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SMALL AMPLITUDE CRITICAL PERIODS IN A CUBIC POLYNOMIAL SYSTEM

Bourama Toni

Abstract. In this paper, we study the local bifurcations of critical periods in the neighbourhood of a nondegenerate centre of a parametrized family of plane cubic polynomial systems. We find that this family has no isochronous centres. We show that at most three local critical periods bifurcate from a centre at the origin. Moreover there exist perturbations which lead to the maximum number of critical periods. To get some of the results we reducing the question to that of transversal intersection of some algebraic surfaces, using computational algebra techniques such as the Theory of of Gröbner Bases.

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1. Introduction

Consider a family of plane polynomial autonomous vector fields $(\chi_\lambda)_{\lambda\in\mathbb{R}^N}$ parametrized by λ in the form

$$(\chi_\lambda) \qquad \chi_\lambda = (-y + P_\lambda(x,y))\frac{\partial}{\partial x} + (x + Q_\lambda(x,y))\frac{\partial}{\partial y},$$

P_λ and Q_λ polynomials in x and y, and with a nondegenerate center at the origin, i.e., a singular point surrounded by a continuous family of periodic trajectories with the system of differential equations normalized so that its linearization at the center has eigenvalues $\pm i$. We discuss the behaviour of the periods of such periodic solutions near the center. A period function $T(h,\lambda)$ associated to the center at the parameter value λ is defined by taking a semi-transversal Σ, Poincaré section,

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX through the center with some parametrization h. $T(h,\lambda)$ is actually the time taken for the trajectory starting at h to next cross Σ, i.e., $T: \quad (0,L) \longrightarrow \mathbb{R}$ (perhaps $L=\infty$) assigns to each $h \in (0,L)$ the minimum period of the trajectory γ_h. $T(h,\lambda)$ is an even analytic function of the coordinate parametrization h, and has the asymptotic development

$$T(h,\lambda) = 2\pi + \sum_{k=1}^{\infty} p_{2k}(\lambda) h^{2k}, \tag{1-1}$$

for $|h|$ and $|\lambda - \lambda_*|$ sufficiently small, where λ_* corresponds to a center.

A number of results related to the period function has been obtained in the past few years, and specially on the monotonicity of the function, and the number of its critical points. See, for instance, [1,2,3] as well as for some historical details. A global study of the number of critical points of the period is a very difficult problem and is generally unsolved. Our interest lies in the local question of the bifurcation of critical points (*critical periods*) of the period function as the parameter λ varies in the neighbourhood of λ_* corresponding to a center. It is the *Small Amplitude Critical Periods problem.*

Our approach is based on the analysis on the Taylor expression of $T(h,\lambda)$ in (1-1) by determining the so-called *period coefficients* $p_{2k}(\lambda), k \geq 1$. These coefficients are polynomials in the components of the bifurcation parameter λ_*. By the Hilbert basis theorem, there exists $K \in \mathbb{N}$ such that the ideal of all coefficients, $\mathcal{I} =< p_2, \cdots, p_{2k}, \cdots >$ over the ring $\mathbb{R}[\lambda, \ldots, \lambda_N]$, that we call the *Bautin-like ideal*, has the finite generator $\mathcal{G} =< p_2, \cdots, p_{2K} >$. See [2] for more details. Determining K is at the core of the method. It also leads to uncover the isochronous centres (all periodic orbits have the same period), i.e., the ones for which all period coefficients $p_{2k}(\lambda), k \geq 1$ vanish. Using this approach the small amplitude critical periods problem has been completely solved for quadratic systems in [2], for cubic symmetric systems in [4], and for Kukles reduced system in [5]. Similarities were found: A greater number of critical periods is attained from the nonisochronous centres, with the maximum number on the symmetric stratum of the parameter values. (See below). To derive general laws such as the maximum number of local critical periods for a polynomial family in terms of the degree of the system, and tackle the global problem, more systems need to be investigated.

In particular, we consider the following quadratic-like family $(\mathcal{C}_\lambda)_{\lambda \in \mathbb{R}^5}$ of plane cubic polynomial systems

$$(\mathcal{C}_\lambda) \qquad \begin{aligned} \dot{x} &= -y + Ax^2 + (B+2D)xy + Cy^2 - y(x^2+y^2), \\ \dot{y} &= x + Dx^2 + (E-2A)xy - Dy^2 + x(x^2+y^2), \end{aligned}$$

parametrized by $\lambda = (A,B,C,D,E) \in \mathbb{R}^5$. It has a center at the origin for the parameter values $\lambda: \quad B=E=0$ (Hamiltonian Stratum) and $\lambda: \quad B=D=0$ (Symmetric Stratum). See [6]. We address and solve completely the *small amplitude critical periods* problem for such a family, and moreover we prove that the upper bound can be precisely attained.

After giving in section 2. some definitions and general results, we prove the followings in section 3.

(1) At most three critical periods can bifurcate from these centres. The family has no nonlinear isochronous centres.

(2) There are perturbations of the bifurcation parameter leading to the maximum number of critical periods. We prove this result by reducing the question to that of transversal intersections of some algebraic surfaces, using computational algebraic geometry techniques such as the Theory of Resultant, Sturm's algorithm ([7]), Gröbner Bases ([8],[9]), and the Transversality theory of algebraic curves ([10]). This is one of the originalities of the paper.

Another originality is the computer assisted method of proofs, such as the method of *reduction modulo a Gröbner basis* via the computer algebra system MapleV.

Actually system $(\mathcal{C}_\lambda)$ without the cubic terms is the pure quadratic system studied in [2], where the maximum number obtained is two, along with the existence of isochronous centres. On the isochronous parameter strata the order of degeneracy of the derivative of the period function is infinite. In other words, our results show that the addition of the cubic terms causes the system to have no center where the order of degeneracy is greater than three, and thus may be seen as a mechanism to destroy the isochronous strata, and to inscrease the number of critical periods.

Moreover, unlike other systems studied in [3,4,5], this maximum number of critical periods is attained on both Hamiltonian and symmetric strata.

2. Some General Results

Definition 2.1.

(1) *The origin is a weak centre of finite order k, if $p_1 = p_2 = \ldots = p_{2k} = p_{2k+1} = 0$ and $p_{2k+2} \neq 0$. It is of infinite order or an isochronous centre, if $p_k = 0$ for $k \geq 1$.*

(2) *We say that k local critical periods bifurcate from the weak centre corresponding to the parameter λ_* if, for every $\alpha > 0$, sufficiently small, there exists a neighbourhood W of λ_* such that for any $\lambda \in W$, $T(h, \lambda)$ has at most k critical points in $U = (0, \alpha)$. Moreover, any neighbourhood of λ_* contains a point λ^1 such that $T(h, \lambda)$ has exactly k critical points in $U = (0, \alpha)$.*

Definition 2.2. *Let $\{\chi_\lambda\}_{\lambda \in \mathbb{R}^N}$ be a family of systems with a centre at the origin and associated period coefficients $p_k(\lambda)$. The family is said to satisfy condition $(\mathcal{P})$ if for any $\lambda_* \in \mathbb{R}^N$ such that $p_2(\lambda_*) = \cdots = p_{2k}(\lambda_*) = 0$, $p_{2k+2}(\lambda_*) \neq 0$ and any neighbourhood $W \subset \mathbb{R}^N$ of λ_* in which $p_{2k+2} \neq 0$, there exists $\lambda_1 \in W$ such that*

$$p_{2k+2}(\lambda_1) \times p_{2k}(\lambda_1) < 0, \quad \text{with } p_2(\lambda_1) = \cdots = p_{2k-2}(\lambda_1) = 0. \tag{2-2}$$

The system χ_{λ_*} is said to satisfy condition $(\mathcal{P}_k)$.

We proved the following in [5]

Proposition 2.3. *Let $\{\chi_\lambda\}_{\lambda \in \mathbb{R}^N}$ be a family of systems with a centre at the origin, satisfying condition $(\mathcal{P})$. And let $\chi_{\lambda *}$ satisfy condition $(\mathcal{P}_k)$. Then for any sufficiently small $\epsilon > 0$, any neighbourhood V of λ_*, and for any $0 \leq n \leq k$, there exists $\lambda_{**} \in V$ such that $\chi_{\lambda **}$ has exactly n periodic orbits with critical periods passing through points $(x, 0)$ for $x \in (0, \epsilon)$.*

Straightforward from the Weierstrass-Malgrange Preparation Theorem, we derive the following theorem applied throughout the study.

Finite Order Bifurcation Theorem. *From weak centres of finite order at the parameter value λ_*, no more than k local critical periods can bifurcate. Moreover there are perturbations with exactly $n \leq k$ local critical periods, if the coefficients $p_2, \dots, p_{2k+2}$ satisfy the condition $(\mathcal{P}_k)$ at λ_*.*

3. Critical periods of $(\mathcal{C}_\lambda)_{\lambda\in\mathbb{R}^5}$

Blows and Rousseau in [6] obtained the following conditions of center.

Theorem 3.1. *System $((\mathcal{C}_\lambda)_{\lambda\in\mathbb{R}^5})$ has a center at the origin for a parameter value λ_* if and only if λ_* is on either of the following stratas:*

$$\begin{cases} B = E = 0, & \textit{(Hamiltonian stratum } (\mathcal{H})) \\ B = D = 0, & \textit{(Symmetric stratum } (\mathcal{S})) \end{cases}$$

Throughout the paper, $(\mathcal{H})$ and $(\mathcal{S})$ indicate respectively the Hamiltonian and the symmetric stratum. We consider a point $\lambda_* \in (\mathcal{I}) = (\mathcal{H}) \cup (\mathcal{S})$, and study the bifurcations of critical periods for perturbations of λ_* which remain in $(\mathcal{I})$.

Under condition $(\mathcal{H})$, $(\mathcal{C}_\lambda)_{\lambda\in\mathbb{R}^5}$ is a Hamiltonian system with a Hamiltonian in the form

$$H(x,y) = \frac{x^2+y^2}{2} + \frac{Dx^3}{3} - Ax^2y - Dxy^2 - \frac{Cy^3}{3} + \frac{(x^2+y^2)^2}{4}. \tag{3-1}$$

Under condition $(\mathcal{S})$, $(\mathcal{C}_\lambda)_{\lambda\in\mathbb{R}^5}$ is symmetric with respect to the y-axis.

Assuming the parameter value λ_* in $(\mathcal{I})$, changing to polar coordinates (r,θ), and eliminating t bring system $(\mathcal{C}_\lambda)$ into the form

$$\frac{dr}{d\theta} = \frac{r^2 f_0(\theta,\lambda_*)}{1 + r g_0(\theta,\lambda_*) + r^2}, \tag{3-2}$$

where

$$f_0(\theta,\lambda_*) = A\cos^3\theta + (A+3D)\cos^2\theta\sin\theta + (E+C-2A)\cos\theta\sin^2\theta - D\sin^3\theta$$
$$g_0(\theta,\lambda_*) = D\cos^3\theta + (D-3A)\cos^2\theta\sin\theta + (3D+B)\cos\theta\sin^2\theta - C\sin^3\theta.$$

In the neighbourhood of the centre at the origin, the period function $T(h,\lambda)$ is computed as follows:

$$T(h,\lambda) = \int_0^{2\pi} \frac{d\theta}{\dot\theta} = \int_0^{2\pi} \frac{d\theta}{1 + r g_0(\theta,\lambda_*) + r^2} = 2\pi + \sum_{k\geq 1} p_{2k}(\lambda) h^{2k}, \tag{3-3}$$

for $|h|$ and $|\lambda - \lambda_*|$ sufficiently small. The solution $\theta \mapsto r(\theta,h,\lambda)$ of the differential equation in (3-2) with initial condition $r(0,\lambda) = h$ is well-known to be locally represented as the convergent power series in h

$$r(\theta,h,\lambda) = \sum_{k\geq 1} u_k(\theta,\lambda) h^k. \tag{3-4}$$

Substituting back into (3-2) yields the following recurrent relation to determine the coefficients $u_k(\theta,\lambda)$:

$$\begin{aligned} \dot{u}_1 =& 0, \quad \text{(from the initial condition)}, \\ \dot{u}_k =& \sum_{1\leq i\leq (k-1)} (f_0 u_i u_{k-i} - (g_0 u_{k-i} + v_{k-i})\dot{u}_i), \end{aligned} \tag{3-5}$$

with $v_1 = 0$, $v_k = \sum_{i=1}^{k-1} u_i u_{k-i}$. Using the integral representation of T in (3-3), we derive the period coefficients p_k by simple integration

$$p_k(\lambda) = \int_0^{2\pi} A_k(\theta,\lambda)d\theta, \tag{3-6}$$

with the terms A_k recursively defined by

$$A_0(\theta,\lambda) \equiv 1, \quad A_k(\theta,\lambda) = -\sum_{1\leq i\leq (k-1)} (g_0 u_{k-i} + v_{k-i})A_i. \tag{3-7}$$

$p_k(\lambda)$ is clearly a polynomial of degree k in (A,B,C,D,E). For $k=2$, we get

$$p_2(\lambda) = -24 + 18A^2 + B^2 + 12AC + 10C^2 + 8BD + 16D^2 - 9AE - CE + E^2. \tag{3-8}$$

We next study the bifurcations of the critical periods from centres corresponding to a parameter value λ_* on each one of the strata $(\mathcal{H})$ and $(\mathcal{S})$.

