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# A Bargaining Procedure with Offer-Dependent Breakdown Probabilities

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## Abstract

*This paper introduces a new solution concept for two-person bargaining. First a sequential bargaining process is proposed where break-down probabilities depend on the current offers. After showing that this process terminates in finitely many steps in a large class of bargaining problems, a one-stage solution concept is introduced which is the optimal offer for both players under given relative power indices. Three numerical examples illustrate the concept and solution methodology. The generalization of the solution for the n-person case is outlined in the last section.*

**A.M.S. Subject Classification Codes:** 90
**Keywords:** *bargaining, dynamic games, solution concept.*

## 1 Introduction

This paper introduces a new solution concept and methodology for resolving

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two-person conflicts. In the literature, there are two major approaches for conflict resolution. The first kind of methods is based on certain conditions which the solution must satisfy. These conditions are given as axioms. In most cases, the existence and uniqueness of a solution satisfying these axioms are proved. Nash's bargaining model has been one of the simplest yet most fruitful paradigms in such concept (Nash 1950). Following the idea of Nash, several authors have suggested modification of his set of axioms. For example, Harsanyi and Selten(1972) have dropped the symmetry assumption, and have proved the existence of a unique solution which maximizes the weighted geometric average of the gains. Kalai and Smorodinsky(1975) have criticized the axiom requiring the independence of irrelevant alternatives. They have shown that in many cases Nash's solution is unfair. They have suggested that this axiom be replaced by individual monotonicity. The resulting solution is the same as a direction based method known from multi-objective programming. This concept has been further extended by Anbarci(1995) as the reference function solution. Many additional solution concepts have been introduced during the last decade. Among the most commonly known concepts we mention here the egalatarian solution of Kalai(1977), the equal sacrifice solution of Chun(1988), the equal area solution of Anbarci(1993), and the super-additive solution of Perles and Maschler(1981). The second type of methods is based on ficticious negotiation processes. Either the equilibrium of the resulting dynamic game is accepted as the solution, or the convergence of the process is proved and the limit is accepted as the solution. Nash(1953) claimed that the equilibrium set coincides with the set of Pareto solutions if bargaining is considered as a two-person noncooperative game. Bargaining can be also considered as a single player decision problem, when the strategy selection of the other player is considered to be random. Each player then maximizes his/her expected profit. If uniform distribution is assumed, then the resulting solution is equivalent to the Nash-product maximizing alternative. In dynamic negotiations, the main problem is to determine the order in which concessions are made. If the principle of Zeuthen(1930) is assumed, then the limit of the dynamic process coincides with the Nash solution. The alternating offer method (Rubinstein, 1982) is one of the most popular bargaining models. It is based on a sequence of alternating offers, and the process is complicated by the introduction of break down probabilities. Under general conditions, it is proved that there is a unique subgame perfect equilibrium. Howard(1992) has introduced an extended version of the alternating offer method, and in the model of Anbarci(1995) the payoffs of the players depend not only on their offers for themselves but also on how generous their offers are.

In this paper, a new solution concept is introduced and examined, which has certain elements from both major concepts. First a dynamic process is introduced, and then the one-step solution concept is discussed, which is an extension of the one-shot bargaining solution of Anbar and Kalai(1978). It is assumed that the solution is Pareto optimal, and it maximizes the expected profit for both players. Instead of assuming that the alternative selection of the competitor is random, offer dependent break down probabilities are introduced. The resulting nonsymmetric solution is based on the relative power indices of the players.

The paper develops as follows. In section 2, the two player concept is explored. A dynamic bargaining process is examined first, and then the existence of a unique solution is verified. In addition, an iteration scheme is proposed to find the solution. In section 3, some examples are given to illustrate the concept and the solution algorithm. Conclusions are drawn in section 4.

# 2 Dynamic Process and Solution Concept in the Two-Person Case

A standard two-person bargaining game $S$ is considered, which is illustrated in Fig. 1. It is assumed that $S$ is compact and comprehensive in $R^2$. We may assume without loss of generality (if necessary, after a suitable positive affine transformation of utility scales) that the disagreement point is the origin. Let the Pareto frontier be given by the graph of a twice continuously differentiable, concave and strictly decreasing function $x_2 = g(x_1)$. The concavity assumption on function $g$ will be relaxed later. The domain of $g$ is assumed to be a finite closed interval, and there is an $x_1 > 0$ such that $g(x_1)$ is defined and $g(x_1) > 0$.

