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Common fixed point theorems for commutings k-uniformly Lipschitzian mappings in metric spaces

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Abstract

In this paper we give a common fixed points existence theorems for any sequence of commuting k-uniformly Lipschitzian mappings (eventualy for $k = 1$ *for any sequence of commutings nonexpansive mappings) defined on a bounded and complete metric space* (X, d) *with uniform normal structure. After that we deduce, by using the T.C.Lim an J.Kulesza result [11], that this result can be generalized for any family of commutings k-uniformly Lipschitzian mappings.*

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1 Introduction

In classical theorems concerning the existence of fixed points for families of mappings, such as the Kukatuni theorem [1] and its well know generalization due to Ryll-Nardzewski [2], the mappings of the family are usually assumed

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to be linear, or at least to be weakly continuous and affine [3]. In the nonlinear theory, stronger geometric structure are utilized. In particuler for a families of nonexpansives mappings, Khamsi proved in [4] that any family of nonexpansives mappings defined on metric space (X, d) with compact and normal convexity structure $\mathcal{F}$, has a common fixed points. In his proof, Khamsi investigated the concept of 1-local retract. In this paper we prove that any sequential family of k-uniformly Lipschitzian mappings defined on bounded metric space with an uniform normal convexity structure $\mathcal{F}$ with constant β, which contains all closed ball of (X, d), has a common fixed point provided that $k^2\beta < 1$. Recall that Khamsi proved in [5] that any nonexpansive mapping defined on a bounded complete metric space with uniform normal structure with constant β has a nonempty fixed point set. For more details on fixed point theory for nonexpansive and k-uniformly Lipschitzian mappings in metric spaces we refere to [8], [9] and [10].

2 Definition and Preliminary

In all the following in this work (X, d) will be a metric space. We use $B(x, r)$ to denote the closed ball centred at $x \in X$ with radius $r > 0$. For a subset A of X we write

$$r_x(A) = \sup_{y \in A} d(x, y) \tag{1}$$

$$r(A) = \inf_{x \in A} r_x(A) \tag{2}$$

$$\delta(A) = \sup_{x \in A} r_x(A) \tag{3}$$

$$cov(A) = \cap_{B \in \mathcal{F}} B \tag{4}$$

where $\mathcal{F}$ is the families of closed ball containing A. A subset A of X is said to be admissible if and only if $A = cov(A)$. In other term A is admissible if it is an intersection of a family of closed ball centred in X.

Definition 1 *Let $\mathcal{F}$ be a nonempty family of subset of X. We say that $\mathcal{F}$ defines a convexity structure on X if and only if it is stable by intersection.*

In this work, we will always assume that $\mathcal{F}$ contains the balls, and therefore we will not recall it. Also we denote by $\mathcal{A}(X)$ the smallest convexity structure on X.

Definition 2 *We say that $\mathcal{F}$ has the property (R) if and only if any decreasing sequence $(X_n)_n$ of nonempty bounded closed subsets of X with $X_n \in \mathcal{F}$ has a nonempty intersection.*

Definition 3 *i) We say that X has uniform normal structure if and only if $r(A) \leq \beta\delta(A)$ for some $0 < \beta < 1$ and every $A \in \mathcal{F}$.*
ii) We say that $\mathcal{F}$ is normal if and only if $r(A) < \delta(A)$ for every $A \in \mathcal{F}$.

Lets recall that a self mapping $T : X \to X$ is said to be k-uniformly Lipschitzian if there exists $k > 0$ such that

$$d(T^i x, T^i y) \leq kd(x, y)$$

for every $i \in \mathbb{N}$ and every x, y in X. An 1-uniformly Lipschitzian maps is also called nonexpansive. For such class of mappings we recall the most important result

Theorem 1 **[5]** *Let (X, d) be a complete bounded metric space. Assume that X has uniform normal structure, then any nonexpansive mapping defined on X has a fixed point.*

In [4], Khamsi gived the definition and a caracterization of 1-local retract subset of a metric space. Now we define