3.1 Hamiltonian stratum.

The parameter value λ_* satisfies the condition $B = E = 0$. We first compute sucessively the period coefficients $p_2(\lambda_*)$, $p_4(\lambda_*)$, $p_6(\lambda_*)$, $p_8(\lambda_*)$. Substituting $B = E = 0$ in formula (3-8) gives

$$p_2(\lambda_*) = b_1^2 + 4C^2 + 8D^2 - 12 \quad \text{with } b_1 = 3A + C. \tag{3-9}$$

Thus

$$p_2(\lambda_*) = 0 \quad \text{if and only if } D^2 = b_4 = \frac{12 - b_1^2 - 4C^2}{8} \geq 0, \tag{3-10}$$

that is, for A and C in $[-\sqrt{3},\sqrt{3}]$. Therefore, out of the ellipse $b_1^2+4C^2+8D^2-12 = 0$, we have necessarily $p_2 \neq 0$. Hence no critical period can bifurcate from the corresponding centre.

From formulas (3-6) and (3-7), and modulo $p_2(\lambda_*)$ we obtain the period coefficient $p_4(\lambda_*)$ as

$$\begin{aligned} p_4(\lambda_*) =& -64 + 120A^2 - 75A^4 + 240AC - 260A^3C + 120C^2 - 330A^2C^2 \\ & - 180AC^3 - 35C^4. \end{aligned} \tag{3-11}$$

Through the linear transformation $(\mathcal{T}_1)$: $\quad (A,C) = (\frac{-7u+v}{8}, \frac{15u-v}{8})$ for u and v such that $|u| \leq 2\sqrt{3}$ and $|v| \leq 22\sqrt{3}$ (using condition (3-10)), the coefficient $p_4(\lambda_*)$ is simplified to

$$\tilde{p}_4(u,v) = -5u^3v + 120u^2 - 64. \tag{3-12}$$

First let us assume $u = 0$, i.e., $A = -C$. Let λ_*^1 be the corresponding parameter value; then $p_4(\lambda_*^1) = -64 < 0$. The centre associated with λ_*^1 is therefore of order 1. Moreover, $p_2(\lambda_*^1)$ is reduced to

$$p_2(\lambda_*^1) = 2C^2 + 2D^2 - 3. \tag{3-13}$$

It suffices to take any value $\tilde{\lambda}_*^1$ next to the exterior of the ellipse $p_2(\lambda_*^1) = 0$, to obtain a perturbation leading to $p_2(\tilde{\lambda}_*^1) > 0$ and $p_4(\tilde{\lambda}_*^1) < 0$. Therefore, condition $(\mathcal{P}_k)$, $k = 1$ is fulfilled. Hence the following lemma.

Lemma 3.2. *No more than one critical period can bifurcate from the centre corresponding to the parameter value λ_*^1.*

We denote by $(\mathcal{C}_{\lambda\in\mathcal{H}})$ the corresponding system and prove the following theorem.

Theorem 3.3.

(1) *System $(\mathcal{C}_{\lambda\in\mathcal{H}})$ has a weak centre of finite order at most three at the origin, i.e., at most three critical periods can bifurcate from this centre.*
(2) *Moreover, there exists a perturbation of the centre of finite order three leading to the maximum number of critical periods.*
(3) *System $(\mathcal{C}_{\lambda\in\mathcal{H}})$ has no non-linear isochronous centre.*

Proof. For $u \neq 0$ in (3-12) and λ_*^2 the corresponding parameter value, we have

$$p_4(\lambda_*^2) = 0 \quad \text{if and only if } v = \frac{120u^2 - 64}{5u^3}. \tag{3-14}$$

Modulo p_2, and through the above transformation $(\mathcal{T}_1)$, the period coefficient $p_6(\lambda_*^2)$ is simplified to

$$p_6(u, v) = 15u^3(63u^2 - 116)v - 850u^6 + 15120u^4 + 1440u^2 - 3328. \tag{3-15}$$

Thus the analysis of the system of equations

$$p_2 = p_4 = p_6 = 0 \quad \text{with } A + C = 0 \tag{3-16}$$

is easily reduced to that of

$$\begin{aligned} v = f_4(u) &= \frac{120u^2 - 64}{5u^3} \\ f_6(u) &= 18944 - 52416u^2 + 37800u^4 - 8505u^6 = 0. \end{aligned} \tag{3-17}$$

System (3-17) has at least one real root, denoted (u_0, v_0), admissible for p_2. Indeed, by Sturm's algorithm, we verify that $f_6(u)$ has a unique real root u_0 in the interval $\left]\frac{74654}{10000}, \frac{74655}{10000}\right)$, with the corresponding v_0 in the interval $\left]\frac{13837}{10000}, \frac{138461}{10000}\right)$. The root (u_0, v_0) is not a zero of the next period coefficient p_8; precisely, modulo p_2, and, through the transformation $(\mathcal{T}_1)$ p_8 is determined, simplified via (3-17) to

$$\begin{aligned} f_8(u) = &- 3559140352 - 446772960u^2 + 15477215400u^4 \\ &- 12057953625u^6 + 266716800u^8. \end{aligned} \tag{3-18}$$

A direct calculation of the resultant of the polynomials $f_6(u)$ and $f_8(u)$ yields a nonzero constant. Therefore, from the theory of Resultant, these polynomials have no common root. Hence there exists a parameter value $\tilde{\lambda}_*^2$ satisfying

$$p_2(\tilde{\lambda}_*^2) = p_4(\tilde{\lambda}_*^2) = p_6(\tilde{\lambda}_*^2) = 0, \quad \text{with } p_8(\tilde{\lambda}_*^2) \neq 0. \tag{3-19}$$

The corresponding weak centre is therefore of finite order $k = 3$; and at most three critical periods can bifurcate from this centre by the Finite Order Bifurcation Theorem.

To show that there exists a perturbation leading to the maximum number of critical periods we consider the algebraic curves

$$\left\{\begin{array}{ll}\tilde{p}_4(u,v)=0, & (\text{equation of curve } (\mathcal{C}_4))\\ \tilde{p}_6(u,v)=0, & (\text{equation of curve } (\mathcal{C}_6))\end{array}\right. \tag{3-20}$$

$(\mathcal{C}_4)$ and $(\mathcal{C}_6)$ intersect transversally at their common root; indeed, the assumption $(\mathcal{A}_u): \quad u\neq 0 \quad$ and $63u^2-116\neq 0$, allows the equations
(3-21)

$$v=g_4(u)=\frac{120u^2-64}{5u^3}=g_6(u)=-\frac{-8505u^6+15120u^4+1440u^2-3328}{15u^3(63u^2-116)}.$$

$63u^2-116=0$ yields the period coefficients

$$p_4=\pm\sqrt{\frac{116}{63}}\times\frac{568}{63}v+\frac{3296}{21}, \quad \text{whereas } p_6=-\frac{368576}{147}<0. \tag{3-22}$$

This implies a corresponding weak centre of order at most two, and therefore leads to a bifurcation of at most two critical periods.

Now assuming $(\mathcal{A}_u)$, the curves g_4 and g_6 in (3-21) cannot be tangent at their common root; this results from the fact that the system of equations

$$F(u)=g_4(u)-g_6(u)=0, \quad T(u)=\frac{dg_4(u)}{du}-\frac{dg_6(u)}{du}=0 \tag{3-23}$$

has no real solution. Indeed, direct calculation of the resultant of $F(u)$ and $T(u)$ gives a nonzero constant. Consequently, the curves $(\mathcal{C}_4)$ and $(\mathcal{C}_6)$ have a transversal intersection that guarantees, in the neighbourhood of the common root (u_0,v_0), the existence of a perturbation λ^3_* such that

$$p_4(\lambda^3_*)\times p_6(\lambda^3_*)<0, \quad \text{with } p_8(\lambda^3_*)\neq 0. \tag{3-24}$$

Finally, in the neighbourhood of such a perturbation we may choose $\tilde{\lambda}^3_*$, to realize condition $(\mathcal{P}_k)$ with $k=3$, that is, the birth of three critical periods.

Hence system $p_2=p_4=p_6=p_8=0$ has no real solution, yielding an obstruction to the existence of an isochrone subclass in the family $(\mathcal{C}_{\lambda\in\mathcal{H}})$. $\square$

3.2 Symmetric stratum.

The parameter value satisfies the condition $B=D=0$. The corresponding system is denoted $(\mathcal{C}_{\lambda\in\mathcal{S}})$. From (3-8) the first period coefficient p_2 is given by

$$p_2(A,C,E)=18A^2+12AC+10C^2+E^2-CE-9AE-24. \tag{3-25}$$

The next three period coefficients are computed and simplified by the method of reduction modulo a Gröbner basis. We get the following.

Lemma 3.4. *From formulas* (3-6) *and* (3-7)*, and using transformation* $(\mathcal{T}_1)$*, the period coefficients* p_4*,* p_6*, and* p_8*, reduced modulo the Gröbner basis of the preceding, are respectively*

$$p_4(A,C,E)=Ag_4(C,E)+f_4(C,E), \quad \textit{[Gröbner basis of } [p_2]\textit{]} \tag{3-26}$$

with

$$g_4(C,E) = -120C - 54E + 32C^2 + 3CE^2 + 80C^3,$$
$$f_4(C,E) = -19CE - 600C^2 + 6E^2 + 320C^4 + 120C^3E + 6C^2E^2 - CE^3 - 252.$$

$$p_6(A,C,E) = Ag_6(C,E) + f_6(C,E), \quad \textit{[Gröbner basis of } [p_2,p_4]\textit{]}, \tag{3-27}$$

with

$$\begin{aligned} g_6(C,E) &= 20784C^2E^3 + 1647E^4C + 3472320C^2E - 2432496CE^2 \\ &\quad - 11815E^3 - 75479040C - 71743968E, \\ f_6(C,E) &= -1387210CE^3 + 12810720CE - 21529280C^3E - 12514400C^2E^2 \\ &\quad - 441360C^4E^2 + 167858E^4 - 45130C^2E^4 - 2399C^5E - 22440240 \\ &\quad + 29352960C^2 + 12012396E^2 + 6988800C^4 - 238280C^3E^3; \end{aligned}$$

$$p_8(A,C,E) = Ag_8(C,E) + f_8(C,E), \quad \textit{[Gröbner basis of } [p_2,p_4,p_6]\textit{]}, \tag{3-28}$$

where g_8 and f_8 are polynomials of respective degree 5 and 6 in the variables C and E.

As in the Hamiltonian case, the results are derived from the analysis of the system of equations:

$$p_2(A,C,E) = p_4(A,C,E) = p_6(A,C,E) = p_8(A,C,E) = 0. \tag{3-29}$$

We first prove

Theorem 3.5. *The hamiltonian stratum intersects the symmetric stratum to generate one local critical period.*

Proof. Using the above reduced period coefficients, we get at $(C,E) = (0,0)$

$$p_2(A,0,0) = 3A^2 - 4, \quad \text{with } p_4(A,0,0) = -252 < 0. \tag{3-30}$$

Let λ_*^3 be the corresponding parameter value with components $C = E = B = D = 0$, that is, at the intersection of the Hamiltonian and symmetric strata. Therefore the corresponding centre is of order at most one. Let us consider such a centre of order one; we construct a perturbation $\tilde{\lambda}_*^3$ of λ_*^3 with exactly one critical period. That is:

$$p_4(\tilde{\lambda}_*^1) \times p_2(\tilde{\lambda}_*^1) < 0,, \quad \text{condition } (\mathcal{P}_1). \tag{3-31}$$

To realize this conditon, we simply perturb the component A into $\tilde{A} = A + \epsilon_1$ with $A\epsilon_1 > 0$. From (3-31), it suffices to show that $p_2(\tilde{\lambda}_*^3) > 0$, that is,

$$3(A+\epsilon_1)^2 - 4 > 0, \quad \text{with } 3A^2 - 4 = 0. \tag{3-32}$$

This follows from

$$2A\epsilon_1 + \epsilon_1 > 0, \quad \text{for } A\epsilon_1 > 0. \tag{3-33}$$

We may always choose ϵ_1 to meet the above requirements. $\square$

A more general analysis leads to the following theorem:

Theorem 3.6.

(1) *System* $(\mathcal{C}_{\lambda\in\mathcal{S}})$ *has no non-linear isochrone centre at the origin.*
(2) *The origin is a weak centre of finite order of at most three, i.e., at most three critical periods can bifurcate from a such a centre.*

Proof. We note from lemma (3.4) that $g_4(C,E)$ and $f_4(C,E)$ are in $\mathbb{R}[C][E]$, i.e., polynomials in the variable E with coefficients in $\mathbb{R}[C]$. Thus, for $C \neq 0$, we easily derive a nonzero resultant of g_4 and f_4 using Sturm's algorithm. In other words, system

$$g_4(C,E) = f_4(C,E) = 0 \tag{3-34}$$

has no real solution.

Now assume $g_4(C,E)$ is nonzero. This yields

$$p_4(A,C,E) = 0 \quad \text{if and only if } A = q_4(C,E) = -\frac{f_4(C,E)}{g_4(C,E)}, \tag{3-35}$$

and allows us to reduce $p_2(A,C,E)$, $p_6(A,C,E)$, and $p_8(A,C,E)$ respectively to $q_2(C,E)$, $q_6(C,E)$, $q_8(C,E)$ polynomials in the variables C and E. The corresponding system of equations

$$\begin{gathered} A = q_4(C,E), \\ q_2(C,E) = q_6(C,E) = q_8(C,E) = 0 \end{gathered} \tag{3-36}$$

has no real solution by calculating the resultants of the pairs (q_2,q_6), (q_6,q_8), and (q_2,q_8). Thus we cannot get simultaneously $q_2 = q_6 = 0$ and $q_2 = q_8 = 0$. Hence the claim.

The origin is therefore a weak centre of finite order at most $k = 3$, i.e., by the Finite Order Bifurcation Theorem, no more than three critical periods can bifurcate from such a centre. In fact, we have shown that, if there exists a parameter value λ_* such that

$$p_2(\lambda_*) = p_4(\lambda_*) = p_6(\lambda_*) = 0, \tag{3-37}$$

then necessarily, $p_8(\lambda_*) \neq 0$. Moreover, as in the Hamiltonian case, this implies the non-existence of an isochrone system in the sub-family $(\mathcal{C}_{\lambda\in\mathcal{S}})$. $\square$

Here also, the maximum number of three critical periods can be attained by perturbation as below.

