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NEGOTIATIONS UNDER UNCERTAINTY

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Abstract. A one-step solution for two-person conflicts is examined when the break-down probabilities depend on the goodness of the offers and the relative power indices of the players. In the one-step solution each player maximizes his/her expected payoff. The uncertainty in the power index is characterized by its CDF and fuzzy membership function. Based on the monotonicity of solution simple algorithms are suggested to obtain the CDF and membership function of the solution in closed forms. A simple example illustrates the methodology.

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1. INTRODUCTION

Before 1950, only few negotiation models had been developed and most were based on economic or accounting theory. Since 1950, negotiation and bargaining became an independent and important research area. For example, Nash's papers laid the foundation for the modern theoretical works in negotiation and bargaining ([9], [10], [11]). His first paper

stated that there is more than one way that two individuals in a bargaining situation could collaborate for mutual benefits. His second paper argued that non-cooperative game theory should be used to study the bargaining and cooperation in games. His third paper suggested the idea to relate the axiomatic solutions to the equilibria of strategic models, which implies that in order to understand the outcomes of bargaining, it is appropriate to model the bargaining as a noncooperative game and solve for its equilibria. This procedure is known as the *Nash program*.

There were several bargaining or negotiation models developed later. The non-symmetric Nash solution was introduced in [5], and solutions satisfying individual

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monotonicity have been introduced in [8]. This solution has been further generalized in [3] as the reference function solution. Among the mostly known solution concepts we mention the egalitarian solution [7], the superadditive solution [12], the equal sacrifice solution [4], and the equal area solution [2]. Nash has shown that the equilibrium set coincides with the set of Pareto solutions if the problem is considered as a two-person noncooperative game. One might consider bargaining as a single player decision problem, when the strategy selection of the other player is considered random. If uniform distribution is assumed and each player maximizes his/her expected payoff, then the optimal selection is equivalent to the Nash solution. Similar equivalence holds for applying the principle of Zeuthen [17] in determining the order in which concessions are made. One of the most popular bargaining methods is the alternating offer process [13], which has been extended in [6]. In the model of [3] the payoffs of the players depend not only on their offers to themselves but also on how generous their offers are.

The bargaining process to be examined in this paper is a combination of the ideas presented in [13] and [3], when the break-down probabilities depend on how good the offers are, and also on the relative power of the participants. In [16] the uncertainty of the Pareto frontier has been examined. In addition, time-variant negotiations have been discussed. In this paper the uncertainty in the relative power index is our main concern. Randomization as well as fuzzification will be used to predict the effect of this uncertainty in the solution of the conflict.

2. THE BARGAINING SOLUTION

A two-person conflict is identified mathematically with a pair $(S, \mathbf{d})$, where $S \subseteq R^2$ is a convex, closed, comprehensive feasible set, and $\mathbf{d} \in S$ is the disagreement payoff vector. Any point $(x_1, x_2) \in S$ can be considered as a simultaneous payoff pair for the two players. Without loss of generality we may assume that $\mathbf{d} = \mathbf{0}$, otherwise we may introduce the new variables $x_i^1 = x_i - d_i$ $(i = 1, 2)$. It is also assumed that there exists a concave, strictly decreasing, continuously differentiable function $g\colon [0, A] \mapsto R$ such that $g(A) = 0$. Since no person accepts an offer worse than the disagreement payoff, we may assume that $S = \{(x_1, x_2) \mid 0 \le x_1 \le A, 0 \le x_2 \le g(x_1)\}$. The Pareto frontier is therefore the graph of function g. Let (x_1^0, x_2^0) and (y_1^0, y_2^0) denote the initial positions of the two players, where $A \ge x_1^0 > y_1^0 \ge 0$, $x_2^0 = g(x_1^0)$, and $y_2^0 = g(y_1^0)$. If player 1 presents an offer (z_1, z_2), then we assume that it will be accepted by player 2 with probability $[(z_2 - x_2^0)/(y_2^0 - x_2^0)]^{P_2}$, where P_2 is the power of player 2. Notice that the selection $z_2 = x_2^0$ leads to certain rejection, and offer $z_2 = y_2^0$ will result in certain acceptance. Between these extreme values the acceptance probability increases if a better offer is presented. The optimal offer of player 1 can be obtained by maximizing his/her expected payoff $E_1(z_1) = z_1 \cdot [(g(z_1) - x_2^0)/(y_2^0 - x_2^0)]^{P_2}$. Similarly, if player 2 presents an offer (z_1, z_2), then it will be accepted with probability $[(z_1 - y_1^0)/(x_1^0 - y_1^0)]^{P_1}$, where P_1 is the power of player 1. Therefore the optimal offer of player 2 is obtained by maximizing his/her expected payoff $E_2(z_1) = g(z_1) \cdot [(z_1 - y_1^0)/(x_1^0 - y_1^0)]^{P_1}$. It can be proven (see [15]) that under the above assumptions both players have a unique payoff maximizing offers with arbitrary values of $P_1, P_2 > 0$. Finally, let α denote the relative power of player 2 above player 1. The solution of the conflict is a point (z_1, z_2) of the Pareto frontier that is optimal for both players with some values of P_1 and P_2 such that $\alpha = P_2/P_1$. It can be proved (see, for example [14]) that the solution is the unique solution of the single variable monotonic equation

