by (4), where $\mathcal{D} = \{B \in \mathcal{L}(E) : AB = BA \text{ and } \|B\| \leq 1\}$, $\|B\| := \sup\{|Bx| : |x| \leq 1\}$. If $B_1, B_2 \in \mathcal{D}$

then

$$\begin{aligned}\|P_\varepsilon(B_1)-P_\varepsilon(B_2)\| &= \|R_1[(A+\varepsilon B_1)^{-1}-(A+\varepsilon B_2)^{-1}]+R_2[(A+\varepsilon B_1)^{-2}\\ &\quad -(A+\varepsilon B_2)^{-2}]+\cdots+R_k[(A+\varepsilon B_1)^{-k}-(A+\varepsilon B_2)^{-k}]+\cdots\|.\end{aligned}$$

Since $(A+\varepsilon B_i)=A(I+\varepsilon A^{-1}B_i)$ and $AB_i=B_iA$, $i=1,2$ where I is the unit matrix, we have

$$\begin{aligned}\|P_\varepsilon(B_1)-P_\varepsilon(B_2)\| \le \|R_1\|\|A^{-1}\|\|(I+\varepsilon A^{-1}B_1)^{-1}-(I+\varepsilon A^{-1}B_2)^{-1}\|\\ +\|R_2\|\|A^{-1}\|^2\|(I+\varepsilon A^{-1}B_1)^{-2}-(I+\varepsilon A^{-1}B_2)^{-2}\|+\cdots\\ +\|R_k\|\|A^{-1}\|^k\|(I+\varepsilon A^{-1}B_1)^{-k}-(I+\varepsilon A^{-1}B_2)^{-k}\|+\cdots.\end{aligned} \tag{5}$$

We have the following estimate

$$\begin{aligned}&\|(I+\varepsilon A^{-1}B_1)^{-k}-(I+\varepsilon A^{-1}B_2)^{-k}\| = \|[(I+\varepsilon A^{-1}B_1)^{-1}]^k-[(I+\varepsilon A^{-1}B_2)^{-1}]^k\|\\ &\quad\le \|(I+\varepsilon A^{-1}B_1)^{-1}-(I+\varepsilon A^{-1}B_2)^{-1}\|\|[(I+\varepsilon A^{-1}B_1)^{-1}]^{k-1}\\ &\qquad +[(I+\varepsilon A^{-1}B_1)^{-1}]^{k-2}(I+\varepsilon A^{-1}B_2)^{-1}+\cdots+[(I+\varepsilon A^{-1}B_2)^{-1}]^{k-1}\|\\ &\quad\le \|(I+\varepsilon A^{-1}B_1)^{-1}-(I+\varepsilon A^{-1}B_2)^{-1}\|\{\|(I+\varepsilon A^{-1}B_1)^{-1}\|^{k-1}\\ &\qquad +\|(I+\varepsilon A^{-1}B_1)^{-1}\|^{k-2}\|(I+\varepsilon A^{-1}B_2)^{-1}\|+\cdots+\|(I+\varepsilon A^{-1}B_2)^{-1}\|^{k-1}\}.\end{aligned}$$

As for $B_1,B_2\in\mathcal{D}$ and $\varepsilon\in(0,\varepsilon_1]$, where $\varepsilon_1<1/\|A^{-1}\|$, we have

$$\begin{aligned}\|(I+\varepsilon A^{-1}B_i)^{-1}\| &= \|I-\varepsilon(A^{-1}B_i)+\varepsilon^2(A^{-1}B_i)^2-\cdots\|\\ &\le 1+\varepsilon\|A^{-1}\|+\varepsilon^2\|A^{-1}\|^2+\cdots=\frac{1}{1-\varepsilon\|A^{-1}\|},\end{aligned}$$

we obtain the inegality

$$\begin{aligned}&\|(I+\varepsilon A^{-1}B_1)^{-k}-(I+\varepsilon A^{-1}B_2)^{-k}\|\\ &\qquad\le \|(I+\varepsilon A^{-1}B_1)^{-1}-(I+\varepsilon A^{-1}B_2)^{-1}\|\frac{k}{(1-\varepsilon\|A^{-1}\|)^{k-1}}.\end{aligned} \tag{6}$$