The state of the negotiation process at each time period is a pair $((x_1, x_2), (y_1, y_2))$ of points on the Pareto frontier, such that $x_1 > y_1$ and $y_2 > x_2$. Here points $(x_1, x_2)$ and $(y_1, y_2)$ are considered as the positions (or current offers) of player 1 and player 2, repectively. It is also naturally assumed that $y_1 \geq 0$

and $x_2 \geq 0$, that is, no player offers less to the other player than the disagreement payoff value.

As the negotiation preceeds, the state changes when any of the players makes a concession. If player 1 makes the concession, then he/she moves to another point $(z_1, z_2)$, which decreases his/her payoff and at the same time, increases the payoff of player 2. Player 2 will either accept the offer with probability $P_2(z_2)$ or break up the bargaining process with probability $1 - P_2(z_2)$, where $P_2 : [x_2, y_2] \mapsto [0, 1]$ is a real valued function. In the case of acceptance the players receive the agreed payoffs, and in the case of breakdown, both players get zero payoffs. Suppose that $P_2(x_2) = 0$ and $P_2(y_2) = 1$, which models the concept that no concession from player 1 will result in certain breakdown and full concession will be certainly accepted by player 2. For the sake of simplicity, assume that

$$P_2(z_2) = (\frac{z_2 - x_2}{y_2 - x_2})^{p_2} \tag{1}$$

where $p_2 > 0$ is a given constant.

Under this assumption, player 1 will select the point that maximizes his/her expected payoff:

$$\begin{array}{l} \text{maximize } z_1(\frac{z_2-x_2}{y_2-x_2})^{p_2} + 0(1 - (\frac{z_2-x_2}{y_2-x_2})^{p_2}) \\ \text{subject to } x_1 \geq z_1 \geq y_1. \end{array} \tag{2}$$

Since $y_2 - x_2$ is fixed and $(z_1, z_2)$ is on the Pareto frontier, this optimization problem is equivalent to maximizing function

$$z_1(g(z_1) - x_2)^{p_2}. \tag{3}$$

Note that the objective value is zero at $z_1 = x_1$, and positive at $z_1 = y_1$. Hence, at the optimum, $y_1 \leq z_1 < x_1$. The optimal solution $z_1$ is either $y_1$ or a stationary point, which is a solution of equation

$$(z_1(g(z_1) - x_2)^{p_2})' = 0. \tag{4}$$

Since at the optimum, $z_1 \neq x_1$ and $g(z_1) - x_2 \neq 0$, this equation can be rewritten as

$$h(z_1) = z_1 p_2 g'(z_1) + g(z_1) - x_2 = 0 \qquad (x_1 > z_1 \geq y_1). \tag{5}$$

It is easy to see that the sign of $h$ equals the sign of the derivative of the objective function (3). Note first that

$$h(x_1) = x_1 p_2 g'(x_1) + g(x_1) - x_2 \leq 0, \tag{6}$$

since $g'(x_1) \leq 0$, and $g(x_1) = x_2$ at the Pareto frontier. Furthermore,

$$h'(z_1) = p_2 g'(z_1) + z_1 p_2 g''(z_1) + g'(z_1) < 0, \tag{7}$$

since $g' < 0$ and $g'' \leq 0$. Therefore, $h$ strictly decreases. In the case of $h(y_1) > 0$, there is a unique stationary point $z_1$ in the interval $(y_1, x_1)$. Otherwise, $z_1 = y_1$ is the best choice. In this case player 1 accepts the offer of player 2 and the negotiation terminates.

In proving the unique optimal choice of player 1, the strict monotonicity of function $h$ is very important. This property is directly assured by assuming that $g$ is concave. We will next show how this condition can be relaxed and the strict monotonicity of $h$ still holds. From equation (7), we conclude that $h'(z_1) < 0$ for all $p_2 > 0$ if and only if

$$z_1 g''(z_1) + g'(z_1) \leq 0. \tag{8}$$

This relation is equivalent to the condition that $z_1 g'(z_1)$ decreases in $z_1$ for $z_1 \in [y_1, x_1]$. This is a much weaker condition than the assumption that $g$ is concave. In Example 3 we will demonstrate a case when condition (8) holds but $g$ is convex.