$$\frac{x_2^0 - g(z_1)}{z_1 g'(z_1)} - \alpha \frac{y_1^0 - z_1}{\frac{g(z_1)}{g'(z_1)}} = 0 \tag{1}$$

inside the interval $[y_1^0, x_1^0]$. It is easy to see that in the special case when $x_2^0 = y_1^0 = 0$ this equation can be further simplified as

$$g(z_1) + \alpha z_1 g'(z_1) = 0. \tag{2}$$

Notice that equation (2) shows that in this special case the bargaining solution is equivalent to a nonsymmetric Nash-solution ([5]).

For the sake of simplicity let $h(z_1, \alpha)$ denote the left-hand side of equation (1). Under the assumptions given in the beginning of this section, h is strictly decreasing in z_1 with any fixed $\alpha > 0$, and is also strictly decreasing in α with arbitrary fixed $z_1 \in (y_1^0, x_1^0]$. Therefore there is a pair of unique functions

$$z_1 = h_1(\alpha) \quad \text{and} \quad \alpha = h_2(z_1),$$

where h_1 and h_2 are inverses of each other. Function h_1 can be computed only numerically; for each value of α, the unique root of equation (1) has to be determined, however function h_2 can be obtained in a closed form:

$$\alpha = h_2(z_1) = \frac{[x_2^0 - g(z_1)]g(z_1)}{z_1[g'(z_1)]^2(y_1^0 - z_1)}. \tag{3}$$

Next we show that both functions h_1 and h_2 are strictly decreasing. Consider first function h_1 and assume that with some $\alpha < \alpha'$, $z_1 = h_1(\alpha) \le z_1' = h_1(\alpha')$. Then $0 = h(z_1, \alpha) > h(z_1, \alpha') \ge h(z_1', \alpha') = 0$, which is an obvious contradiction. Since h_2 is the inverse of h_1, it is also strictly decreasing.

The above discussion is next illustrated by a simple example.

Example 1. Assume that two players want to share a pie, that is, $g(x_1) = 1 - x_1$ with $A = 1$. Assume that the initial positions are $(y_1^0, y_2^0) = (0, 1)$ and $(x_1^0, x_2^0) = (1, 0)$. Then equation (1) has the special form

$$\frac{-1 + z_1}{-z_1} - \alpha \frac{-z_1}{\frac{1 - z_1}{-1}} = 0$$

which can be simplied as

$$(1 - z_1)^2 = \alpha z_1^2.$$

Therefore

$$z_1 = \frac{1}{1 + \sqrt{\alpha}} \quad \text{and} \quad \alpha = \frac{(1 - z_1)^2}{z_1^2}.$$

3. STOCHASTIC ANALYSIS

Assume now that α is a random variable with a continuous cumulative distribution function F. Then $z_1 = h_1(\alpha)$ is also a random variable, and its cumulative distribution function can be obtained as follows:

$$\begin{aligned} G(Z) &= P(z_1 \le Z) = P(h_1(\alpha) \le Z) = P(\alpha \ge h_2(Z)) \\ &= 1 - F(h_2(Z)). \end{aligned} \tag{4}$$

Hence $G(Z)$ is obtained in a closed form giving the cumulative distribution function of the solution z_1 of the conflict. The expectation and variance of z_1 can then be determined in a routine method.