Theorem 3.7. *For a centre of order three as in theorem* (3-6)*, we may construct a perturbation with the maximum number three of critical periods. Moreover, condition* $(\mathcal{P}_k)$ *ascertains that, for every* $n \leq 3$*, there exists a perturbation with exactly* n *periods.*

Proof. Consider a weak centre of order three. Keeping the same notations, we first simplify the period coefficients $p_{2k}(A,C,E)$, for $k = 1,2,3,4$ via the following transformation $(\mathcal{T}_2)$

$$(A,C,E) = \left(\frac{2b_3 + 5b_5}{3}, b_5, b_1 + 3b_3 + 8b_5\right). \tag{3-38}$$

We obtain successively $p_2(b_1, b_3, b_5) = b_1^2 - b_3^2 + 16b_5^2 - 24$, and

$$p_4(b_1, b_3, b_5) = b_1 g_4(b_3, b_5) + f_4(b_3, b_5), \quad \text{modulo Gröbner basis of } [p_2], \tag{3-39}$$

with

$$\begin{aligned} g_4(b_3, b_5) &= -36b_3 - 198b_5 + 220b_5^3 + 71b_3b_5^2 + 4b_3^2b_5 \\ f_4(b_3, b_5) &= -1146b_3b_5 + 1188b_3b_5^3 + 12b_3^3b_5 + 225b_3^2b_5^2 - 132b_3^2 - 54. \end{aligned} \tag{3-40}$$

$$p_6(b_1, b_3, b_5) = b_1 g_6(b_3, b_5) + f_6(b_3, b_5), \quad \text{modulo Gröbner basis of } [p_2, p_4], \tag{3-41}$$

with g_6 and f_6 polynomials of degree 6 in b_3 and b_5, and so on for p_8 modulo the Gröbner basis of $[p_2, p_4, p_6]$. Having done this, we reduce the analysis of the system $p_2 = p_4 = p_6 = 0$ to a question of transversal intersection of some algebraic surfaces. More precisely, we consider the following surface equations:

(3-42)

$$p_2(b_1, b_3, b_5) = 0, \quad (\mathcal{S}_2), \quad p_4(b_1, b_3, b_5) = 0, \quad (\mathcal{S}_4), \quad p_6(b_1, b_3, b_5) = 0, \quad (\mathcal{S}_6).$$

From the equation of $(\mathcal{S}_4)$ we derive $b_1 = -\frac{f_4(b_3,b_5)}{g_4(b_3,b_5)}$. Substitution in $(\mathcal{S}_2)$ and $(\mathcal{S}_6)$ yields the following curves

$$\begin{cases} p_2(b_1, b_3, b_5) = \bar{p}_2(b_3, b_5) = 0, & (\tilde{\mathcal{C}}_2) \\ p_6(b_1, b_3, b_5) = \bar{p}_6(b_3, b_5) = 0, & (\tilde{\mathcal{C}}_6). \end{cases} \tag{3-43}$$

The algebraic curves $(\tilde{C}_j), j = 2, 6$, are non-singular and intersect transversally at a point $M_* = (b_3^*, b_5^*)$ in the rectangle

$$R_* = R_*^1 \times R_*^2 = \left[\frac{72}{100}, \frac{78}{100}\right] \times \left[-\frac{4}{100}, -\frac{2}{100}\right]. \tag{3-44}$$

Indeed, letting $J_b^1 = J_b^1(b_1, b_3, b_5)$ be the determinant of the Jacobian matrix $\left(\frac{\partial(p_2,p_4,p_6)}{\partial(b_1,b_3,b_5)}\right)$, it follows, using resultants, that system

$$p_2 = p_4 = p_6 = J_b^1 = 0 \tag{3-45}$$

has no real solution. Therefore, the surfaces (S_j), $j = 2, 4, 6$ are transversal. By Sturm's algorithm, each one of the polynomials

$$f_2^1(b_5) = \bar{p}_2\left(\frac{72}{100}, b_5\right), \quad f_2^2(b_5) = \bar{p}_2\left(\frac{78}{100}, b_5\right) \tag{3-46}$$

has a single real root in the interval $It^1 = (-\frac{4}{100}, -\frac{2}{100}]$, whereas polynomials

$$f_6^1(b_5) = \bar{p}_2\left(\frac{72}{100}, b_5\right), \quad f_6^2(b_5) = \bar{p}_2\left(\frac{78}{100}, b_5\right) \tag{3-47}$$

do not vanish over this interval. On the other hand, the interval $It^2 = (\frac{72}{100}, \frac{78}{100}]$ contains a single common real root of the polynomials

$$g_6^1(b_5) = \bar{p}_6\left(b_3, \frac{-2}{100}\right), \quad g_6^2(b_5) = \bar{p}_6\left(b_3, \frac{4}{100}\right), \tag{3-48}$$

and none for

$$g_2^1(b_5) = \bar{p}_2\left(b_3, \frac{-2}{100}\right); \quad g_2^2(b_5) = \bar{p}_2\left(b_3, \frac{4}{100}\right). \tag{3-49}$$

Moreover, the curves $\bar{p}_2 = 0$ and $\bar{p}_6 = 0$ are non-singular, for the polynomials $\bar{p}_2$ and $\bar{p}_6$ have nonvanishing discriminants respectively in the intervals It^2 and It^1. Finally, setting $J_b^2(b_3, b_5)$ as the determinant of the Jacobian matrix $\left(\frac{\partial(\bar{p}_2, \bar{p}_4)}{\partial(b_3, b_5)}\right)$, and again using Sturm's algorithm , one shows that this determinant does not vanish on the sides on the rectangle R_*, on the one hand; on the other hand, from discriminant(J_b^2, b_3) and discriminant(J_b^2, b_5), respectively polynomials in the variables b_3 and b_5, it results that the curve $J_b^2(b_3, b_5) = 0$ has no component in R_*. Thus the curves $(\tilde{C}_j), j = 2, 6$ are non-singular with a transversal intersection in the surface (S_4).

Substituting $b_1 = -\frac{f_4(b_3, b_5)}{g_4(b_3, b_5)}$ in the expression $J_b^1(b_1, b_3, b_5)$ gives a polynomial $H(b_3, b_5)$ in b_3 and b_5. As above, computing the resultants of the pairs $(\bar{p}_2, \bar{p}_6)$, $(\bar{p}_2, H)$ and $(\bar{p}_6, H)$ leads us to conclude that, on the algebraic surfaces, we do not get simultaneously

$$\bar{p}_2(b_3, b_5) = \bar{p}_6(b_3, b_5) = 0, \quad \text{and } \bar{p}_6(b_3, b_5) = H(b_3, b_5) = 0. \tag{3-50}$$

Therefore, the three algebraic surfaces (S_j), $j = 2, 4, 6$ are in general position, that is, transversal in the space $\{(b_1, b_3, b_5), b_j \in \mathbb{R}^3, j = 1, 3, 5\}$. This transversality argument guarantees the existence of a perturbation of the weak centre of order three to produce exactly three local critical periods. $\square$

4. Concluding Remarks

Comparing our results with [2], we may conclude that the cubic terms $(-y(x^2 + y^2), x(x^2 + y^2))$, provide a mechanism to destroy the isochrone stratas found in the case of quadratic systems in Bautin's normal form studied therein, on one hand, and, on the other hand, to increase by one the maximum number of critical periods. Such a mechanism might be generalized.

Our approach, efficiently integrating techniques from computational algebraic geometry, ideal theory, and transversality theory of algebraic curves, might be applied with much success to the global study, and as well as the more difficult case of the general return time function, i.e., period function of a polycycle using its Dulac asymptotic expansion.

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Solutions to Stationary Phase-Field Equations

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Abstract

In this paper we consider time-independent versions of the Phase-Field Equations introduced by O. Penrose and P. Fife. To this end we will investigate a system of semi-linear elliptic equations. We will show existence of solutions and give results on the multiplicity and the stability of solutions. This will be achieved by applying variational methods and bifurcation methods. We will also point out the physical significance of these results.

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Variational Methods, Phase Transitions.

1 Introduction

In recent years several authors investigated systems of non-linear parabolic equations known as phase field equations. These equations arise in studying temperature driven phase changes (cf., for example, [1, 3, 4, 9, 12, 16]) and usually consist of a non-linear parabolic equation of the form

$$\phi_t = \Delta\phi - s_\phi(x,\phi) + \lambda(x,\phi)u \tag{1}$$

for the phase parameter ϕ coupled with a linear or non-linear heat equation for the temperature u. The coupling term is of the form

$$\lambda(x,\phi)\phi_t, \tag{2}$$

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and represents the latent heat of the phase transition. The existence and uniqueness of solutions to such systems has been studied extensively (cf. [2, 4, 7, 8, 9, 11, 14, 16]), but to the best of our knowledge the steady state solutions to these equations have not been thoroughly investigated. In this paper we will consider stationary solutions to the system

$$\phi_t = \Delta\phi - s_\phi(x,\phi) + \lambda(x,\phi)u \tag{3}$$

$$\frac{u_t}{u^2} = \Delta u - \lambda(x,\phi)\phi_t \tag{4}$$

on a bounded domain $\Omega \subset \mathbf{R}^n$ with a sufficiently smooth boundary $\partial\Omega$. As in the papers on the time-dependent equations, we will restrict ourselves to the case that $n \leq 3$. However, most of the analysis can be carried out in higher dimensions as well.

We will use the same boundary conditions as in previous papers, namely

$$\frac{\partial\phi}{\partial\nu} = 0, \tag{5}$$

$$\frac{\partial u}{\partial\nu} = \left(u - u^2\theta_\Gamma\right), \tag{6}$$

however, one could also consider a linearized version of (6). In addition to these conditions we need to make the following assumptions:

- **(H 1)** The function $s(x,\cdot) \in C^3(A) \cap C(\overline{A})$ for an open interval $A \subseteq \mathbf{R}$, independent of $x \in \Omega$.
- **(H 2)** $s(x,\cdot)$ is a double-well potential for every $x \in \Omega$.
- **(H 3)** There exist positive constants α and β such that $\alpha \leq \theta_\Gamma \leq \beta$. Furthermore, $\theta_\Gamma \in H^1(\partial\Omega)$.
- **(H 4)** $\lambda(x,\phi) = a(x)\phi + b(x)$ with $a(x) \geq 0$ and $b(x) > 0$ smooth functions.

These are sufficient conditions for the existence of solutions to (3)–(6) with sufficiently smooth initial conditions. For the purpose of this paper the condition **(H 4)** could be eased to $\lambda_\phi > 0$ without significantly changing the arguments. For further analysis of the time-dependent problems we refer to the above mentioned papers as well as [6] for an overview of the problems. We would also like to mention [5] for a proof that such a model approximates a Stefan problem.

The function $-s_\phi(x,\phi)$ has three zeros (as a function of ϕ). Hence, the function $-s_\phi(x,\phi) + \lambda(x,\phi)u(x)$ will have at least one and at most three zeros (for fixed x). Zeros at which the derivative $-s_{\phi\phi}(x,\phi) + \lambda_\phi(x,\phi)u(x)$ is negative represent a single phase, zeros with positive derivative (with respect to ϕ) represent a point at which both phases are present.

In this paper we are interested in the time-independent stationary problem. In particular, we will investigate the system of non-linear elliptic equations

$$-\Delta\phi = -s_\phi(x,\phi) + \lambda(x,\phi)u, \tag{7}$$
$$-\Delta u = 0, \tag{8}$$

together with the above mentioned boundary conditions (5)– (6). This system is essentially decoupled, and (8) is just a homogeneous Laplace equation with non-linear boundary conditions. Aside from stating a maximum principle for u, we will concentrate on the study of solutions to (7) for a given function u satisfying (8).

In section 2 of the paper we will use variational techniques to prove a general existence result for (7) with arbitrary smooth $u(x)$. This is very similar to a result proved in [4] for a specific choice of the function s.

In section 3 we will give a more specific result for functions u which are close to a constant. This will also show that solutions to (7) are in general not unique. The results of this section rely on techniques from general bifurcation theory. The physical significance of these results will be summarized at the end of the section.

In section 4 we will analyze the linear stability of solutions to the time-independent system. In this section we will treat the entire system including the nonlinear heat equation. We will give a sufficient condition for stability and give an example of an unstable solution which consist entirely of "mushy regions", i. e. a solution for which at every point in the domain both phases exist.

Throughout the paper we will extensively use the Hilbert spaces $X = L^2(\Omega)$ and $Y = H^2(\Omega)$.

2 Variational Approach — Existence of Solutions

The main result of this section is the generalization of a result that first appeared in [4]. We will use a variational approach to prove the general existence of a solution to (7). To begin we have to state a maximum principle for solutions of (8).

Lemma 2.1 *Let $u(x)$ be a smooth solution of (8) with boundary conditions (6). Then $u(x)$ takes its extrema on the boundary of Ω and*

$$\frac{1}{\beta} \le u(x) \le \frac{1}{\alpha}$$

for all $x \in \overline{\Omega}$.

We omit the simple proof of this lemma and continue with the investigation of (7). We will assume that $u(x)$ is a solution to (8) with boundary conditions (6).