Interchanging the role of the two players, the same concept is applied to player 2. Function $g$ is replaced by its inverse function, which will be denoted by $G$. Obviously $G$ is also twice continuously differentiable, strictly decreasing and concave if the same properties hold for $g$. Similarly to the case of player 1, the necessary and sufficient condition for the monotonicity of the equation giving the unique optimal choice for player 2 with all $p_1 > 0$ is that $x_2 G'(x_2)$ decreases in $x_2$ (or increases) in $x_1$. Notice that

$$z_1 = G(z_2). \tag{9}$$

By differentiating,

$$G'(z_2) g'(z_1) = 1, \tag{10}$$

that is

$$z_2 G'(z_2) = \frac{g(z_1)}{g'(z_1)}. \tag{11}$$

It is easy to see that this function is increasing if and only if

$$g''(z_1) \leq \frac{g'(z_1)^2}{g(z_1)}. \tag{12}$$

This is also a weaker condition than assuming the concavity of $G$. Combining the conditions for the two players leads to the following observation. The necessary and sufficient condition for having monotone equations for both players for all $p_1, p_2 > 0$ is the following:

$$g''(z_1) \leq min\{\frac{g'(z_1)^2}{g(z_1)}, -\frac{g'(z_1)}{z_1}\}. \tag{13}$$

The right hand side is positive for all $z_1$, so this condition allows even small positive values of $g''$. That is, $g$ might be even slightly convex.

Based on the above described expected payoff maximizing rule the bargaining process can be defined in several different ways. One possible approach is the following. Assume that player 1 makes the move in each odd(even) step, and player 2 makes offer in each even(odd) step. A pair of such steps is considered as a round. It is also assumed that at each step, the offer making player selects his/her optimal alternative. Notice that at each step the players optimize their expected payoffs occuring in that step only, and they do not consider payoffs of the later steps. This approach is realistic in cases when there is a large uncertainty in future payoff values or even in continuing the negotiations. In such cases each player wants to get immediately as much payoff as possible. We mention that in dynamic economic modeling this assumption is commonly accepted (see, for example, Okuguchi and Szidarovszky, 1990). We will next prove that this process is always finite.

**Theorem 1.** Assume that $g$ is twice continuously differentible, $g'(z_1) < 0$ and relation (13) holds for $z_1 \in [y_1^0, x_1^0]$. Then the above alternating offer procedure is always finite.

**Proof.** Suppose the process is infinite. Let $(x_1^{(1)}, x_2^{(1)})$, $(x_1^{(2)}, x_2^{(2)}), \cdots$ denote the sequential positions of player 1. At round $k$, when player 1 makes the next offer, the maximizer of his/her expected payoff must be an interior point between $(x_1^{(k)}, x_2^{(k)})$ and $(y_1^{(k)}, y_2^{(k)})$, such that $h(x_1^{(k+1)}) = 0$. After some algebra and the use of the mean value theorem of derivatives, we have

$$x_1^{(k+1)} = \frac{g'(c_k)(x_1^{(k)} - x_1^{(k+1)})}{p_2 g'(x_1^{(k+1)})}, \tag{14}$$

where $x_1^{(k+1)} \leq c_k \leq x_1^{(k)}$.

Since $g'$ is continuous, it is bounded, say, $0 < c_1 \leq |g'(x)| \leq c_2$, which implies that

$$x_1^{(k+1)} \leq \frac{c_2}{p_2 c_1}(x_1^{(k)} - x_1^{(k+1)}). \tag{15}$$

Denote $c_2/(p_2 c_1)$ by $K$, then

$$x_1^{(k+1)} \leq \frac{K}{K+1} x_1^{(k)}. \tag{16}$$

This implies that $x_1^{(k+1)}$ tends to zero. This is an obvious contradiction, since for all $k \geq 2$, $x_1^{(k)} \geq y_1^{(1)} > 0$. Thus the proof is complete.