Example 2. Consider again the conflict of Example 1 and assume that the probability density function of α is triangular, as shown in Figure 1. Then simple integration shows that the cumulative distribution function is as follows:

$$F(\alpha) = \begin{cases} 0, & \text{if } \alpha \leq 1 \\ \frac{(\alpha-1)^2}{2}, & \text{if } 1 < \alpha \leq 2 \\ 1 - \frac{(3-\alpha)^2}{2}, & \text{if } 2 < \alpha \leq 3 \\ 1, & \text{if } \alpha > 3 \end{cases}$$

This function is illustrated in Figure 2. In our case $h_2(Z) = \frac{(1-Z)^2}{Z^2}$, therefore equation (4) implies that:

$$G(Z) = \begin{cases} 0, & \text{if } Z \leq \frac{\sqrt{3}-1}{2} \\ \frac{(2Z^2+2Z-1)^2}{2Z^4}, & \text{if } \frac{\sqrt{3}-1}{2} \leq Z < \sqrt{2}-1 \\ 1 - \frac{(1-2Z)^2}{2Z^4}, & \text{if } \sqrt{2}-1 \leq Z < \frac{1}{2} \\ 1, & \text{if } Z \geq \frac{1}{2} \end{cases}$$

This is a continuous cumulative distribution function showing that the Z values from the open interval $\left(\frac{\sqrt{3}-1}{2}, \frac{1}{2}\right)$ might occur with positive probabilities. It is also interesting to note that $G(\sqrt{2}-1) = \frac{1}{2}$. The graph of G is shown in Figure 3.