Thus, for all $B_1,B_2\in\mathcal{D}$ and $\varepsilon\in(0,\varepsilon_1]$, from (5) we obtain the estimate

$$\begin{aligned}&\|P_\varepsilon(B_1)-P_\varepsilon(B_2)\| \le \{\|R_1\|\|A^{-1}\|+\|R_2\|\|A^{-1}\|^2\frac{2}{1-\varepsilon\|A^{-1}\|}+\cdots\\ &\quad +\|R_k\|\|A^{-1}\|^k\frac{k}{(1-\varepsilon\|A^{-1}\|)^{k-1}}+\cdots\}\|(I+\varepsilon A^{-1}B_1)^{-1}-(I+\varepsilon A^{-1}B_2)^{-1}\|\\ &\qquad \le S_1\|(I+\varepsilon A^{-1}B_1)^{-1}-(I+\varepsilon A^{-1}B_2)^{-1}\|,\end{aligned} \tag{7}$$

where $S_1 = (1-\varepsilon\|A^{-1}\|)\sum_{k=1}^{\infty} k\|R_k\|\frac{\|A^{-1}\|^k}{(1-\varepsilon\|A^{-1}\|)^k}$. As $\|A^{-1}\| < \gamma$ then $\gamma^{-1} < 1/\|A^{-1}\|$. Choose ε_2 sufficiently small such that $\gamma^{-1}+\varepsilon < 1/\|A^{-1}\|$ for $\varepsilon \in (0,\varepsilon_2]$. We obtain

$$\frac{\gamma^{-1}\|A^{-1}\|}{1-\varepsilon\|A^{-1}\|} < 1 \quad \text{for } \varepsilon \in (0,\varepsilon_2]. \tag{8}$$

The conditions (3) and (8) insure the convergence of the series S_1.

The following estimate is needed

$$\begin{aligned} &\|(I+\varepsilon A^{-1}B_1)^{-1} - (I+\varepsilon A^{-1}B_2)^{-1}\| \\ &= \|(I-\varepsilon(A^{-1}B_1)+\varepsilon^2(A^{-1}B_1)^2-\cdots)-(I-\varepsilon(A^{-1}B_2)+\varepsilon^2(A^{-1}B_2)^2-\cdots)\| \\ &\le \varepsilon\|A^{-1}B_1 - A^{-1}B_2\| + \varepsilon^2\|(A^{-1}B_1)^2-(A^{-1}B_2)^2\| + \cdots \\ &\quad +\varepsilon^k\|(A^{-1}B_1)^k-(A^{-1}B_2)^k\| + \cdots. \end{aligned} \tag{9}$$

The mean value theorem yields

$$\begin{aligned} &\|(A^{-1}B_1)^k-(A^{-1}B_2)^k\| \le \|A^{-1}\|^k\|B_1^k - B_2^k\| \\ &\le \|A^{-1}\|^k \sup\{\|k[(1-\theta)B_1+\theta B_2]^{k-1}\| : 0 \le \theta \le 1\}\|B_1-B_2\| \\ &\le k\|A^{-1}\|^k\|B_1-B_2\|. \end{aligned} \tag{10}$$

From (7), (9) and (10) it follows that for all $B_1, B_2 \in \mathcal{D}$ and $\varepsilon \in (0, \min\{\varepsilon_1,\varepsilon_2\}]$, we have

$$\|P_\varepsilon(B_1) - P_\varepsilon(B_2)\| \le \varepsilon\|A^{-1}\|S_1S_2\|B_1-B_2\|, \tag{11}$$

where $S_2 = \sum_{k=0}^{\infty}(k+1)\varepsilon^k\|A^{-1}\|^k$ is a convergent series. Therefore the inequality (11) implies that if $\varepsilon \in (0,\varepsilon_3]$, where $\varepsilon_3 < \frac{1}{\|A^{-1}\|S_1S_2}$, then the mapping P_ε is a contraction on $\mathcal{D}$ and thus it has a unique fixed point in $\mathcal{D}$. So if we set $x_n(\varepsilon) = (A+\varepsilon B)^n x_0$, where B is the unique fixed point of P_ε, the sequence $x = \{x_n(\varepsilon)\}_{n=-\infty}^{\infty}$ is a solution of (2) satisfying the condition $x_0(\varepsilon) = x_0$. As $\|A^{-1}\| < 1$, choose ε_4 sufficiently small, such that $\|(A+\varepsilon B)^{-p}\| \le \|(A+\varepsilon B)^{-1}\|^p < 1$ for all positive integer p and $\varepsilon \in (0,\varepsilon_4]$. This implies that $\sup\{|x_n(\varepsilon)| : -\infty < n \le 0\} \le |x_0| < \infty$, i.e. $x = \{x_n(\varepsilon)\}_{n=-\infty}^{\infty} \in F$.