Another bargaining process can be defined in the following way. At the initial step each player finds his/her expected payoff maximizing offer $(x_1^{(1)}, x_2^{(1)})$ and $(y_1^{(1)}, y_2^{(1)})$. If $x_1^{(1)} \leq y_1^{(1)}$, then bargaining terminates and each player receives his/her demanded payoff (or any point of the Pareto frontier between $(x_1^{(1)}, x_2^{(1)})$ and $(y_1^{(1)}, y_2^{(1)})$ can be accepted as the final outcome). Otherwise negotiation continues in the same manner as in step 1, starting from these new offers. It is easy to see that under the conditions of Theorem 1 this process is also finite.

The bargaining processes outlined above require the knowledge of the power indices of both players, which is usually hard to quantify. A one-step solution concept is next introduced, when only the ratio $\alpha = p_2/p_1$ has to be known by the participants. This ratio $\alpha$ can be considered as the relative power of player 2 over player 1. A point $(z_1^*, z_2^*)$ is considered as the solution of the bargaining problem, when it is the optimal choice for both players, starting from the initial offers with some parameters $p_1$ and $p_2$ such that $p_2/p_1 = \alpha$. We will next prove that such point exists and is unique.

**Theorem 2.** Under the conditions of the previous theorem, for all $\alpha > 0$, there is a unique point $(z_1^*, z_2^*)$ on the Pareto frontier between the initial offers $(x_1, x_2)$ and $(y_1, y_2)$, which is the optimal choice for both players with some power indices $p_1$ and $p_2$ such that $p_2/p_1 = \alpha$.

**Proof.** A point $(z_1, z_2)$ is optimal for both players if and only if it is the solution of equations

$$z_1 p_1 g'(z_1) + g(z_1) - x_2 = 0 \tag{17}$$

and

$$z_2 p_2 G'(z_2) + G(z_2) - y_1 = 0. \tag{18}$$

Relations $p_2/p_1 = \alpha$, $z_2 = g(z_1)$, and $z_2 G'(z_2) = g(z_1)/g'(z_1)$ imply that these equations are equivalent to a single equation for the single unknown $z_1$:

$$\frac{x_2 - g(z_1)}{z_1 g'(z_1)} - \alpha \frac{y_1 - z_1}{\frac{g(z_1)}{g'(z_1)}} = 0. \tag{19}$$

One can easily verify that the left hand side is strictly decreasing, it is positive at $y_1$ and negative at $x_1$. Consequently, there is a unique solution.

The monotonicity of equation (19) allows us to use special iteration methods (such as bisection, or secant method) to find the solution (Szidarovszky and Yakowitz,1978).

Using the monotonicity of function $h$, one may easily verify the following monotonicity properties:

1. For fixed values of $\alpha$ and $x_2$, $z_1$ strictly increases in $y_1$. Similarly, for fixed $\alpha$ and $y_1$, $z_1$ strictly decreases in $x_2$. In other words, better initial position of any player will result in his/her higher payoff at the solution.

2. For fixed values of $x_2$ and $y_1$, $z_1$ strictly decreases in $\alpha$. It represents the fact that as $\alpha$ increases, the solution moves closer to the position of player 2.

# 3 Numerical Examples

In this section, three examples are presented. The first one shows a linear case, the second example presents a strictly concave case, and in the third example, a slightly convex $g$ is examined.

## Example 1

Select $g(x) = -ax + b$, where $a > 0, b > 0$. Suppose $a = 0.5$ and $b = 1$ are given. The initial positions of player 1 and player 2 are $(1.6, 0.2)$ and

$(0.6, 0.7)$, respectively(see Fig. 2). Assume that the power index of player 1 is 0.4 and that of player 2 is 0.6.

If the two players make concessions alternatively, then the process is finite and the solution depends on the order in which concessions are made. As shown in Tables 1 and 2, the payoff of each player is smaller in the case when he/she makes the first offer assuming that the process does not terminate with disagreement and at each step the offering player selects a point where his/her expected payoff is maximized.

Consider next the one-step solution with $\alpha = 1.5$. Equation(19) has a unique solution $(0.9243, 0.5378)$. Notice that this solution is located between the previous two solutions. Therefore, it can be considered as a compromise solution when no "first concession" is made.