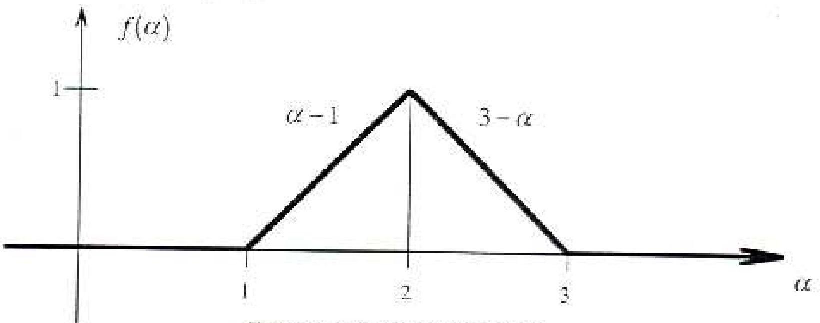

Figure 1. Density Function of α

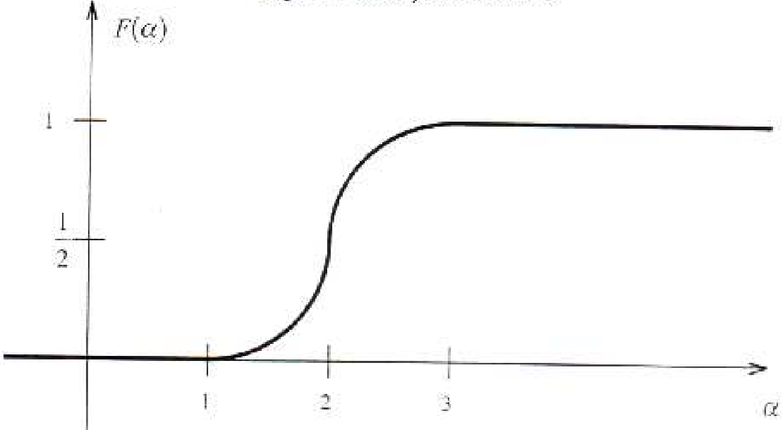

Figure 2. Cumulative Distribution Function of α

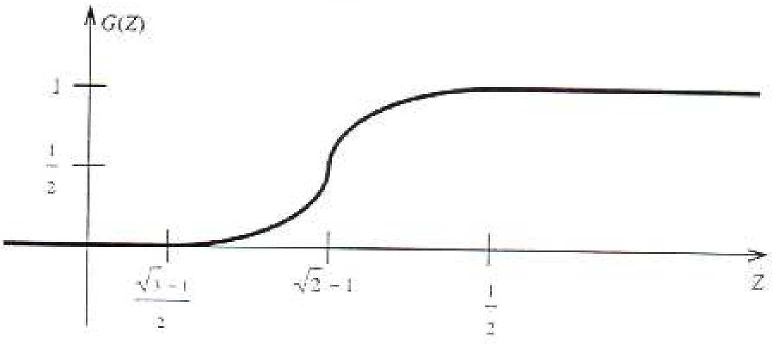

Figure 3. Cumulative Distribution Function of the Solution

4. FUZZY ANALYSIS

In the approach of fuzzification the membership function of parameter α has to be specified as the characterization of its uncertainty. In most applications a triangular or trapezoidal membership function is selected. The membership function of the solution can be then determined in the following way. For each membership level h, let $\alpha_{\min}(h)$ and $\alpha_{\max}(h)$ denote the minimal and maximal α values with membership values at least h. Since Z strictly decreases in α, $h_1(\alpha_{\max}(h))$ and $h_1(\alpha_{\min}(h))$ give the corresponding minimal and maximal Z values for the same membership level h.

Example 3. Consider again the conflict of the previous examples and assume now that α has a triangular membership function in Figure 4.

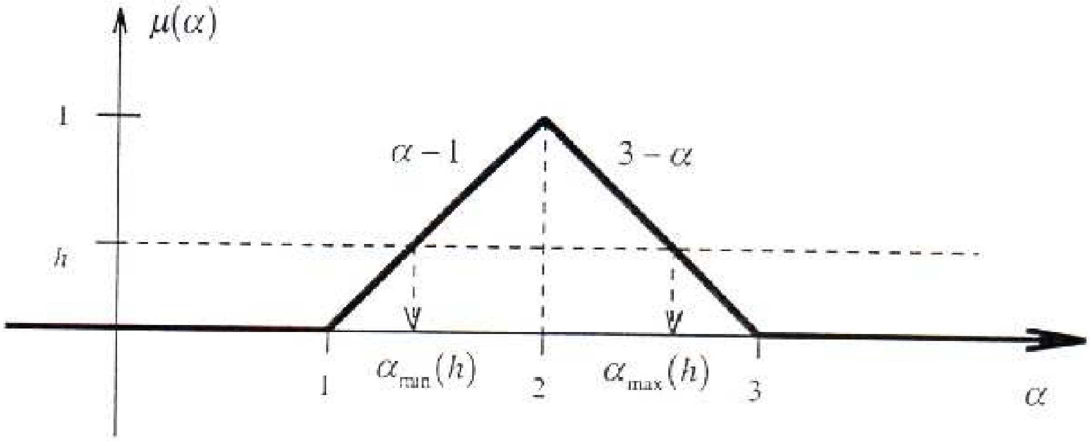

Figure 4. Membership Function $\mu(\alpha)$

For each membership level $h \in (0, 1)$ we have

$$\alpha_{\min}(h) = 1 + h \quad \text{and} \quad \alpha_{\max}(h) = 3 - h$$

therefore

$$Z_{\min}(h) = h_1(\alpha_{\max}(h)) = \frac{1}{1 + \sqrt{3 - h}}$$

and

$$Z_{\max}(h) = h_1(\alpha_{\min}(h)) = \frac{1}{1 + \sqrt{1 + h}}.$$

The resulted membership function $\mu(Z)$ is shown in Figure 5. This membership function is the fuzzy characteristic of the uncertainty of the result of the conflict.