We start by considering the equation

$$-s_\phi(x,\phi) + u(x)\lambda(x,\phi) = 0, \tag{9}$$

which can be solved for ϕ at any point $x \in \Omega$. The equation will have at least one and at most three solutions, which we denote by

$$\phi_1(x) \leq \phi_2(x) \leq \phi_3(x).$$

The functions $\phi_j(x)$ will be continuous in x, But their domains Ω_j might not be the entire domain Ω. However, for every $x \in \Omega$ at least one of the three functions exists and will be continuous. To continue we define

$$\underline{\phi}(x) = \min\left\{\phi_1(x), \phi_2(x), \phi_3(x)\right\}, \tag{10}$$

where the minimum is taken over the functions ϕ_j which exist at the point x. Similarily define

$$\overline{\phi}(x) = \max\left\{\phi_1(x), \phi_2(x), \phi_3(x)\right\}. \tag{11}$$

We are now ready to state a maximum principle for the solutions of (7).

Lemma 2.2 *Define*

$$c_1 = \inf_{x\in\Omega} \underline{\phi}(x),$$

and

$$c_2 = \sup_{x\in\Omega} \overline{\phi}(x),$$

where $\underline{\phi}(x)$ and $\overline{\phi}(x)$ are defined as above. Then for any smooth solution $\phi(x)$ of (7) we have:

$$c_1 \leq \phi(x) \leq c_2. \tag{12}$$

Proof: First we observe that since $u(x)$ is a bounded continuous function, the same holds for $\phi_j(x)$, hence c_1 and c_2 are finite numbers.

Now suppose $\phi(x)$ is a smooth solution to (7), and that ϕ has a maximum at $x_0 \in \Omega$. We have:

$$0 \leq -\Delta\phi(x_0) = -s_\phi(x_0, \phi(x_0)) + u(x_0)\lambda(x_0, \phi(x_0)). \tag{13}$$

Since the right hand side of this inequality becomes negative if $\phi(x_0)$ is larger than the largest root of the right-hand side it follows that:

$$\phi(x) \leq \phi(x_0) \leq \max\left\{\phi_1(x_0), \phi_2(x_0), \phi_3(x_0)\right\} \leq c_2.$$

Since we have $\partial_\nu \phi|_{\partial\Omega} = 0$, we have that ϕ must be concave down near a maximum at $x_0 \in \partial\Omega$ as well. For sufficiently smooth boundary and ϕ we have

$$\lim_{x \to x_0} \Delta\phi(x) \leq 0,$$

and (13) holds on the boundary as well. Analogously we prove the statement for the lower bound of $\phi(x)$.

$\square$

To continue we formulate our equation as a variational problem. (7) is the Euler-Lagrange equation for the following variational problem. We want to find a function $\phi \in H^1(\Omega)$ for which the functional

$$\mathcal{F}(\phi) = \int_\Omega \left(\frac{1}{2} |\nabla \phi|^2 + s(x,\phi) - u(x)\Lambda(x,\phi) \right) dx, \tag{14}$$

where $\Lambda_\phi(x,\phi) = \lambda(x,\phi)$, has a critical point. The functional $\mathcal{F}(\phi)$ represents of course the free energy of the system (cf. [12, 4], for example).

Any solution $\phi \in H^1(\Omega)$ to this variational problem is also a solution of the original equation. The additional regularity required for ϕ to be a classical solution follows from elliptic regularity theory.

We proceed by proving, that there exists a solution to the variational problem.

Theorem 2.1 *Let $C = \inf_{\phi \in H^1(\Omega)} \mathcal{F}(\phi)$. Then $|C| < \infty$ and there exists a $\phi^* \in H^1(\Omega)$ such that*

$$C = \mathcal{F}(\phi^*),$$

i. e. ϕ^* is a critical point of $\mathcal{F}(\phi)$.

Proof: As stated in the previous lemma, we have that

$$s_\phi(x,\phi) - u(x)\lambda(x,\phi) > 0 \quad \text{for} \quad \phi > c_2,$$

and

$$s_\phi(x,\phi) - u(x)\lambda(x,\phi) < 0 \quad \text{for} \quad \phi < c_1.$$

It follows that the function $s(x,\phi) - u(x)\Lambda(x,\phi)$ is bounded below, i. e. there is a $C_0 > 0$ such that

$$s(x,\phi) - u(x)\Lambda(x,\phi) > -C_0.$$

This implies immediately that

$$\begin{aligned} \mathcal{F}(\phi) = \int_\Omega \left(\frac{1}{2} |\nabla \phi|^2 + s(x,\phi) - u(x)\Lambda(\phi) \right) dx \quad &\geq \quad \int_\Omega \left(s(x,\phi) - u(x)\Lambda(\phi) \right) dx \\ &\geq \quad -\mu(\Omega) C_0 \\ &= \quad -C_1 > -\infty. \end{aligned}$$

Hence, $\inf_{\phi\in H^1(\Omega)} \mathcal{F}(\phi) = C \geq -C_1 > -\infty$. Therefore, there exists a minimizing sequence $\phi_n \in H^1(\Omega)$ such that

$$\mathcal{F}(\phi_n) \to C \quad \text{as} \quad n \to \infty.$$

Without loss of generality we may assume that $\mathcal{F}(\phi_n) \leq C+1$. To continue, we replace this minimizing sequence with one that is uniformly bounded as follows.

$$\hat{\phi}_n = \begin{cases} c_1 & \text{for} \quad \phi_n < c_1 \\ \phi_n & \text{for} \quad c_1 \leq \phi_n \leq c_2 \\ c_2 & \text{for} \quad \phi_n > c_2 \end{cases},$$

where c_1 and c_2 are the bounds of Lemma 2.2. It follows that $\hat{\phi}_n \in H^1(\Omega)$. Furthermore, the definitions of c_1 and c_2 imply that $s(x,\phi) - u(x)\Lambda(x,\phi)$ is an increasing function of ϕ for $\phi \geq c_2$ and a decreasing function of ϕ for $\phi \leq c_1$ (see Lemma 2.2). Thus we have

$$\mathcal{F}(\hat{\phi}_n) \leq \mathcal{F}(\phi_n),$$

i. e. $\left\{\hat{\phi}_n\right\}_{n=1}^{\infty}$ is also a minimizing sequence. Furthermore, since

$$\int_\Omega \frac{1}{2}\left|\nabla\hat{\phi}_n\right|^2 dx = \mathcal{F}(\hat{\phi}_n) - \int_\Omega \left(s(x,\hat{\phi}_n) - u(x)\Lambda(x,\hat{\phi}_n)\right) dx \leq C + 1 + C_2$$

for an appropriate constant C_2, the sequence is uniformly bounded in $H^1(\Omega)$. Since $H^1(\Omega)$ is a Hilbert space there exists a weakly convergent subsequence of this sequence, which we also denote by $\left\{\hat{\phi}_n\right\}_{n=1}^{\infty}$. Let $\tilde{\phi}$ denote the weak limit of the subsequence. It follows, that

$$\int_\Omega \frac{1}{2}\left|\nabla\tilde{\phi}\right|^2 dx \leq \liminf_{n\to\infty} \int_\Omega \frac{1}{2}\left|\nabla\hat{\phi}_n\right|^2 dx. \tag{15}$$

Since $c_1 \leq \hat{\phi}_n \leq c_2$, the same holds for $\tilde{\phi}$. We can use the mean-value theorem to get

$$\begin{aligned} &\left|\int_\Omega \left(s(x,\hat{\phi}_n) - u(x)\Lambda(x,\hat{\phi}_n) - s(x,\tilde{\phi}) + u(x)\Lambda(x,\tilde{\phi})\right) dx\right| \\ &\qquad\qquad \leq \int_\Omega \sup_{c_1\leq\xi\leq c_2} |s_\phi(x,\xi) - u(x)\lambda(\xi)| \left|\hat{\phi}_n - \tilde{\phi}\right| dx \\ &\qquad\qquad \leq C_3 \left\|\hat{\phi}_n - \hat{\phi}\right\|_{L^2(\Omega)}, \end{aligned}$$

for some constant C_3. But by the Sobolev imbedding theorem, $\hat{\phi}_n$ converges strongly to $\hat{\phi}$ in $L^2(\Omega)$, which in turn implies

$$\int_\Omega \left(s(x,\hat{\phi}_n) - u(x)\Lambda(x,\hat{\phi}_n)\right) dx \to \int_\Omega \left(s(x,\tilde{\phi}) - u(x)\Lambda(x,\tilde{\phi})\right) dx$$

as $n \to \infty$. Together with (15) this implies

$$\mathcal{F}(\tilde{\phi}) \leq \liminf_{n\to\infty} \mathcal{F}(\hat{\phi}_n) = C,$$

i. e. $\tilde{\phi}$ is a solution to the variational problem.

$\square$

Remark:

This section shows that for any function u, there is a solution ϕ to the elliptic equation (7). For (ϕ, u) to be a solution of the system, u must also be a harmonic function which satisfies the boundary condition (6). For suitable boundary conditions, the inverse temperature u can vary over a quite large range of positive values. This implies that in some regions of Ω the function $-s_\phi(x,\phi) + \lambda(x,\phi)u(x)$ has only the largest root, whereas in other regions the function has only the smallest root. This insures the existence of solutions for which both phases are present in different areas of Ω. Lemma 2.2 gives a simple a priori bound for smooth solutions to the equation.

3 Bifurcation Theory and Single Phase Solutions

In this section we study the solutions to (7) more closely. We will restrict ourselves to the generic case when the function s does not explicitely depend on x. Furthermore, we will assume that

$$\lambda(\phi) = a\phi + b,$$

for constants $a \geq 0$ and $b > 0$.

We will apply a result from generic bifurcation theory to show the multiplicity of specific solutions, which are nearly constant. We start by stating the obvious result for constant temperatures $u(x) = \overline{u}$.

Lemma 3.1 *Let $\hat{\phi}$ be a solution of the equation*

$$-s_\phi(\hat{\phi}) + \overline{u}\lambda(\hat{\phi}) = 0. \tag{16}$$

Then $\phi(x) \equiv \hat{\phi}$ is clearly a solution of (7).

The proof of this is obvious. This result also shows, that we cannot expect uniqueness for solutions of this equation, since the algebraic equation (16) may have up to three distinct solutions.

We continue by showing that there exist solutions for $u \in X = L^2(\Omega)$ provided that u is close enough to a constant $\overline{u}$. To do this we let Σ denote the spectrum of the negative Laplacian on Ω with Neumann boundary conditions. It is well known that Σ is a pure point spectrum consisting of countably many non-negative numbers. Furthermore, Σ is nowhere dense in $\mathbf{R}^+$. Using this one can state:

Theorem 3.1 *Let $\overline{u}$ be constant and $\hat{\phi}$ be one of the solutions to*

$$-s_\phi(\hat{\phi}) + \overline{u}\lambda(\hat{\phi}) = -s_\phi(\hat{\phi}) + \overline{u}\left(a\hat{\phi} + b\right) = 0.$$

Furthermore, assume that

$$\overline{u}a - s_{\phi\phi}(\hat{\phi}) \notin \Sigma.$$

Then there exist positive numbers ξ_0 and η_0, such that for every $u \in X$ with $\|u - \overline{u}\|_X < \eta_0$ there exists a unique $\phi \in Y = H^2(\Omega)$ with $\left\|\phi - \hat{\phi}\right\|_Y < \xi_0$ that solves (7).

Proof: We will apply a result from genral bifurcation theory, namely Theorem 8.C of [15]. Observe that

$$\begin{aligned} -\Delta\phi &= -s_\phi(\phi) + au\phi + bu \\ -\Delta\hat{\phi} &= -s_\phi(\hat{\phi}) + a\overline{u}\hat{\phi} + b\overline{u} \end{aligned}$$

implies

$$\begin{aligned} -\Delta\left(\phi - \hat{\phi}\right) &= -s_\phi(\phi) + s_\phi(\hat{\phi}) + au\left(\phi - \hat{\phi}\right) + a\hat{\phi}\left(u - \overline{u}\right) + b\left(u - \overline{u}\right) \\ &= -s_\phi(\phi) + s_\phi(\hat{\phi}) + s_{\phi\phi}(\hat{\phi})\left(\phi - \hat{\phi}\right) + a\left(u - \overline{u}\right)\left(\phi - \hat{\phi}\right) \\ &\quad - \left(s_{\phi\phi}(\hat{\phi}) - a\overline{u}\right)\left(\phi - \hat{\phi}\right) + \left(a\hat{\phi} + b\right)\left(u - \overline{u}\right) \end{aligned}$$

Let $\psi = \phi - \hat{\phi}$ and $\theta = u - \overline{u}$. After bringing $\left(s_{\phi\phi}(\hat{\phi}) - a\overline{u}\right)\psi$ to the left the last equation becomes

$$-\Delta\psi - \left(a\overline{u} - s_{\phi\phi}(\hat{\phi})\right)\psi = G(\psi, \theta). \tag{17}$$

Since $a\overline{u} - s_{\phi\phi}(\hat{\phi}) \notin \Sigma$, the linear operator $L = -\Delta - (a\overline{u} - s_{\phi\phi}(\hat{\phi}))$ on the left has an inverse operator $L^{-1} : X \to Y$. Furthermore, the operator

$$G : Y \times X \to X$$

mapping

$$G : (\psi, \theta) \mapsto G(\psi, \theta)$$

has the following properties:

- $G(0,0) = 0$
- If $\|\psi\|_Y < \xi$ and $\|\theta\|_X < \eta$ for positive numbers ξ and η, then

$$\|G(\psi,\theta)\|_X \leq g(\xi,\eta), \tag{18}$$

where $g(\xi,\eta) = O(\xi^2) + O(\eta)$ as $\xi \to 0$ and $\eta \to 0$. Indeed, by applying the mean-value theorem twice we get

$$\begin{aligned}
|G(\psi,\theta)| &\leq \left|-s_\phi(\phi) + s_\phi(\hat{\phi}) + s_{\phi\phi}(\hat{\phi})\psi\right| + |a\theta\psi| + \left|\left(a\hat{\phi} + b\right)\theta\right| \\
&\leq \left|s_{\phi\phi}(\hat{\phi})\psi - s_{\phi\phi}(\tau)\psi\right| + |a\theta\psi| + \left|\left(a\hat{\phi} + b\right)\theta\right| \\
&\leq C_1 |\psi|^2 + |a\theta\psi| + \left|\left(a\hat{\phi} + b\right)\theta\right|
\end{aligned}$$