Definition 4 *A subset A is said to be k-local retract if for any family $(B_i)_i$ of closed ball centred in A such that $\cap_{i\in I} B(x_i, r_i) \neq \emptyset$, we have $A \cap_{i\in I} B(x_i, kr_i) \neq \emptyset$.*

It is imediate that uniform normal structure is not hereditary. However for 1-local retract subset we have

Lemma 1 *Let (X, d) be a metric space. Suppose that $\mathcal{A}(X)$ is an uniform normal convexity structure with constant $\beta < 1$. If Y is a 1-local retract subset of X, then $\mathcal{A}(Y)$ is an uniform normal structure with the same constant β.*

The proof is based on the next lemma

Lemma 2 [4] *Let (X, d) be a metric space and A a nonempty bounded subset of X. Then*
1) $cov(A) = \cap_{x \in X} B(x, r_x(A))$.
2) $r_x(A) = r_x(covA)$ for every x in X.
3) $\delta(A) = \delta(covA)$.
4) $r(covA) \leq r(A)$.
5) If (X, d) has the (n, ∞) property and is convex then $\frac{\delta(A)}{2} \leq r(covA) \leq \frac{n-1}{n}\delta(A)$.

Recall that (X, d) is said to have the (n, ∞) property if for any family $(B_i)_{i \in I}$ of closed ball of X such that $\cap_{i \in J} B_i \neq \emptyset$ for any finite subfamily J of I with card(J) less than n, we have $\cap_{i \in I} B_i \neq \emptyset$.
A metric space (X, d) is said to be convex if for all x, y in X and $\alpha \in [0, 1]$ there exists $z \in X$ suh that

$$d(z, x) = \alpha d(x, y) \text{ and } d(z, y) = (1 - \alpha)d(x, y).$$

Proof of lemma 1.
We assume that A is not reduce to a singleton. By 4) of lemma 2, we have $r(covA) \leq r(A)$. Since $A \in \mathcal{A}(Y)$ then $A = \cap_{i \in I} B(x_i, r_i) \cap Y$ with $x_i \in Y$. Hence $covA \subset \cap_{i \in I} B(x_i, r_i)$. Let $z \in cov(A)$ and define $r = r_z(A)$, then $z \in B = \cap_{x \in A} B(x, r) \cap \cap_{i \in I} B(x_i, r_i)$ is in $\mathcal{A}(X)$. Since Y is a 1-local retract of X then $B \cap Y \neq \emptyset$. Let $w \in B \cap Y$, so $w \in A = \cap_{i \in I} B(x_i, r_i) \cap Y$ and $w \in \cap_{x \in A} B(x, r)$. We deduce that $r_w(A) \leq r$. Hence $r(A) \leq r = r_z(A)$.
Since z is arbitrary in $cov(A)$ we obtain from (2) that $r(A) \leq r(cov(A))$. But $cov(A) \in \mathcal{A}(X)$ which is uniform normal, then

$$r(A) \leq r(cov(A)) \leq \beta\delta(cov(A)) = \beta\delta(A)$$

from property 4) of lemma 2. □

3 Fixed Point for k-Uniformly Lipschitzian Mappings

In the next theorem we obtain fixed point theorem for k-uniformly Lipschitzian mapping by utilizing the existence theorem of nonexpansive mapping [4]. In our knowledge This connection was never utilized. Moreover theorem 2 contains the result of theorem 1.

Theorem 2 *Let (X, d) be a complete bounded metric space. Assume that X has an uniform normal structure with constant $\beta < 1$. Then any k-uniformly Lipschitzian mapping $T : X \to X$ has a fixed point if $k^2\beta < 1$.*

Proof.
We need the followings two lemmas

Lemma 3 *Under the sames hypothesis of theorem 2 and for $T : X \to X$ k-uniformly Lipschitzian let*

$$d'(x, y) = \sup_{i=0,1,...} d(T^i x, T^i y).$$

Then
1) (X, d') is a bounded complete metric space.
2) T is d'-nonexpansive, ie

$$d'(Tx, Ty) \leq d'(x, y) \quad \forall x, y \in X.$$

Lemma 4 *Under the sames hypothesis of theorem 2 and for $T : X \to X$ k-uniformly Lipschitzian, the family of all admissibles subset of (X, d') is an uniform normale convexity structure with constant c ($c \leq k^2\beta$).*