## Example 2

Select $g(x) = \sqrt{1 - x^2}$ where $0 \leq x \leq 1$ (see Figure 3). Suppose the initial positions are given as $(0.8, 0.6)$, $(0.6, 0.8)$, and the power index is 20 for both players. Tables 3 and 4 give the solutions when the alternating offer strategy is used. Obviously the problem is symmetric. The one-step solution is $(\sqrt{2}/2, \sqrt{2}/2)$ which is also symmetric in the two players.

We will next repeat the computations when the power index of player 2 is doubled. The solution in the alternating process remains the same. By solving equation(19) again, the one-step solution is $(0.68155, 0.7318)$. It is easy to see that the payoff of the second player has been increased with this higher power index.

## Example 3

Assume now that $g(x) = (2 - x)^2 - 1$ for $0 \leq x \leq 1$ and the initial positions of player 1 and player 2 are (0.1,2.61) and (0.9,0.21), respectively (see Figure 4). Tables 5 and 6 give the solutions when the alternating offer process is used.

Solving equation (19) with $\alpha = 0.5$ leads to the one-step solution (0.46165,1,3665). Notice that function $g$ is strictly convex but it satisfies all conditions of Theorem 2.

Table 1: Result of Example 1 when player 1 makes the first concession

| round(k) | $(x_1^{(k)}, x_2^{(k)})$ | $(y_1^{(k)}, y_2^{(k)})$ |
|---|---|---|
| 0 | $(1.6, 0.2)$ | $(0.4, 0.8)$ |
| 1 | $(1, 0.5)$ | $(6/7, 4/7)$ |
| 2 | $(6/7, 4/7)$ | ——— |

Table 2: Result of Example 1 when player 2 makes the first concession

| round(k) | $(y_1^{(k)}, y_2^{(k)})$ | $(x_1^{(k)}, x_2^{(k)})$ |
|---|---|---|
| 0 | $(0.4, 0.8)$ | $(1.2, 0.2)$ |
| 1 | $(6/7, 4/7)$ | $(1, 0.5)$ |
| 2 | $(1, 0.5)$ | ——— |

Table 3: Result of Example 2 when player 1 makes the first concession

| round(k) | $(x_1^{(k)}, x_2^{(k)})$ | $(y_1^{(k)}, y_2^{(k)})$ |
|---|---|---|
| 0 | $(0.8, 0.6)$ | $(0.6, 0.8)$ |
| 1 | $(0.6, 0.8)$ | ——— |

Table 4: Result of Example 2 when player 2 makes the first concession

| round(k) | $(y_1^{(k)}, y_2^{(k)})$ | $(x_1^{(k)}, x_2^{(k)})$ |
|---|---|---|
| 0 | $(0.6, 0.8)$ | $(0.8, 0.6)$ |
| 1 | $(0.8, 0.6)$ | ——— |

Table 5: Result of Example 3 when player 1 makes the first concession

| round(k) | $(x_1^{(k)}, x_2^{(k)})$ | $(y_1^{(k)}, y_2^{(k)})$ |
|---|---|---|
| 0 | $(0.9, 0.21)$ | $(0.1, 2.61)$ |
| 1 | $(0.575, 1.030)$ | $(0.358, 1.696)$ |
| 2 | $(0.375, 1.640)$ | $(0.375, 1.640)$ |

Table 6: Result of Example 3 when player 2 makes the first concession

| round(k) | $(y_1^{(k)}, y_2^{(k)})$ | $(x_1^{(k)}, x_2^{(k)})$ |
|---|---|---|
| 0 | $(0.1, 2.61)$ | $(0.9, 0.21)$ |
| 1 | $(0.358, 1.696)$ | $(0.575, 1.030)$ |
| 2 | $(0.549, 1.107)$ | $(0.549, 1.107)$ |

# 4 Discussions and Conclusions

In this paper new solution concepts were introduced to solve 2-person conflict situations. A dynamic process was first examined when the players make concessions alternatively based on offer-dependent break-down probabilities. We have also examined a variant of this bargaining process when the players make simultaneous concessions. We proved that these processes are always finite. A one-step solution concept was next introduced, in which the relative power index of player 2 over player 1 is assumed to be given. The solution is the best alternative selection for both players with some power indices such that their ratio is given. The existence of the unique solution was verified in the paper.