After squaring this last inequality, integrating the result over Ω and applying the Sobolev imbedding theorem (18) follows immediately. Furthermore, one sees that one can choose

$$g(\xi,\theta) = C_1\xi^2 + C_2\eta$$

- If $\|\psi\|_Y \leq \xi$ and $\|\theta\|_X \leq \eta$ for positive ξ and η, then

$$\|G(\psi_1,\theta) - G(\psi_2,\theta)\|_X \leq h(\xi,\eta)\, \|\psi_1 - \psi_2\|_Y\,, \tag{19}$$

where $h(\xi,\eta) \to 0$ as $\xi \to 0$ and $\eta \to 0$. To prove this we observe that the mean value theorem implies

$$\begin{aligned}
|G(\psi_1,\theta) - G(\psi_2,\theta)| &\leq \left|-s_\phi(\phi_1) + s_\phi(\phi_2) + s_{\phi\phi}(\hat{\phi})\,(\psi_1 - \psi_2)\right| + |a\theta| \quad |\psi_1 - \psi_2| \\
&\leq \left|s_{\phi\phi}(\phi^*) - s_{\phi\phi}(\hat{\phi})\right| \quad |\psi_1 - \psi_2| + |a\theta| \quad |\psi_1 - \psi_2| \\
&\leq \left(\left|s'_{\phi\phi}(\phi^{**})\left(\phi^* - \hat{\phi}\right)\right| + |a\theta|\right) |\psi_1 - \psi_2|\,.
\end{aligned}$$

Squaring and integrating this inequality gives the desired result with

$$h(\xi,\eta) = C_3\xi + C_4\eta.$$

Therefore, all the hypotheses of Theorem 8.C of [15] are satisfied, and the result follows by applying this theorem.

□

This theorem has the following immediate consequences.

Remarks:

- If
$$-s_\phi(\phi) + \overline{u}\lambda(\phi) < 0,$$
then it is automatically not a part of the spectrum Σ and the theorem applies. Physically this case corresponds with single phase solutions.

- Clearly the theorem cannot be applied if $\hat{\phi}$ is a multiple root of
$$-s_\phi(\phi) + \overline{u}\lambda(\phi) = 0,$$
since in this case
$$\overline{u} - s_{\phi\phi}(\hat{\phi}) = 0 \in \Sigma.$$
However, this is not a generic case and will not be studied here.

In order to more closely investigate this result we need the following classical lemma which gives an estimate of $\|L^{-1}\|$.

Lemma 3.2 *Let $L = -\Delta - \nu$ on Ω together with Neumann boundary conditions. Then the operator $L^{-1} : X \to Y$ satisfies*

$$\|L^{-1}\| \leq C_0 \left(1 + \frac{|\nu|}{\inf\limits_{\lambda\in\Sigma} |\nu - \lambda|}\right),$$

where C_0 is a constant which does not depend on ν. Moreover, for negative values of ν $\|L^{-1}\|$ is uniformly bounded.

Proof: The proof of this lemma is a straight forward application of some results from Fourier analysis and can be found in most standard texts on Spectral Theory. We will only briefly outline it. For an arbitrary $f \in X$ let v be the solution of
$$Lv = f.$$
We can expand both v and f in terms of the eigenfunctions φ_j of the negative Laplacian with Neumann boundary conditions. Using $v = \sum_{j=0}^{\infty} v_j \varphi_j$ and $f = \sum_{j=0}^{\infty} f_j \varphi_j$ we see that
$$(\lambda_j - \nu)\, v_j = f_j,$$
where λ_j is the j-th eigenvalue of the negative Laplacian. Plancherel's theorem implies that

$$\|v\|_X^2 = \sum_{j=0}^{\infty} v_j^2 = \sum_{j=0}^{\infty} \frac{f_j^2}{(\lambda_j - \nu)^2} \leq \frac{1}{\inf\limits_{\lambda\in\Sigma} |\nu - \lambda|^2} \|f\|_X^2 .$$

In order to get an estimate on the $Y-$norm of the solution we observe that

$$-\Delta v = \nu v + f,$$

which immediately implies

$$\|v\|_{H^2(\Omega)} \leq C_0 \left(|\nu| \, \|v\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right),$$

which immediately implies the first part of the Lemma. Since all eigenvalues of the negative Laplacian are non-negative it follows that for

$$\inf_{\lambda \in \Sigma} |\nu - \lambda| = |\nu|,$$

for negative values of ν, i.e. L^{-1} is uniformly bounded for such values of ν.

□

In the setting of Theorem 3.1, negative values of ν correspond to the smallest and largest simple root of $-s_\phi(\phi) + \overline{u}\lambda(\phi)$. Physically these roots correspond to single phase steady states. We can now formulate the following theorem.

Theorem 3.2 *Let $0 < \alpha \leq \theta_\Gamma \leq \beta$ be the bounds on the boundary conditions for the inverse temperature u. Furthermore, let C_i, $i = 1, 2, 3, 4$ be the constants in the proof of Theorem 3.1. Then there exists at least one solution in the sense of Theorem 3.1 if*

$$\min\left\{ \left(\frac{1}{2\|L^{-1}\|C_1} \right)^4, \left(\frac{\left(1 - \frac{C_3}{2C_1}\right)}{\|L^{-1}\|C_4} \right)^2 \right\} > \mu(\Omega) \left(\frac{1}{\alpha^2} - \frac{1}{\beta^2} \right), \tag{20}$$

where $\mu(\Omega)$ denotes the usual Lebesgue measure of Ω.

Proof: Let $u(x)$ be an arbitrary positive solution to equation (8) with boundary values as in the statement of the theorem. Define

$$\overline{u} = \frac{1}{\mu(\Omega)} \int_\Omega u(x) \, dx = \frac{1}{\mu(\Omega)} \|u\|_{L^1(\Omega)}.$$

It follows that

$$\|u - \overline{u}\|_X^2 = \int_\Omega |u(x) - \overline{u}|^2 \, dx = \|u\|_X^2 - \frac{1}{\mu(\Omega)} \|u\|_{L^1(\Omega)}.$$

Using the bounds on u from Lemma 2.1 we obtain

$$\|u - \overline{u}\|_X^2 \leq \mu(\Omega) \left(\frac{1}{\alpha^2} - \frac{1}{\beta^2} \right). \tag{21}$$

So we can ensure existence of a solution if η_0 in the statement of Theorem 3.1 satisfies:

$$\eta_0^2 \geq \mu(\Omega) \left(\frac{1}{\alpha^2} - \frac{1}{\beta^2} \right). \tag{22}$$

¿From the proof of Theorem 8.C of [15] we have that ξ_0 and η_0 are chosen to satisfy

$$\left\|L^{-1}\right\|\left(C_1\xi_0^2 + C_2\eta_0\right) \leq \xi_0, \tag{23}$$

and

$$\left\|L^{-1}\right\|\left(C_3\xi_0 + C_4\eta_0\right) < 1. \tag{24}$$

¿From the first equation we get that

$$\eta_0 \leq \frac{\xi_0}{C_2}\left(\frac{1}{\|L^{-1}\|} - C_1\xi_0\right) \leq \frac{1}{\left(2\|L^{-1}\|C_1\right)^2}.$$

At the ξ_0 value corresponding to this maximum the second condition gives

$$\eta_0 < \frac{1 - \frac{C_3}{C_1}}{\|L^{-1}\|C_4}.$$

Combining these results we get the sufficient condition from the theorem.

□

Remarks:

- The graph of $-s_\phi(\phi)+\overline{u}\lambda(\phi)$ has at least one root. Theorem 3.2 states that for temperature fields which are nearly constant, there will always be at least one solution. If there is only one solution, this solution corresponds to a single phase.
- The other possibility is that $-s_\phi(\phi)+\overline{u}\lambda(\phi)$ has three solutions for which inequality (20) is satisfied. Then equation (7) has also three solutions, one of which corresponds to a root of $-s_\phi(\phi)+\overline{u}\lambda(\phi)$, for which both phases are present at every point. At this root the derivative of $-s_\phi(\phi)+\overline{u}\lambda(\phi)$ with respect to ϕ is positive. This solution makes no physical sense. Areas in which both phases are present are commonly called mushy regions.
- The conditions (20) in the statement of Theorem 3.2 are not optimal. Indeed, the conditions (22)–(24) in the proof imply that (ξ_0, η_0) lie in a region in the $\xi\eta$-plane bounded by a parabola (23), a straight decreasing line (24), and a horizontal line (22). One can easily see that this area is not empty if and only if

$$\frac{1}{\|L^{-1}\|C_4} - \frac{C_3}{C_4}\hat{\xi} > \sqrt{\mu(\Omega)\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)},$$

where $\hat{\xi}$ is the smallest root of

$$\frac{\xi}{C_2}\left(\frac{1}{\|L^{-1}\|} - C_1\xi\right) - \sqrt{\mu(\Omega)\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)}.$$

The last theorem also has some implications on the stability of solutions. Solutions (ϕ, u) for which the quantity $u(x) - s_{\phi\phi}(\phi)$ is negative correspond to a single phase. Theorem 8.C of [15] can always be applied (even if u is not a constant) and shows that the map

$$u(x) \mapsto \phi(x),$$

which assigns a single phase solution ϕ to a function $u(x)$ is continuous, i. e. single phase solutions are stable in the sense that small changes in temperature field $u(x)$ will imply small changes in the phase field ϕ. However, this is not the case for solutions for which $u(x) - s_{\phi\phi}(\phi)$ is negative. In this case continuity cannot be guaranteed, and a rather small change of $u(x)$ could cause a relatively large change of ϕ. These latter solutions correspond to values of the phase parameter at which both phases are present at the same location. Regions in which this happens are so called mushy regions. Solutions which consist of entirely mushy regions are inconsistent with the experimental experience. It is therefore to be expected that such solutions should be unstable.

4 Linearization and Asymptotic Stability

This section is devoted to the investigation of the asymptotic stability of the solutions to the stationary system (7) and (8). Whereas in the last section we considered the map

$$u(x) \mapsto \phi(x),$$

we now assume that the pair (u_0, ϕ_0) is a solution to the stationary phase field system and we want to analyze its stability. Furthermore, we allow λ tobe explicitely dependent on x.

We start by linearizing the parabolic system (3)-(4) about a solution $(\phi_0(x), u_0(x))$ to the elliptic system. Substituting

$$\begin{aligned} \phi(x,t) &= \phi_0(x) + \varepsilon\phi_1(x,t), \\ u(x,t) &= u_0(x) + \varepsilon u_1(x,t), \end{aligned}$$

in the parabolic equations and the boundary conditions one gets – after using the properties of $(\phi_0(x), u_0(x))$ – a linear parabolic system

$$\phi_{1t} = \Delta\phi_1 - (s_{\phi\phi}(x,\phi_0) - \lambda_\phi(x,\phi_0)u_0)\,\phi_1 + \lambda(x,\phi_0)u_1, \tag{25}$$

$$\frac{u_{1t}}{u_0^2} = \Delta u_1 - \lambda(x,\phi_0)\phi_{1t}, \tag{26}$$

with the following boundary conditions

$$\frac{\partial\phi_1}{\partial\nu} = 0, \tag{27}$$

$$\frac{\partial u_1}{\partial\nu} = u_1\,(1 - 2u_0\theta_\Gamma)\,. \tag{28}$$

In addition we assume that the solutions $(\phi_1(x,t), u_1(x,t))$ satisfy the initial conditions

$$\begin{aligned} \phi_1(x,0) &= \phi_1^0(x), && (29)\\ u_1(x,0) &= u_1^0(x). && (30) \end{aligned}$$

Using the terminology of [13], we say that the stationary solution (ϕ_0, u_0) is **asymptotically stable**, if the solutions (ϕ_1, u_1) of the linearized system satisfy

$$\lim_{t\to\infty} \left(\|\phi_1(t)\|_X + \|u_1(t)\|_X\right) = 0,$$

for all initial values with $\left\|\phi_1^0\right\|_X \leq 1$ and $\left\|u_1^0(x)\right|_X \leq 1$. Since the non-linear parabolic system is not semi-linear, we cannot directly apply the usual methods (see, for example, Chapter 11 of [13]). However, only slight modifications are necessary.

In the following theorem we give sufficient conditions for asymptotic stability. This will show that single phase solutions are asymptotically stable.

Theorem 4.1 *Suppose that $(\phi_0, u_0)$is a solution to the stationary problem (7)-(8) with*

1. $s_{\phi\phi}(x,\phi_0) - \lambda_\phi(x,\phi_0) > 0$ *for all* $x \in \overline{\Omega}$.
2. $u_0 > \frac{1}{2\theta_\Gamma}$ *for all* $x \in \partial\Gamma$.

Then the solution is asymptotically stable.

Proof: We only sketch the proof of this theorem and omit some of the details. The theory of systems of linear parabolic equations implies that (25)-(26) has a unique classical solution.