Proof of lemma 3.
1-1) d' is a metric on X. Indeed
1-1-a) For every x, y in X we have
$d'(x, y) = 0$ is equivalent to $d(T^i x, T^i y) = 0$ for every $i = 0, 1, 2, ...$
Specifically for $i = 0$, it imply that $d(x, y) = 0$. Since d is a metric on X, then $x = y$.
1-1-b) For every $i = 0, 1, 2, ...$, and every x, y, z in X, we have

$$d(T^i x, T^i y) \leq d(T^i x, T^i z) + d(T^i z, T^i y),$$

since d is a metric on X.
By passing to the supremum on $i \in \mathbb{N}$, we obtain that

$$d'(x, y) \leq d'(x, z) + d'(z, y).$$

1-1-c) It is imediate that $d'(x, y) = d'(y, x)$ for all x, y in X.

1-2) Since T is k-uniformly Lipschitzian on X, and by definition of d', we have the inequality

$$d(x,y) \leq d'(x,y) \leq kd(x,y)$$

for all x, y in X. It follows from this inequality that (X, d') is a bounded complete metric space since (X, d) is it.
2) For every x, y in X we have

$$\begin{aligned} d'(Tx,Ty) &= \sup\{d(T^{i+1}x, T^{i+1}y)/i = 0,1,2,...\} \\ &\leq \sup\{d(T^i x, T^i y)/i = 0,1,2,...\} \\ &= d'(x,y). \quad \square \end{aligned}$$

Proof of lemma 4.
Let A be an admissible subset for d', then

$$A = \cap_{x\in X} B'(x, r'_x(A)) \subset cov(A) = \cap_{x\in X} B(x, r_x(A)).$$

On the other hand it follows from the definition of d' that

$$d(z,y) \leq d'(z,y) \leq kd(z,y) \ \ \forall z, y \in X.$$

Hence

$$r'_z(A) \leq kr_z(A) \ \ \forall z \in X. \tag{5}$$

By passing in (5) to the infimum on $z \in \cap_{x\in X} B'(x, r'_x(A))$ we get

$$\inf_{z\in \cap_{x\in X} B'(x, r'_x(A))} r'_z(A) \leq k \inf_{z\in \cap_{x\in X} B'(x, r'_x(A))} r_z(A)$$

Which implies that

$$\begin{aligned} r'(A) = \{\inf r'_z(A)/z \in A = \cap_{x\in X} B'(x, r'_x(A))\} &\leq k \ \ \{\inf r_z(A)/z \in \cap_{x\in X} B'(x, r'_x(A))\} \\ &\leq k \ \inf\{\sup_{x\in A} d(z,x)/d(z,x) \leq \frac{r_x(A)}{k}\} \\ &\leq k \ \ \inf\{k \sup_{x\in A} d(z,x)/d(z,x) \leq r_x(A)\} \end{aligned}$$

since

$$\cap_{x\in X} B(x, \frac{r_x(A)}{k}) \subset \cap_{x\in X} B'(x, r'_x(A)).$$

Therefore

$$r'(A) \leq k^2 r(cov(A)) \leq k^2 \beta\delta(cov(A)) = k^2\beta\delta(A) \leq k^2\beta\delta'(A). \quad \square$$

Proof of theorem 2.
It follows imediatelly from theorem 1, property 2) of lemma 3, and lemma 4.