Three examples illustrated the methodology. We selected a linear, a concave and a convex function in order to show all possible cases.

The computation of the solution is based on solving a special nonlinear equation. The equation is monotonic, and therefore routine methods are available for obtaining the solutions. The $N$-person extension of the solution concepts and methods will be the subject of a future paper.

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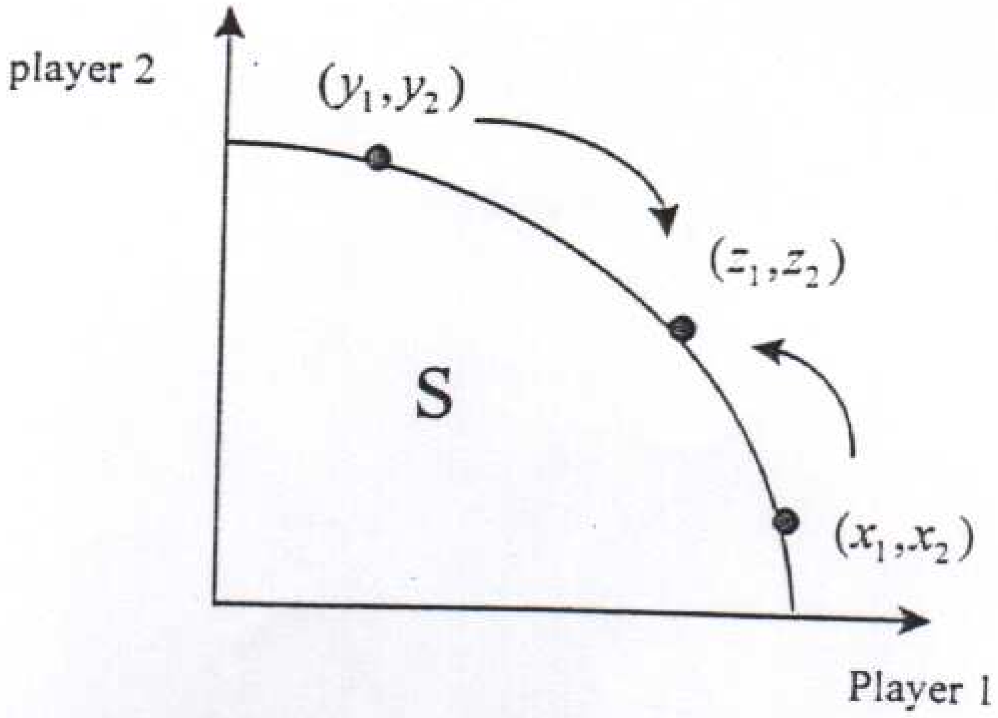