To continue we multiply (25) by ${\phi_1}_t$ and (26) by u_1. Adding the results and integrating over Ω yields:

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\frac{u_1(t)}{u_0}\right\|_X^2 + \|\nabla\phi_1(t)\|_X^2 + \int_\Omega \left(s_{\phi\phi}(x,\phi_0) - \lambda_\phi(x,\phi_0)\right)\phi_1^2(x,t)\,dx\right) = -C(t), \tag{31}$$

where

$$\begin{aligned} C(t) &= \|{\phi_1}_t(t)\|_X^2 + \|\nabla u_1(t)\|_X^2 - \int_{\partial\Omega} u_1(t)\frac{\partial u_1(t)}{\partial\nu}\,d\gamma \\ &= \|{\phi_1}_t(t)\|_X^2 + \|\nabla u_1(t)\|_X^2 - \int_{\partial\Omega} u_1^2(t)\left(1 - 2u_0\theta_\Gamma\right)\,dx \\ &\geq 0. \end{aligned}$$

In the next step we integrate (31) to get

$$-\int_0^t C(s)\,ds = \frac{1}{2}\left(\left\|\frac{u_1}{u_0}\right\|_X^2 + \|\nabla\phi_1\|_X^2 + \int_\Omega \left(s_{\phi\phi}(x,\phi_0) - \lambda_\phi(x,\phi_0)\right)\phi_1^2(x)\,dx\right)\Bigg|_{s=0}^t.$$

If there exists a $\delta > 0$ such that $C(t) \geq \delta$ for all $t > 0$, there is a number $\tau > 0$ such that

$$\int_0^\tau C(s)\,ds = \frac{1}{2}\left(\left\|\frac{u_1^0}{u_0}\right\|_X^2 + \left\|\nabla\phi_1^0\right\|_X^2 + \int_\Omega \left(s_{\phi\phi}(x,\phi_0) - \lambda_\phi(x,\phi_0)\right)(\phi_1^0)^2(x)\,dx\right),$$

i.e.

$$\frac{1}{2}\left(\left\|\frac{u_1(\tau)}{u_0}\right\|_X^2 + \|\nabla\phi_1(\tau)\|_X^2 + \int_\Omega \left(s_{\phi\phi}(x,\phi_0) - \lambda_\phi(x,\phi_0)\right)\phi_1^2(x,\tau)\,dx\right) = 0,$$

and hence we have $\phi_1(x,\tau) = 0$ and $u_1(x,\tau) = 0$.

In the other case observe that the right-hand-side of (4) is bounded below by

$$-\frac{1}{2}\left(\left\|\frac{u_1^0}{u_0}\right\|_X^2 + \left\|\nabla\phi_1^0\right\|_X^2 + \int_\Omega \left(s_{\phi\phi}(x,\phi_0) - \lambda_\phi(x,\phi_0)\right)(\phi_1^0)^2(x)\,dx\right),$$

which immedately implies that $\int_0^\infty C(s)\,ds$ exists. Thus, we have $\lim_{t\to\infty} C(t) = 0$. This implies that $\phi_1(x,t) \to \hat{\phi}(x)$ and $u_1(x,t) \to \hat{u}(x)$ as $t \to \infty$ for some functions $\hat{\phi}$ and $\hat{u}$ which satisfy

$$\begin{aligned}
\Delta\hat{\phi} &= \left(-s_{\phi\phi}(x,\phi_0) + \lambda_\phi(x,\phi_0)\right)\hat{\phi} + \lambda(x,\phi_0)\hat{u} \\
\Delta\hat{u} &= 0,
\end{aligned}$$

and the boundary conditions (27) and (28). The result now follows from immediately from the following lemma.

$\square$

Lemma 4.1 *Suppose* $(\hat{\phi},\hat{u})$ *is a smooth solution to*

$$\begin{aligned}
-\Delta\hat{\phi} &= \left(-s_{\phi\phi}(x,\phi_0) + \lambda_\phi(x,\phi_0)u_0\right)\hat{\phi} + \lambda(x,\phi_0)\hat{u}, \\
-\Delta\hat{u} &= 0,
\end{aligned}$$

with the same boundary conditions as above. Furthermore, assume that the hypotheses of the theorem hold.

Then $(\hat{\phi},\hat{u}) = (0,0)$

Proof: Suppose $\hat{u}$ has a positive maximum, then this maximum is on the boundary and

$$\frac{\partial\hat{u}}{\partial\nu} \geq 0.$$

But we have

$$\frac{\partial\hat{u}}{\partial\nu} = \hat{u}\left(1 - 2u_0\theta_\Gamma\right) < 0,$$

and therefore we have

$$\hat{u} \leq 0, \qquad \forall x \in \overline{\Omega}.$$

The analogous argument for negative minima shows that $\hat{u} \geq 0$ and therefore

$$\hat{u} = 0, \qquad \forall x \in \overline{\Omega}.$$

Using this in the first equation we get

$$\left\| \nabla \hat{\phi} \right\|_X^2 = \int_\Omega \left(-s_{\phi\phi}(x, \phi_0) + \lambda_\phi(x, \phi_0) u_0 \right) \hat{\phi}^2 \, dx \leq 0,$$

which immediately implies $\hat{\phi} = 0$.

$\square$

We finish this section by giving a specific example of an unstable solution. For this let Ω be a region with smooth boundary in $\mathbf{R}^n$ and let

$$s_\phi - \lambda u = \phi^3 - \phi + 1 - u$$

and

$$\theta_\Gamma = 1.$$

It follows that $u_0 = 1$ satisfies the stationary equation for u and $\phi_0 = 0$ satisfies the stationary equation for ϕ. The linearized system becomes:

$$\begin{aligned} \phi_{1t} &= \Delta\phi_1 + \phi_1 - u_1 \\ u_{1t} &= \Delta u_1 - \phi_{1t} \end{aligned}$$

with boundary conditions

$$\begin{aligned} \frac{\partial \phi_1}{\partial \nu} &= 0, \\ \frac{\partial u_1}{\partial \nu} &= -u_1 \end{aligned}$$

For the initial values $u_1^0(x) = 0$ and $\phi_1^0(x) = 1$ the unique solution to this system is given by $u_1(x,t) = 0$ and $\phi_1(x,t) = 1$, i.e. $\phi(x,t) \not\to 0$ as $t \to \infty$.

This implies that ϕ_0 is an unstable stationary solution. This solution does however make no physical sense, since it would imply the existence of both phases at every point in the domain.

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IRREDUCIBLE MAPPINGS AND HU-TERMINAL CONTINUA

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Abstract. We introduce the notion of an *irreducible* mapping between continua. We prove in the present paper that each atomic mapping is irreducible and each irreducible mapping is monotone. It is also shown that irreducible mappings between hereditarily unicoherent continua map an HU-terminal continuum of the domain onto an HU-terminal continuum of the range. Our examples demonstrate that neither monotone nor open mappings have this property in general. We show that the inverse image of an HU-terminal continuum of a nondegenerate range continuum under an atomic mapping is an HU-terminal continuum of the domain continuum. Some relevant counterexamples are indicated.

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1. Introduction and notation.

It is known that each atomic mapping between continua is monotone (see [8, 3.7, p. 13], [4, Theorem 1, p. 49]). It is natural to try to find some family of mappings which is somewhere between the families of all atomic and all monotone mappings. We introduce in the present paper the notion of an *irreducible* mapping between continua. In Section 2 we prove that each atomic mapping is irreducible, that

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each irreducible mapping is monotone and that both these inclusions are proper (Corollary 2.9).

In Section 3 we will proceed with the study of HU-terminal continua (see [2, p. 380]). In his paper J. J. Charatonik posed a (still open) question ([2, Question 22, p. 382]):

1.1. Question. *What mappings among hereditarily unicoherent continua map HU-terminal continua of the domain onto HU-terminal continua of the range?*

Partially answering this question, we show that irreducible mappings between hereditarily unicoherent continua map HU-terminal continua of the domain onto HU-terminal continua of the range (Theorem 3.3). Thus we see that atomic mappings preserve the concept of an HU-terminal continuum, which is not in general preserved by monotone mappings (Corollary 3.5). We show that the inverse image of an HU-terminal continuum of a nondegenerate range continuum under an atomic mapping is an HU-terminal continuum of the domain continuum (Theorem 3.7).

In Section 4 we will look more closely at the open mappings. It is natural to ask if open mappings preserve HU-terminality. Our example demonstrates that open mappings do not in general preserve the concept of an HU-terminal continuum.

For the convenience of the reader we repeat the relevant material. All spaces considered in the paper are assumed to be metric and all mappings are continuous. A *continuum* means a compact connected space. Given a space X and its subset S, we denote by $\operatorname{cl} S$ the closure of S.

Let X be a continuum and $A \subset X$. Then X is said to be *irreducible about* A provided no proper subcontinuum of X contains A. A continuum X is said to be *irreducible* provided that X is irreducible about $\{p, q\}$ for some $p, q \in X$ (then X is said to be *irreducible between* p *and* q).

Let X be a continuum, and let A and B be nonempty compact subsets of X. We say that a subcontinuum C of X is *irreducible from* A *to* B, written $C = \operatorname{irr}(A, B)$, provided that $C \cap A \neq \emptyset$, $C \cap B \neq \emptyset$, and no proper subcontinuum of C intersects both A and B. If A or B is a singleton set, we omit the set notation; thus, e.g. we write $C = \operatorname{irr}(p, B)$, $C = \operatorname{irr}(p, q)$.

Let X and Y be continua. A mapping $f : X \to f(X) = Y$ is said to be:
– *atomic* provided that for each subcontinuum $K \subset X$ with nondegenerate image $f(K)$ the equality $f^{-1}(f(K)) = K$ holds;
– *irreducible* provided that for each nondegenerate irreducible subcontinuum $I \subset Y$ the inverse image $f^{-1}(I)$ is irreducible;
– *monotone* provided that for each subcontinuum Q of Y the inverse image $f^{-1}(Q)$ is connected;
– *interior at a point* $p \in X$ provided that for each open neighborhood U of p in X the point $f(p)$ is an interior point of the image $f(U)$ in Y;
– *open* provided that for each open subset U of X its image $f(U)$ is an open subset of Y.

Obviously a mapping is open if and only if it is interior at each point of its domain. It is known that each atomic mapping between continua is monotone (see [8, 3.7, p. 13], [4, Theorem 1, p. 49]).

Given continua X and Y, we will denote by

$$\mathsf{Atomic}, \quad \mathsf{Irreducible}, \quad \mathsf{Monotone} \quad \text{and} \quad \mathsf{Open}$$

the family of all atomic, irreducible, monotone and open mappings $f : X \to f(X) = Y$, respectively.

Let $\mathcal{M}$ be a class of mappings between continua. A mapping $f : X \to Y$ is said to be *hereditarily* $\mathcal{M}$ provided that for every subcontinuum $Z \subset X$ the partial mapping $f \restriction Z : Z \to f(Z) \subset Y$ is in $\mathcal{M}$.

In the next example we construct a mapping between continua which is irreducible but not hereditarily irreducible.

1.2. Example. Let $\varphi : \mathbb{R} \to \mathbb{R}^2$ be defined by $\varphi(t) = (\cos(t), \sin(t))$. Put $X = \{\varphi(t) \in \mathbb{R}^2 : t \in \mathbb{R}\}$. The function $f : X \to f(X) = X$ is defined by
(i) $f(\varphi(t)) = \varphi(2t)$ for $t \in [0, \pi]$;
(ii)$f(\varphi(t)) = (1, 0)$ for $t \in (\pi, 2\pi)$.
Then we observe

Claim 1. f is irreducible.

Proof. Indeed, any nondegenerate irreducible subcontinuum C of X is a proper subcontinuum and is homeomorphic to the closed unit interval. Then $f^{-1}(C)$ is again a proper subcontinuum of X due to the construction of f. It follows that $f^{-1}(C)$ is homeomorphic to the closed unit interval. Therefore $f^{-1}(C)$ is irreducible and Claim 1 is proved.

Claim 2. f is not hereditarily irreducible.

Proof. Fix a nondegenerate subcontinuum $C = \varphi([0, \pi]) \subsetneq X$ and put $g = f \restriction C$. Then $g(C) = X$ and $g : C \to X$ is not irreducible, for $g^{-1}(\varphi([0, 1]))$ is not connected (equals $(-1, 0) \cup \varphi([0, 2])$) in C. Claim 2 is proved. □

2. Irreducible mappings.

2.1. Proposition. *Let $f : X \to Y$ be an atomic mapping. Let $K \subset Y$ be a nondegenerate continuum irreducible about a set $L \subset K$. If $f(P) = L$ for some $P \subset X$, then $f^{-1}(K)$ is irreducible about P.*

Proof. Let $f : X \to Y$ be an atomic mapping. Let $K \subset Y$ be a nondegenerate continuum irreducible about a set $L \subset K$. Fix some set $P \subset X$ such that $f(P) = L$. We show that $f^{-1}(K)$ is irreducible about P.

Clearly L (and P as well) is not a singleton. It is known that every atomic mapping is monotone (see [8, 3.7, p. 13], [4, Theorem 1, p. 49]), hence $M = f^{-1}(K)$ is a continuum in X (we have a nondegenerate continuum K). Clearly $P \subset M$ because $L = f(P) \subset K$. Take some continuum N such that $P \subset N \subset M$.

We conclude that $f(N) = K$. (Indeed, obviously $f(N)$ is a continuum containing L, because $L = f(P)$. Hence $K \subset f(N)$, since K is irreducible about L. Conversely, $N \subset M$ implies $f(N) \subset f(M) = K$.)

Since f is atomic and $f(N) = K$ is a nondegenerate continuum we obtain that $N = f^{-1}(f(N)) = f^{-1}(K) = M$ (we know that N is nondegenerate since $P \subset N$, f is atomic). Then $M = f^{-1}(K)$ is irreducible about P and the proof is complete. □

2.2. Theorem. *Every atomic mapping $f : X \to f(X) = Y$ from a continuum X onto a continuum Y is irreducible.*

Proof. Let $f : X \to f(X) = Y$ from a continuum X onto a continuum Y be an atomic mapping. Consider a nondegenerate subcontinuum $K \subset Y$ which is irreducible about $\{p, q\}$. Then $f^{-1}(K)$ is irreducible about any two points a, b in X such that $f(a) = p$ and $f(b) = q$ due to Proposition 2.1. Hence f is irreducible. $\square$

We show that there exists an irreducible mapping which is neither open nor atomic in the following example.