By theorem 2, we have $Fix(T) \neq \emptyset$ for every k-uniformly Lipschitzian mapping T defined on a bounded complete metric space (X, d) with uniform normal convexity structure $\mathcal{F}$ with constant $\beta < \frac{1}{k^2}$. More than $Fix(T)$ is k-local retract of X, ie for every closed balls $B(x_i, r_i)_{i\in I}$ we have

$$\cap_{i\in I} B(x_i, r_i) \neq \emptyset \text{ implies } \cap_{i\in I} B(x_i, kr_i) \cap Fix(T) \neq \emptyset.$$

Now we are able to show that

Theorem 3 *Let $T_n : X \to X$; $n = 0, 1, 2, ...$ be a family of commutings k-uniformly Lipsitzian mappings. Suppose that X has an uniform normal convexity structure $\mathcal{F}$ with constant $\beta < \frac{1}{k^2}$. Then $\cap_{n\in N} Fix(T_n) \neq \emptyset$ and is k-local retract of X.*

Proof of theorem 3.
The first part of the theorem follows imediatelly from theorem 2. For the second part, let $(B_i)_{i\in I}$ be a family of closed balls centred in $\cap_{n\in N} Fix(T_n)$ such that $(B_i)_{i\in I} \neq \emptyset$.
We have

$$B_d(x_i, r_i) \subset B_{d'}(x_i, kr_i) \subset B_d(x_i, kr_i)$$

Hence

$$\cap_{i\in I} B_{d'}(x_i, kr_i) \neq \emptyset$$

and since $(T_n)_n$ are nonexpansives mappings on (X, d'), it follows from theorem 2 that

$$\cap_{n\in N} Fix(T_n) \cap \cap_{i\in I} B_{d'}(x_i, kr_i) \neq \emptyset$$

which implies that

$$\emptyset \neq \cap_{n\in N} Fix(T_n) \cap \cap_{i\in I} B_{d'}(x_i, kr_i) \neq \emptyset \subset \cap_{n\in N} Fix(T_n) \cap \cap_{i\in I} B_d(x_i, kr_i) \neq \emptyset \quad \square.$$

The problem whether the conclusion of theorem 3 holds for any commuting family $(T_i)_{i\in I}$ of k-uniformly Lipschitzians mappings $(k > 1)$ was still open for several years. However by using the result of T.C.Lim and J.Kulesza [11] in which they shows that the weak compactness and weak countably compactness are equivalent if the metric space have normal structure, we prove that

Theorem 4 *Let (X, d) be a bounded complete metric space with an uniform normal convexity structure $(\beta < 1)$. Then any commuting family $T_i : X \to X,\ i \in I$ of k-uniformly Lipschitzians mappings has a common fixed point provided that $k^2\beta < 1$.*

Proof.
Since (X, d') has uniform normal structure with constant c $(c < k^2\beta)$, then by the well known theorem of Khamsi [5], $\mathcal{A}(X_{d'})$ is coutably compact.
Hence by the T. C. Lim and J. Kulesza result, it follows that $\mathcal{A}(X_{d'})$ is in fact compact. On the others hand , since each $T_i, i \in I$ is d'-nonexpansive (lemma 4), it follows that the result of theorem 3 is a direct consequence of Khamsi theorem's in which he shows that any commutings family of non-expansive mappings defined on a bounded metric space for which $\mathcal{A}(X_{d'})$ is compact and normal, has a common fixed point. $\square$

Remark that the result of theorem 4, was deduced from T. C. Lim and J. Kulesza theorem's and the uniform convexity of (X, d') (lemma 4); but the problem whether the compacity and normality of (X, d) imply the compacity and normality of (X, d'), still open.

4 Applications

It was proved by Nachbin [12] and Kelley [13] that all Banach spaces which have the $(2, \infty)$ property are those of form $C(E)$ where E is a compact Stonian, for example l_∞ and L_∞. Then by theorem 4 and property 5) of lemma 2, we have

corolary 1 *The unit balls of l_∞, L_∞, and $C(E)$ where E is a compact Stonian have the common fixed point property for every commutings family $T_i : X \to X,\ i \in I$ of k-uniformly Lipschitzians mappings provided that $k < \sqrt{2}$.*