We shall prove the uniqueness of the solution in F of (2).

Consider the space

$$\tilde{F} := \{x = \{x_n\}_{n=-\infty}^{0} : x_n \in E,\ \sup\{|x_n| : -\infty < n \le 0\} < \infty\}$$

which is a Banach space with the norm $\|x\| := \sup\{|x_n| : -\infty < n \le 0\}$.

Let $x_n = A^n y_n$, $-\infty < n \le 0$. Then by substituting into (2), we obtain

$$y_{n+1} = y_n + \varepsilon A^{-(n+1)}(R_0 A^n y_n + R_1 A^{n-1} y_{n-1} + \cdots + R_k A^{n-k} y_{n-k} + \cdots).$$

Observe that

$$\begin{array}{rcl} y_0 - y_{-1} & = & \varepsilon(R_0 A^{-1} y_{-1} + R_1 A^{-2} y_{-2} + \cdots + R_k A^{-1-k} y_{-1-k} + \cdots), \\ y_{-1} - y_{-2} & = & \varepsilon A(R_0 A^{-2} y_{-2} + R_1 A^{-3} y_{-3} + \cdots + R_k A^{-2-k} y_{-2-k} + \cdots), \\ \vdots & & \\ y_{-(p-1)} - y_{-p} & = & \varepsilon A^{p-1}(R_0 A^{-p} y_{-p} + R_1 A^{-p-1} y_{-p-1} + \cdots + R_k A^{-p-k} y_{-p-k} + \cdots). \end{array}$$

By adding these equations, we get

$$\begin{array}{rcl} y_0 - y_{-p} & = & (y_0 - y_{-1}) + (y_{-1} - y_{-2}) + \cdots + (y_{-(p-1)} - y_{-p}) \\ & = & \varepsilon A^{p-1}(R_0 A^{-p} y_{-p} + R_1 A^{-p-1} y_{-p-1} + \cdots + R_k A^{-p-k} y_{-p-k} + \cdots) \\ & & + \cdots + \varepsilon A(R_0 A^{-2} y_{-2} + R_1 A^{-3} y_{-3} + \cdots + R_k A^{-2-k} y_{-2-k} + \cdots) \\ & & +\varepsilon(R_0 A^{-1} y_{-1} + R_1 A^{-2} y_{-2} + \cdots + R_k A^{-1-k} y_{-1-k} + \cdots). \end{array}$$

A return to the initial variable gives

$$\begin{array}{rcl} A^{-p} x_0 - x_{-p} & = & \varepsilon A^{-1}(R_0 x_{-p} + R_1 x_{-p-1} + \cdots + R_k x_{-p-k} + \cdots) \\ & & + \cdots + \varepsilon A^{-p+1}(R_0 x_{-2} + R_1 x_{-3} + \cdots + R_k x_{-2-k} + \cdots) \\ & & +\varepsilon A^{-p}(R_0 x_{-1} + R_1 x_{-2} + \cdots + R_k x_{-1-k} + \cdots). \end{array}$$

Define the mapping

$$Q_\varepsilon : \tilde{F} \to \{x = \{x_n\}_{n=-\infty}^{0}, x_n \in E\},$$

by

$$\begin{array}{rcl} (Q_\varepsilon x)_{-p} & = & \varepsilon A^{-1}(R_0 x_{-p} + R_1 x_{-p-1} + \cdots + R_k x_{-p-k} + \cdots) + \cdots \\ & & +\varepsilon A^{-p}(R_0 x_{-1} + R_1 x_{-2} + \cdots + R_k x_{-1-k} + \cdots), \end{array}$$

for all positive integer p.