2.3. Example. Let $X = [0, 2]$, $Y = [0, 1]$. The mapping $f : X \to f(X) = Y$ is defined for $x \in X$ by $f(x) = \min(x, 1)$. From this we have $f((1, 2)) = \{1\}$. Hence f is not open. We see that f is not atomic because $[1/2, 2] = f^{-1}f([1/2, 3/2]) \neq [1/2, 3/2]$. Since the inverse images of intervals in $[0, 1]$ are intervals in $[0, 2]$, it follows that f is an irreducible mapping, which is neither open nor atomic. $\square$

We can formulate the previous results in

2.4. Corollary. Atomic$\subsetneq$ Irreducible.

Proof. See Theorem 2.2 and Example 3.2. $\square$

We recall that a continuum X is said to be a *simple triod* if X is the union of three straight line segments pa, pb and pc any two of which have the point p in common only. Obviously, a simple triod is homeomorphic to the letter T).

We say that a topological space S is a *spiral* if it is homeomorphic to the real half line $[0, +\infty)$.

A continuous mapping $f : X \to f(X) = Y$ is called *strongly irreducible* if $f^{-1}(I)$ is irreducible in X for every irreducible $I \subset Y$ (i.e. singletons in Y are allowed). The following example shows that an atomic mapping need not to be strongly irreducible.

2.5. Example. Let X be the disjoint union of a simple triod T and a spiral S approximating T. The continuum X is illustrated in Figure 1.

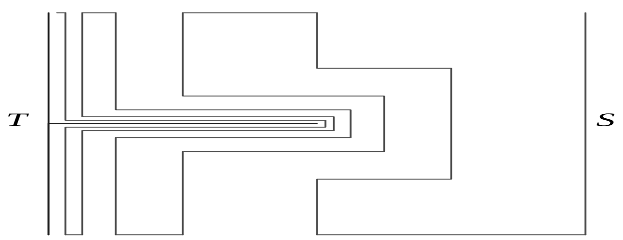

FIGURE 1 (EXAMPLE 2.5.)

Let $g : S \to [1, \infty)$ be a homeomorphism. Define $f : X \to [0,1]$ by requiring $f(x) = 0$ for $x \in T$ and $f(x) = 1/g(x)$ for $x \in S$. Then f is not strongly irreducible, since $f^{-1}(0)$ is the simple triod T which is not an irreducible continuum. But f is atomic: Indeed, let C be a subcontinuum of X such that $f(C)$ is not a singleton. Clearly $C = S'$ or $C = S' \cup T$, for some $S' \subset S$. Since f is injective on S, $f^{-1}(f(S')) = S'$ and since $f^{-1}(f(T)) = T$, we have $f^{-1}(f(C)) = C$. We have shown that f is atomic.

The mapping f is an atomic mapping which is not strongly irreducible. □

Recall that a sequence of subsets of a metric space is called a *null-sequence* if the diameters of its members tend to zero.

For the convenience of the reader we include a useful statement (see [9, Corollary 5.5, p. 74]) which will be used in the following theorem.

2.6. Statement. *Let X be a nondegenerate continuum. Then, X contains a nondegenerate proper subcontinuum. Furthermore: If A is a proper subcontinuum of X and U is an open subset of X such that $A \subset U$, then there is a subcontinuum B of X such that*

$$A \subset B \neq A \quad \text{and} \quad B \subset U.$$

2.7. Theorem. Irreducible $\subset$ Monotone.

Proof. Let $f : X \to f(X) = Y$ be an irreducible mapping. We prove that f is monotone. We first prove a reduced form of the definition :

Claim. For each nondegenerate subcontinuum Q of Y the inverse image $f^{-1}(Q)$ is connected.

Proof. To deduce this, take a nondegenerate subcontinuum Q of Y. Suppose, contrary to this, that $f^{-1}(Q)$ is not connected.

Fix a, b in different components of $f^{-1}(Q)$ such that $f(a) \neq f(b)$. To fix such two points we proceed as follows :

(i) if for each component C of $f^{-1}(Q)$ the image $f(C)$ is a singleton in Q, then we have at least two disjoint components C_a and C_b such that $f(C_a) \neq f(C_b)$; then we choose $a \in C_a$ and $b \in C_b$.

(ii) if for some component C_a of $f^{-1}(Q)$ the image $f(C_a)$ is not a singleton in Q, then we find a component C_b of $f^{-1}(Q)$ disjoint with C_a; we choose any $b \in C_b$ and find $a \in C_a$ such that $f(a) \neq f(b)$.

Then $I = \operatorname{irr}(f(a), f(b))$ is a nondegenerate irreducible subcontinuum of Y contained in Q. Hence $f^{-1}(I)$ is an irreducible continuum of X contained in $f^{-1}(Q)$. Consequently, a and b are in the same component of $f^{-1}(Q)$, contrary to the choice of a, b. This contradicts our assumption. Claim is proved.

We are now in a position to show that f is monotone. Let $Q = \{p\}$ be a degenerate subcontinuum of Y. We show that the inverse image $f^{-1}(Q)$ is connected.

We first construct in Y a null sequence of nondegenerate subcontinua $K_1 \supset K_2 \supset \cdots \supset K_n \cdots \ni p$. Set $K_1 = Y$. We now proceed by induction. We construct K_{n+1} as a nondegenerate proper subcontinuum of K_n of diameter less then $1/n$ containing p (see Statement 2.6).

Then $f^{-1}(K_n)$ is a monotone sequence of nondegenerate subcontinua of X according to Claim. Hence $A = \cap f^{-1}(K_n)$ is a subcontinuum of X as well due to [9, Theorem 1.8, p. 6]. We observe that $A = f^{-1}(Q)$. Hence $f^{-1}(Q)$ is connected. □

The following example was presented by J. J. Charatonik in [2, p. 381].

2.8. Example. Let a continuum X be a simple triod, i.e. the union of three straight line segments pa, pb and pc any two of which have the point p in common only. The mapping $f : X \to f(X) \subset X$ is defined by requiring f to be the identity on $pa \cup pb$, $f(pc) = \{p\}$.

Clearly f is monotone, but f is not irreducible, since its range $f(T)$ is a space homeomorphic to the closed unit interval, which is irreducible, but its domain (the simple triod T) is not. $\square$

We have proved the following

2.9. Corollary. $\mathsf{Atomic} \subsetneq \mathsf{Irreducible} \subsetneq \mathsf{Monotone}$.

Proof. See Corollary 2.4, Theorem 2.7 and Example 2.8. $\square$

3. HU-terminality.

A continuum X is said to be *hereditarily unicoherent* provided that the intersection of any two its subcontinua is connected.

J. J. Charatonik in [2, p. 380-381] compared the concept of a terminal continuum (see [1, Definition 4.1. p. 34], [1, Example 1.2(a), p. 7], [2, p. 377], [5] and Nadler's book [9, 1.54, p. 107]) with the concept of a terminal continuum as defined by G. R. Gordh in [6]. Since G. R. Gordh restricts his considerations to subcontinua of hereditarily unicoherent continua only, J. J. Charatonik renamed the concept of G. R. Gordh as HU-terminal (see [6, p. 458], [2, p. 380-381]).

A subcontinuum K of a hereditarily unicoherent continuum X is said to be an HU-*terminal continuum* of X provided that K is contained in an irreducible subcontinuum of X and for every irreducible subcontinuum I of X containing K there is a point $x \in X$ such that I is irreducible about the union $K \cup \{x\}$ (see [6, p. 458], [2, p. 380-381]).

Given hereditarily unicoherent continua X and Y, we will denote by

$$\mathsf{HU\text{-}terminal}$$

the family of all mappings $f : X \to f(X) = Y$ which map each HU-terminal continuum of X onto an HU-terminal continuum of Y.

3.1. Proposition. *Let $f : X \to f(X) = Y$ be a monotone mapping. Let J be a subcontinuum of Y and $P \subset J$. Let $f^{-1}(J)$ be a continuum irreducible about Q and $f(Q) = P$. Then J is irreducible about P.*

Proof. Let $f : X \to f(X) = Y$ be a monotone mapping. Let J be a subcontinuum of Y and $P \subset J$. Let $f^{-1}(J)$ be a continuum irreducible about Q and $f(Q) = P$. We show that J is irreducible about P.

To prove this let $I \subset J$ be a continuum containing P. Then $f^{-1}(I)$ is a subcontinuum of $f^{-1}(J)$ (inverse images under a monotone mapping) containing Q, which implies $f^{-1}(I) = f^{-1}(J)$, since $f^{-1}(J)$ be a continuum irreducible about Q. But then $I = J$ and J is irreducible about P. $\square$

3.2 Proposition. *Let $f : X \to f(X) = Y$ be an irreducible mapping. If I is a subcontinuum of X irreducible about $\{p, q\}$, then $f(I)$ is irreducible about $\{f(p), f(q)\}$.*

Proof. Let $f : X \to f(X) = Y$ be an irreducible mapping. Let I be a subcontinuum of X irreducible about $\{p, q\}$. We prove that $f(I)$ is irreducible about $\{f(p), f(q)\}$.

The case $p = q$ is simple. We may assume that $p \neq q$. Take any continuum $J \subset f(I)$ containing $f(p), f(q)$. Then $f^{-1}(J)$ is a continuum in X (irreducible mappings are monotone due to Theorem 2.7) containing $\{p, q\}$. Hence $I \subset f^{-1}(J)$. But the inclusion cannot be proper since $J \subset f(I)$. We conclude that $f^{-1}(J) = I$, which implies $J = f(I)$. We have showed that $f(I)$ is irreducible about $\{f(p), f(q)\}$. $\square$

3.3. Theorem. *Let X and Y be hereditarily unicoherent continua. Let $f : X \to f(X) = Y$ be irreducible. If $K \subset X$ is an HU-terminal continuum of X, then $f(K) \subset Y$ is an HU-terminal continuum of Y.*

Proof. Let X and Y be hereditarily unicoherent continua. Let $f : X \to f(X) = Y$ be irreducible. Let $K \subset X$ be an HU-terminal continuum of X. We show that $f(K) \subset Y$ is an HU-terminal continuum of Y.

(i) We know that K is contained in some irreducible continuum I of X. Hence $f(K)$ is contained in the irreducible continuum $f(I)$ due to Proposition 3.2.

(ii) If $f(K)$ is contained in a nondegenerate irreducible continuum J of Y, then $f^{-1}(J)$ is a nondegenerate irreducible subcontinuum of X (the mapping f is irreducible) and contains K. Since K is an HU-terminal continuum of X, the inverse image $f^{-1}(J)$ is irreducible about $K \cup \{x\}$ for some $x \in f^{-1}(J)$. So by Proposition 3.1 the continuum J is irreducible about $f(K) \cup \{f(x)\}$ (the mapping f is irreducible, hence monotone due to Theorem 2.7).

If $f(K)$ is contained in a degenerate irreducible continuum J of Y, then J is irreducible about $f(K) \cup \{f(x)\}$ for some $x \in K$.

We have showed that $f(K) \subset Y$ is an HU-terminal continuum of Y. $\square$

The following example establishes a relation between the classes Monotone and HU-terminal.

3.4. Example. Let a continuum X be the union of three straight line segments pa, pb and pc any two of which have the point p in common only, where $a = (-1, 0)$, $b = (1, 0)$, $c = (0, 1)$ and $p = (0, 0)$. The continuum X is a simple triod. Let $Y = pa \cup pb$.

The mapping $g : X \to g(X) = Y \subset X$ is defined by $g((x, y)) = (x + y, 0)$ (i.e. g is the identity on $pa \cup pb$, linear on pc, $g(c) = b$ and g is sending pc to pb.)

The mapping g is not monotone, because the inverse image of b is not connected (equals $\{b, c\}$). Note that a subcontinuum K of Y is an HU-terminal continuum of Y if and only if $K \cap \{a, b\} \neq \emptyset$. Note also that each HU-terminal continuum of X contains at least one point of $\{a, b, c\}$. Since the mapping g maps $\{a, b, c\}$ onto $\{a, b\}$ we see that g belongs to the class HU-terminal.

J. J. Charatonik in [2, p. 381] noticed that the mapping f defined in Example 2.8 (defined for $x \in X$ by $f((x, y)) = (x, 0)$, i.e. f is the identity on $pa \cup pb$, $f(pc) = \{p\}$) is monotone and that f maps the HU-terminal continuum $\{c\}$ of X to p, which is not an HU-terminal continuum of Y.

Consequently, $f \in$ Monotone $\setminus$ HU-terminal and $g \in$ HU-terminal $\setminus$ Monotone. $\square$

We can summarize the properties of HU-terminal in

3.5. Corollary. *Let X and Y be hereditarily unicoherent continua. Then*

(i) Irreducible$\subsetneq$ HU-terminal*;*
(ii) Atomic$\subsetneq$ HU-terminal*;*
(iii) HU-terminal$\not\subset$ Monotone*;*
(iv) Monotone$\not\subset$ HU-terminal.

Proof. See Theorem 2.2, Theorem 3.3 and Example 3.4. □

For irreducible mappings $f : X \to f(X) = Y$, the inverse image of an HU-terminal continuum of Y is not necessarily an HU-terminal continuum of X. We construct an example with this property.

3.6 Example. Let $a = (0,1)$, $a' = (0,-1)$, $a'' = (-1,-1)$, $A = aa'$, $L_A = a'a''$; $b = (1,1)$, $b' = (1,-1)$, $b'' = (2,-1)$, $B = bb'$, $L_B = b'b''$ and

$$S = \left\{(x,y) : \quad 0 < x < 1, \quad y = \sin \frac{1}{x(1-x)}\right\}.$$

Put $X = A \cup L_A \cup B \cup L_B \cup S$. The continuum X is illustrated in Figure 2.