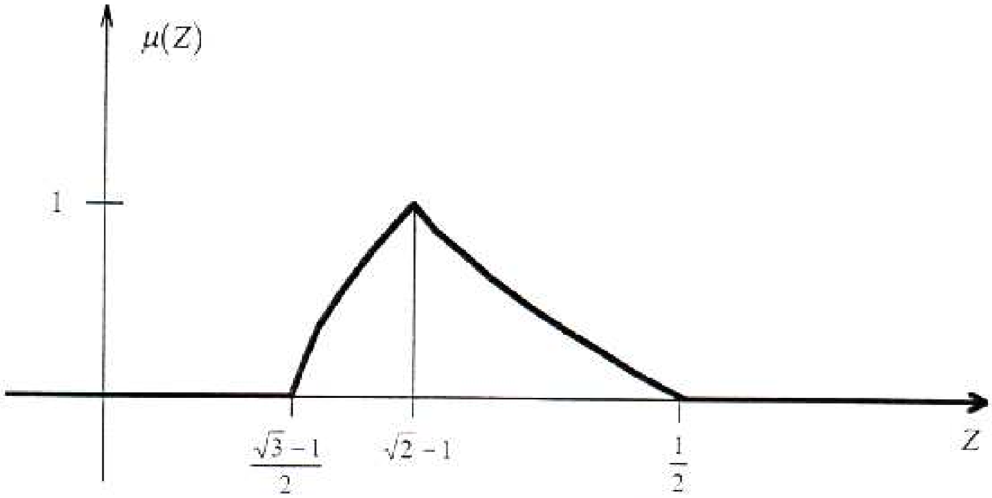

Figure 5. Membership Function $\mu(Z)$

5. CONCLUSIONS

The uncertainty in the relative power index was modelled by randomization and fuzzification. In the first case the cumulative distribution function of the uncertain parameter is assumed to be known, and the same for the solution of the conflict is then determined in a closed form expression. The resulted distribution function illustrates the uncertainty of the solution, and allows us to compute such characteristics as expectation and variance. In the case of fuzzification it is assumed that the membership function of the uncertain parameter is given, and the membership funciton of the solution is then determined by using the transformation model. The resulted membership function illustrates the uncertainty of the solution, and by using defuzzification (such as center of mass) a crisp solution is obtained which takes the uncertainty of the model parameter.

REFERENCES

1 Anbar, D. and Kalai, E.: A One-Shot Bargaining Problem, *International J. of Game Theory* **7** (1978), 13–15.
2 Anbarci, N.: Noncooperative Foundations of the Area Monotonic Solution, *Quarterly J. of Econ.* **108** (1993), 245–258.
3 Anbarci, N.: Reference Functions and Balanced Concessions in Bargaining, *Canadian J. of Economics* **28** (1995), 675–682.
4 Chun, Y.: The Equal-Loss Principle for Bargaining Problems, *Econ. Letters* **26** (1998), 103–106.
5 Harsanyi, J.C. and Selten, R.: A Generalized Nash Solution for Two-Person Bargaining Games with Incomplete Information, *Management Science* **18**, Part II (1972), 80–106.

6 Howard, J.V.: A Social Choice Rule and Its Implementation in Perfect Equilibrium, *J. of Econ. Theory* **56** (1992), 142–159.
7 Kalai, E.: Proportional Solutions to Bargaining Situations: Interpersonal Utility Comparisons, *Econometrica* **45** (1977), 1623–1630.
8 Kalai, E. and Smorodinsky, M.: Other Solutions to Nash's Bargaining Problem, *Econometrica*, **43** (1975), 513–518.
9 Nash, J.F.: The Bargaining Problem, *Econometrica* **18** (1950), 155–162.
10 Nash, J.F.: Non-cooperative Games, *Ann. Math.* **54** (1951), 286–295.
11 Nash, J.F.: Two-person Cooperative Games, *Econometrica* **28** (1953), 128–140.
12 Perles, M.A. and Maschler, M.: A Super-additive Solution for Nash's Bargaining Game, *International J. of Game Theory* **10** (1981), 163–193.
13 Rubinstein, A.: Perfect Equilibrium in a Bargaining Model, *Econometrica* **50** (1982), 97–110.
14 Szidarovszky, F.: A Relation Between Stochastic Bargaining with Offer-dependent Break-down Probabilities and the Non-symmetric Nash Solution, Proceedings of the 27th Annual Meeting of the Western Decision Science Institute, April 7–11, 1998, Reno, Nevada, pp. 485–587.
15 Szidarovszky, F.: Bargaining with Offer-dependent Break-down Probabilities, *Appl. Math. and Comp.* **90** (1998), 117–127.
16 Szidarovszky, F.: Negotiations with Changing Conditions and Payoffs, Proceedings of the 28th Annual Meeting of the Western Decision Science Institute, April 6–10, 1999 (accepted).
17 Zeuthen, F.: *Problems of Monopoly and Economic Welfare*, Routledge, London, 1930.