If $x = \{x_n\}_{n=-\infty}^{0} \in \tilde{F}$, we have for all positive integer p the estimate

$$\begin{aligned}
|(Q_\varepsilon x)_{-p}| &\leq \varepsilon \|x\| \{ \|A^{-1}\| (\|R_0\| + \|R_1\| + \cdots + \|R_k\| + \cdots) + \cdots \\
&\qquad + \|A^{-1}\|^p (\|R_0\| + \|R_1\| + \cdots + \|R_k\| + \cdots) \} \\
&\leq \frac{\varepsilon \|x\|}{1 - \|A^{-1}\|} (1 + \gamma^{-1} + \cdots + \frac{\gamma^{-k}}{k^{1-\alpha}} + \cdots) \\
&\leq \frac{\varepsilon \|x\|}{1 - \|A^{-1}\|} \int_0^\infty e^{-t} t^{\alpha-1} dt = \frac{\varepsilon \|x\|}{1 - \|A^{-1}\|} \Gamma(\alpha),
\end{aligned}$$

where $\Gamma(\alpha)$ is Euler's Gamma-function. Thus, for $x \in \tilde{F}$, we have $Q_\varepsilon x \in \tilde{F}$. Since Q_ε is linear, it satisfies

$$\|Q_\varepsilon x_1 - Q_\varepsilon x_2\| = \|Q_\varepsilon (x_1 - x_2)\| \leq \frac{\varepsilon \Gamma(\alpha)}{1 - \|A^{-1}\|} \|x_1 - x_2\|$$

for all $x_1, x_2 \in \tilde{F}$. This implies that if $\varepsilon \in (0, \varepsilon_5]$, where $\varepsilon_5 < \dfrac{1 - \|A^{-1}\|}{\Gamma(\alpha)}$, then the mapping Q_ε is a contraction on $\tilde{F}$ and thus has a unique fixed point $x^* \in \tilde{F}$. Since Q_ε is linear, we conclude that $x^* = 0$.

Let $x = \{x_n\}_{n=-\infty}^{\infty}$, $y = \{y_n\}_{n=-\infty}^{\infty} \in F$ be two solutions of (2), satisfying the condition $y_0 = x_0$ and let $x^* = \{y_n - x_n\}_{n=-\infty}^{0}$. The sequence x^* is in $\tilde{F}$ and is the fixed point of Q_ε and therefore $x^* = 0$, i.e. $y_n = x_n$, for all $n \leq 0$. This prove the uniqueness of the solution of (2) for all $n \leq 0$.

On the other hand, from (2) and for $n > 0$ we have

$$|x_n - y_n| = \varepsilon \left| \sum_{i=0}^{n-1} A^{n-i-1} [R_0(x_i - y_i) + R_1(x_{i-1} - y_{i-1}) + \cdots + R_k(x_{i-k} - y_{i-k}) + \cdots] \right|.$$

As $y_k = x_k$ for all integer $k \leq 0$, the above equality gives

$$\begin{aligned}
|x_n - y_n| &\leq \varepsilon \{ \|A\|^{n-1} |x_0 - y_0| + \|A\|^{n-2} [|x_1 - y_1| + \gamma^{-1} |x_0 - y_0|] \\
&\qquad + \cdots + [|x_{n-1} - y_{n-1}| + \gamma^{-1} |x_{n-2} - y_{n-2}| + \cdots + \frac{\gamma^{-(n-2)}}{(n-2)^{1-\alpha}} |x_0 - y_0|] \}.
\end{aligned}$$

From this inequality, it follows that

$$|x_1 - y_1| \leq \varepsilon |x_0 - y_0| = 0,$$

$$|x_2 - y_2| \leq \varepsilon \{ \|A\| [|x_0 - y_0| + |x_1 - y_1|] \} = 0,$$

etc. Therefore, by induction we have that $y_n = x_n$ for all $n > 0$.

Consequently, the solution in F of (2) is uniquely defined by the sequence $x = \{x_n(\varepsilon)\}_{n=-\infty}^{+\infty} = \{(A+\varepsilon B)^n x_0\}_{n=-\infty}^{+\infty}$.

Finally, we shall prove that for any positive integer N, there exists a positive constant C, depending on ε_0, x_0 and N, where $\varepsilon_0 = \min\{\varepsilon_i : 1 \leq i \leq 5\}$, such that $\sup\{|x_n(\varepsilon) - A^n x_0| : -N \leq n \leq N\} \leq C\varepsilon$.