Figure 1: The general 2-person bargaining game

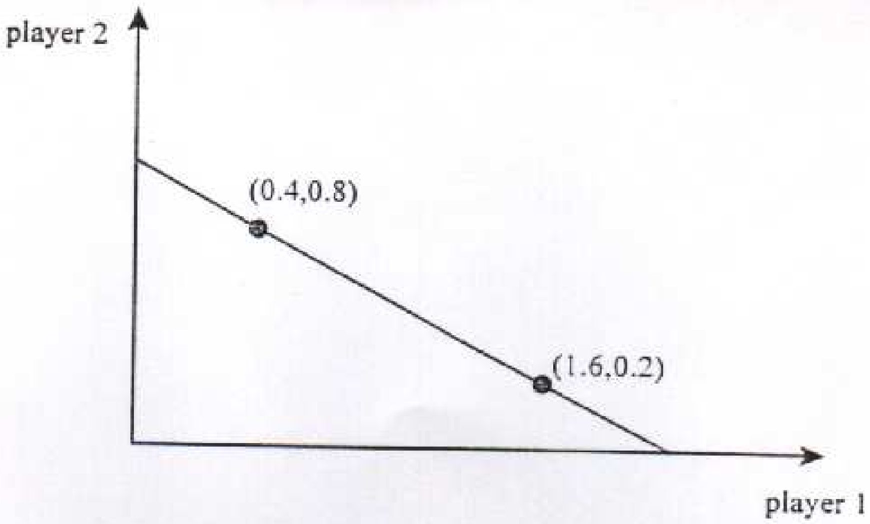


Figure 2: Pareto frontier and initial positions in Example 1

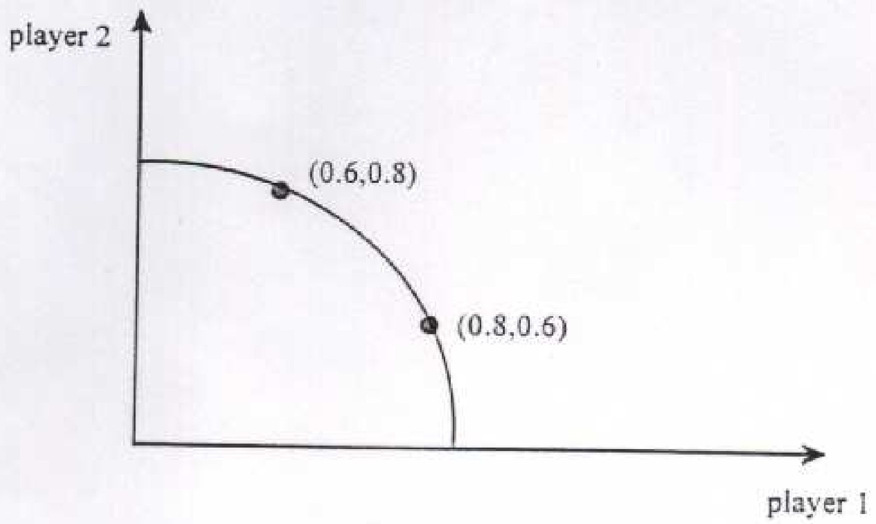


Figure 3: Pareto frontier and initial positions in Example 2

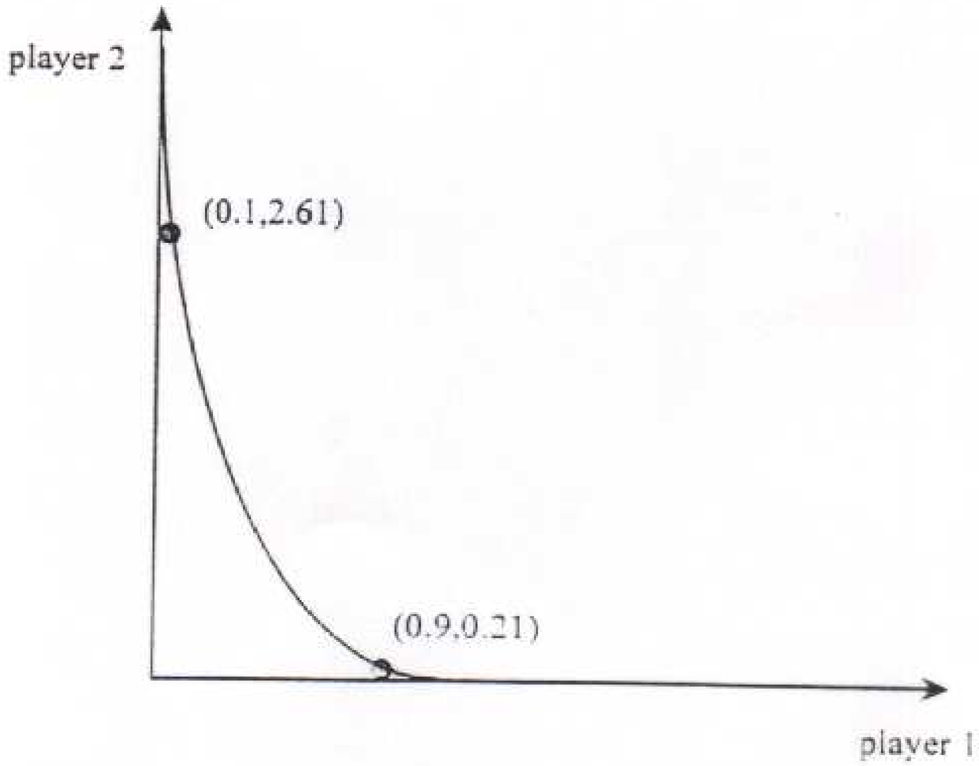


Figure 4: Pareto frontier and initial positions in Example 3