After that, Lindenstrauss [14] proved that l_1 has a $(3,\infty)$ property. Therefore

corolary 2 *The unit ball of l_1 has the common fixed point property for every commutings family $T_i : X \to X$, $i \in I$ of k-uniformly Lipschitzians mappings provided that $k < \sqrt{\frac{3}{2}}$.*

Also, we deduce from theorem 4 and property 5) of lemma 2, the following corolary

corolary 3 *If (X,d) is a Banach space with the (n,∞) property, and if $k < \sqrt{\frac{n}{n-1}}$, then its unit ball has the common fixed point property for every commutings family $T_i : X \to X$, $i \in I$ of k-uniformly Lipschitzian mappings.*

More recently, Prus [15] proved that all Banach spaces L_p $(1 < p < +\infty)$ have uniform normal structure with constant $\beta = min(2^{\frac{1}{p}}, 2^{\frac{1}{q}})$, where $q = p(p-1)^{-1}$ is the conjugat of p.
Hence, it follows that

corolary 4 *The unit balls of L_p has the common fixed point property for every commutings family $T_i : X \to X$, $i \in I$ of k-uniformly Lipschitzians mappings provided that*
$k < \sqrt{min(2^{\frac{1}{p}}, 2^{\frac{1}{q}})}$.

Now we recall the definition of the most geometrical caracterization of l_∞, L_∞, and $C(E)$ where E is a compact Stonian

Definition 5 *A metric space (X,d) is said to be hyperconvex iff for any family*
$\{B(x_i,r_i), i \in I\}$ of closed ball of (X,d) such that

$$d(x_i,x_j) \leq r_i + r_j$$

for every $i,j \in I$, has a nonempty intersection.

Remark.
1) Every hyperconvex metric space is complete, and if A is an admissible subset of (X,d) then also (A,d) is a hyperconvex metric space (see [8]).
2) Every hyperconvex space is convex. Indeed:
For all x,y in X and for any $\alpha \in [0,1]$, let u,v in X. We have

$$\alpha[d(x,u) + d(x,v)] + (1-\alpha)[d(y,u) + d(y,v)] \geq d(u,v).$$

The hyperconvexity of (X,d) imply that

$$\cap_{u\in X} B(u, \alpha d(x,u) + (1-\alpha)d(y,u)) \neq \emptyset.$$

Hence for every x, y in X and for every $\alpha \in [0,1]$, there exists $z \in X$ such that

$$z \in \cap_{u\in X} B(u, \alpha d(x,u) + (1-\alpha)d(y,u)),$$

ie

$$d(u,z) \leq \alpha d(x,u) + (1-\alpha)d(y,u)$$

for all $u \in X$. Therefore

$$d(x,z) = (1-\alpha)d(x,y) \quad \text{and} \quad d(y,z) = \alpha d(x,y). \qquad \square$$

Also by theorem 4 and proprerty 5) of lemma 2, we obtain the following theorem

Theorem 5 *Let (X,d) be a bounded hyperconvex metric space. Then any family of commutings k-uniformly Lipschitzian mappings defined on X has a common fixed point if $k < \sqrt{2}$.*

Proof.
(X,d) is a bounded hyperconvex metric space, then from the above remark, it is complete. Lets prove that (X,d) has the $(2,\infty)$ property. Indeed:
Let $\{B(x_i,r_i), i \in I\}$ be a family of closed balls of (X,d), such that

$$B(x_i,r_i) \cap B(x_j,r_j) \neq \emptyset \ \ \forall i,j \in I (i \neq j).$$

Then we have

$$d(x_i,x_j) \leq d(x_i,x) + d(x_j,x) \leq r_i + r_j$$

where $x \in B(x_i,r_i) \cap B(x_j,r_j)$.
Hyperconvexity of (X,d) imply that $\cap_{i\in I} B(x_i,r_i) \neq \emptyset$. (X,d) is then a convex metric space with the $(2,\infty)$ property. Therefore, by property 5) of lemma 2, $\mathcal{A}(X)$ is an uniform convexity structure with constant $\beta = \frac{1}{2}$. Hence, theorem 4 complete the proof of theorem 5. $\square$

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