Let n be an integer such that $0 \leq n < N$. By use of the binomial formula, we deduce the following estimate

$$\begin{aligned} |x_n(\varepsilon) - A^n x_0| &\leq |[(A+\varepsilon B)^n - A^n]x_0| \leq \|(A+\varepsilon B)^n - A^n\||x_0| \\ &\leq \varepsilon|x_0|\{C_n^{n-1}\|A\|^{n-1}\|B\| + \varepsilon C_n^{n-2}\|A\|^{n-2}\|B\|^2 + \cdots + \varepsilon^{n-1}\} \\ &\leq c_1\varepsilon, \end{aligned}$$

where

$$c_1 = c_1(\varepsilon_0, x_0, N) = |x_0|\{C_N^{N-1}\|A\|^{N-1} + \varepsilon_0 C_N^{N-2}\|A\|^{N-2} + \cdots + \varepsilon_0^{N-1}\},$$

and

$$C_n^p = \frac{n!}{p!(n-p)!} \quad \text{for } 0 \leq p \leq n \leq N.$$

For $-N \leq n < 0$, we set $n = -k$. Then we have

$$\begin{aligned} |x_n(\varepsilon) - A^n x_0| &\leq |[(A+\varepsilon B)^n - A^n]x_0| \leq \|(A+\varepsilon B)^n - A^n\||x_0| \\ &= |x_0|\|(A+\varepsilon B)^{-k} - A^{-k}\| = |x_0|\|[(A+\varepsilon B)^{-1}]^k - [A^{-1}]^k\| \\ &\leq |x_0|\|(A+\varepsilon B)^{-1} - A^{-1}\|\{[(A+\varepsilon B)^{-1}]^{k-1} \\ &\qquad + [(A+\varepsilon B)^{-1}]^{k-2}A^{-1} + \cdots + [A^{-1}]^{k-1}\} \\ &\leq |x_0|\|A^{-1}\|\|(I+\varepsilon A^{-1}B)^{-1} - I\|\{\|(A+\varepsilon B)^{-1}\|^{k-1} \\ &\qquad + \|A^{-1}\|\|(A+\varepsilon B)^{-1}\|^{k-2} + \cdots + \|A^{-1}\|^{k-1}\}. \end{aligned} \tag{12}$$

As we have seen above, $\|(A+\varepsilon B)^{-1}\|^k < 1$, for all positive integer k and $\varepsilon \in (0, \varepsilon_0] \subset (0, \varepsilon_4]$. On the other hand, we have

$$\begin{aligned} \|(I+\varepsilon A^{-1}B)^{-1} - I\| &= \|(I - (\varepsilon A^{-1}B) + (\varepsilon A^{-1}B)^2 - \cdots) - I\| \\ &\leq \varepsilon\|A^{-1}\|\{1 + \varepsilon\|A^{-1}\| + \varepsilon^2\|A^{-1}\|^2 + \cdots\} \\ &\leq \varepsilon\frac{\|A^{-1}\|}{1 - \varepsilon\|A^{-1}\|}. \end{aligned}$$

Therefore, from (12) we obtain

$$\begin{aligned}
|x_n(\varepsilon) - A^n x_0| &\leq \varepsilon |x_0| \frac{\|A^{-1}\|^2}{1-\varepsilon\|A^{-1}\|}\{1 + \|A^{-1}\| + \cdots + \|A^{-1}\|^{k-1}\} \\
&\leq \varepsilon |x_0| \frac{\|A^{-1}\|^2}{1-\varepsilon\|A^{-1}\|} \cdot \frac{1}{1-\|A^{-1}\|} \\
&\leq c_2 \varepsilon,
\end{aligned}$$

where

$$c_2 = c_2(\varepsilon_0, x_0) = |x_0| \frac{\|A^{-1}\|^2}{1-\varepsilon_0\|A^{-1}\|} \cdot \frac{1}{1-\|A^{-1}\|}.$$

Hence, the following inequality holds for $C = \min\{c_1, c_2\}$

$$\sup\{|x_n(\varepsilon) - A^n x_0| : -N \leq n \leq N\} \leq C\varepsilon.$$

This completes the proof of the theorem.

References

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