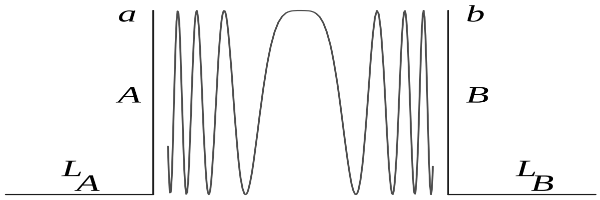

FIGURE 2 (EXAMPLE 3.6)

Consider $f : X \to f(X) = Y$, which maps L_A to a', L_B to b' and is an identity elsewhere. Clearly a is an HU-terminal continuum of $f(X) = A \cup S \cup B$, but $a = f^{-1}(a)$ is not an HU-terminal continuum of X. Moreover, f is clearly irreducible. □

For irreducible mappings, the inverse image of $\operatorname{irr}(a,b)$ is not necessarily irreducible about $f^{-1}(a) \cup f^{-1}(b)$. Indeed, consider X, f, a and b from the previous example. Then $f(X) = \operatorname{irr}(a,b)$, but X is not irreducible about $f^{-1}(a) \cup f^{-1}(b)$.

As we stated in Proposition 2.1, atomic mappings have the mentioned property. That allows us to prove that the inverse image under an atomic mapping of an HU-terminal continuum is an HU-terminal continuum.

3.7. Theorem. *Let $f : X \to f(X) = Y$ be an atomic mapping between two nondegenerate hereditarily unicoherent continua X and Y. Let $K \subset Y$ be an HU-terminal continuum of Y. Then $f^{-1}(K)$ is an HU-terminal continuum of X.*

Proof. Let $f : X \to f(X) = Y$ be an atomic mapping between two nondegenerate hereditarily unicoherent continua X and Y. Let $K \subset Y$ be an HU-terminal continuum of Y. We prove that $f^{-1}(K)$ is an HU-terminal continuum of X.

(i) Since K is an HU-terminal continuum of a nondegenerate continuum Y, K is contained in some nondegenerate irreducible subcontinuum L of Y. Then $f^{-1}(K)$ is a continuum, because atomic mappings are monotone (see [8, 3.7, p. 13], [4, Theorem 1, p. 49]). Clearly $f^{-1}(K)$ is contained in $f^{-1}(L)$, which is irreducible (atomic mappings are irreducible - see Theorem 2.2).

(ii) Now let the continuum $f^{-1}(K)$ is contained in some subcontinuum M of X which is irreducible about $\{p, q\}$. Then $f(M)$ is irreducible about $\{f(p), f(q)\}$ by Proposition 3.2. The continuum K is an HU-terminal continuum of Y, $K \subset f(M)$, therefore there is a point $a \in Y$, such that $f(M)$ is irreducible about $K \cup \{a\}$.

If $f(M)$ is a singleton, then $K = f(M)$ and $f^{-1}(K) = M$. Hence $f^{-1}(K)$ is irreducible about $M \cup \{b\}$ for any $b \in M$.

If $f(M)$ is not a singleton, by Proposition 2.1 we see that $f^{-1}(f(M)) = M$ is irreducible about $f^{-1}(K) \cup \{b\}$, where b is such that $f(b) = a$.

Therefore we have proved that $f^{-1}(K)$ is an HU-terminal continuum of X. □

The atomic mapping sending a simple triod to a singleton shows that Theorem 3.7 does not hold if Y is a degenerate continuum, because a simple triod is not irreducible.

3.8. Corollary. *Let $f : X \to f(X) = Y$ be an atomic mapping between two nondegenerate hereditarily unicoherent continua X and Y. Then $K \subset X$ is an HU-terminal continuum of X if and only if $f(K)$ is an HU-terminal continuum of Y.*

Proof. See Corollary 3.5 and Theorem 3.7. □

Notice that the mapping f in Example 2.3 is not atomic, but $K \subset X$ is an HU-terminal continuum of X if and only if $f(K)$ is an HU-terminal continuum of Y.

4. Open mappings.

We have seen in Example 2.3 that there exists an irreducible mapping which is not open. In the next example we show how to find an open mapping which is not in $\mathsf{HU\text{-}terminal}$. This construction is adapted from [3, Example 7.1, p. 29].

4.1. Example. Let $I = \{(0, y) \in \mathbb{R} : y \in [-1, 1]\}$ and

$$T = \{(0, y) \in \mathbb{R} : y \in [-1/3, 1/3]\} \cup \{(x, 0) \in \mathbb{R} : x \in [0, 1]\} \quad .$$

Set $v = (0, 0)$, $s = (0, -1)$, $t = (0, 1)$ and for each $n \in \mathbb{N}$ put $a_n = (1/n, -1/5n)$, $b_n = (1 + 1/n, -1/5n)$, $c_n = (1 + 1/n, 1/5n)$, $d_n = (1/n, 1/5n)$, $e_n = (1/n, 1)$, and finally

$$L_n = sa_n \cup a_nb_n \cup b_nc_n \cup c_nd_n \cup d_ne_n \quad .$$

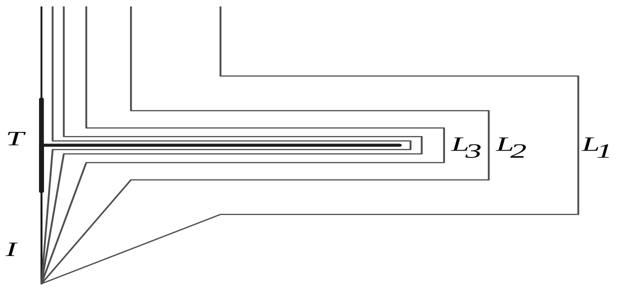

FIGURE 3 (EXAMPLE 4.1)

We will denote by X the space

$$I \cup T \cup \bigcup_{n \in \mathbb{N}} L_n \quad .$$

The continuum X is illustrated in Figure 3.

In the sequel, p_ε stands for the line $\{(x, \varepsilon) : x \in \mathbb{R}\}$ and P stands for the strip $\{(x, y) : x \in \mathbb{R}, y \in (-1/3, 1/3)\}$. We find a sequence $\{r_n\} \subset (X \cap P)$ such that $\mathrm{cl}\,(\{r_n\}) = T$ and $r_n \in L_n$ for $n \in \mathbb{N}$. This is easy. It is sufficient to use the separability of T.

Let us denote by $f : X \to I$ the mapping satisfying

(i) $f(T) = \{v\}$, $f(s) = \{s\}$;

(ii) $f(X \cap p_1) = \{t\}$;

(iii) f is injective on $I \setminus T$;

and for each $n \in \mathbb{N}$

(iv) f is injective on L_n;

(v) $f(L_n \cap p_{1/3}) = (0, 1/3n)$;

(vi) $f(L_n \cap p_{-1/3}) = (0, -1/3n)$;

(vii) $f(r_n) = v$.

CLAIM 1. $f \in \mathsf{Open}$.

Proof. We prove this as follows :

- f is interior at each point of $L_n \setminus I$ due to (iv), $n \in \mathbb{N}$;
- f is interior at each point of $I \setminus T$ due to (i) - (iii);
- f is interior at each point of T due to the construction of $\{r_n\}$. (Indeed, each open neighborhood U of each point $x \in T$ contains some r_n. Since f is open at $L_n \setminus I$, we see that $f(U)$ is open.)

CLAIM 2. $f \notin \mathsf{HU\text{-}terminal}$.

Proof. It is a simple matter to check that the point $r = (1, 0)$ is an HU-terminal continuum of X. We examine the irreducible subcontinua of X containing r. It is immediate that any such subcontinuum is irreducible about $\{r, q\}$ for some $q \in X$.

We now observe that $f(X) = I$. But $p = f(r)$ is not an HU-terminal continuum of I since I is irreducible about $\{s, t\}$ and for any $x \in I$ the continuum I is not irreducible about any $\{p, x\}$.

CLAIM 3. $f \notin \mathsf{Irreducible}$.

Proof. It is a simple matter to check that the inverse image of $I \cap T$ is not connected in X. □

We may summarize the previous by the following corollary.

4.2. Corollary. *Let X and Y be hereditarily unicoherent continua. Then*

(i) $\mathsf{HU\text{-}terminal} \not\subset \mathsf{Open}$;
(ii) $\mathsf{Open} \not\subset \mathsf{HU\text{-}terminal}$;
(iii) $\mathsf{Irreducible} \not\subset \mathsf{Open}$;
(iv) $\mathsf{Open} \not\subset \mathsf{Irreducible}$.

Proof. See Example 2.3, Theorem 3.3 and Example 4.1. □

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DERIVATIVE OPERATOR ON A HYPERGEOMETRIC FUNCTION OF THREE COMPLEX VARIABLES

M.S.Metwally

Abstract. In recent years stydies have been don on the effect of some differential operators on the entire function of several complex variables. In this paper a study of the effect of one of these operators (for example $D^{(N)}$-diferential operator) on the hypergeometric function of three complex variables. The effect of this operator has shown some contiguous function relations. A hypergeometric differential equation with its general solution these type of hypergeometric function hane been obtained.

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Key Words and Phrases. hypergeometric function, contiguous function relations, hypergeometric differential equation, $D^{(N)}$-operator

1.Introduction

The hypergeometric function $H(a,b,b^{'};c;z_1,z_2,z_3)$ of three complex variables z_1, z_2 and z_3, is defined and studied in ([2], [4], [7]) in the form

$$H(a,b,b^{'};c;z_1,z_2,z_3)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+p}(b)_{m+n}(b^{'})_{n+p}}{(c)_{m+n+p}}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\frac{z_3^p}{p!} \qquad (1.1)$$

$$=\sum_{m,n,p}^{\infty}\alpha_{m,n,p}z_1^m z_2^n z_3^p.$$

The following form gives the radius of regularity of this function

$$\rho=\limsup_{m+n+p\to\infty}\left\{\frac{|\alpha_{m,n,p}|}{\sigma_{m,n,p}}\right\}^{\frac{1}{m+n+p}}=1 \qquad (1.2)$$

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where

$$\sigma_{m,n,p} = \begin{cases} (\frac{m+n+p}{m})^{\frac{m}{2}}(\frac{m+n+p}{n})^{\frac{n}{2}}(\frac{m+n+p}{p})^{\frac{p}{2}} & ; m,n,p \neq 0 \\ 1 & ; m,n,p = 0 \end{cases} \tag{1.3}$$

for $a, b, b^{'}$ and c are neither zero or nor negative integers. It is easy to see that the hypergeometric function $H(a, b, b^{'}; c; z_1, z_2, z_3)$ is regular in the closed hypersphere $\bar{S}_r; r = 1$ where is given by $\bar{S}_r = |z_1|^2 + |z_2|^2 + |z_3|^2 = 1$.

The operator D, as given in [3] in the case of two complex variables, takes the form

$$D = \begin{cases} d_1 + d_2 + d_3, & m,n,p \geq 1 \\ 1, & otherwise \end{cases} \tag{1.4}$$

where $d_1 = z_1\frac{\partial}{\partial z_1}$, $d_2 = z_2\frac{\partial}{\partial z_2}$ and $d_3 = z_3\frac{\partial}{\partial z_3}$.

Operating with D on the hypergeometric function $H(a, b, b^{'}; c; z_1, z_2, z_3)$ of three complex variables yields

$$\begin{aligned} DH(a,b,b^{'};c;z_1,z_2,z_3) &= [z_1(b^{'} + a - c + d_1) + z_2(b' + b - c + d_2) \\ &+ z_3(a + b^{'} - c + d_3)]H(a,b,b^{'};c;z_1,z_2,z_3) \\ &+ \frac{1}{c}\{z_1[(a-c) - d_2][(b-c) + d_3] \\ &+ z_2[(b^{'} - c) - d_1][(b-c) + d_3] \\ &+ z_3[(b^{'} - c) - d_1][(a-c) + d_2]\}H(a,b,b^{'};c+1;z_1,z_2,z_3), \end{aligned} \tag{1.5}$$

alternatively, equation (1.5) can be written in the form

$$\begin{aligned} DH(a,b,b^{'};c;z_1,z_2,z_3) &= z_1\frac{ab}{c}H(a+1,b+1,b^{'};c+1;z_1,z_2,z_3) \\ &+ z_2\frac{bb^{'}}{c}H(a,b+1,b^{'}+1;c+1;z_1,z_2,z_3) \\ &+ z_3\frac{ab^{'}}{c}H(a+1,b,b^{'}+1;c+1;z_1,z_2,z_3) \end{aligned} \tag{1.6}$$

since $Dz_1^m z_2^n z_3^p = (m+n+p)z_1^m z_2^n z_3^p$.

2. The Contiguous function relations

For the hypergeometric function $H(a, b, b^{'}; c; z_1, z_2, z_3)$, the following contiguous function relations can be defined

$$
\begin{aligned}
(D+a)H(a,b,b^{'};c;z_1,z_2,z_3) =& aH(a+1,b,b^{'};c;z_1,z_2,z_3) \\
&+ d_2H(a,b,b^{'};c;z_1,z_2,z_3), \qquad (2.1)\\
(D+b)H(a,b,b^{'};c;z_1,z_2,z_3) =& bH(a,b+1,b^{'};c;z_1,z_2,z_3) \\
&+ d_3H(a,b,b^{'};c;z_1,z_2,z_3), \qquad (2.2)\\
(D+b^{'})H(a,b,b^{'};c;z_1,z_2,z_3) =& b^{'}H(a,b,b^{'}+1;c;z_1,z_2,z_3) \\
&+ d_1H(a,b,b^{'};c;z_1,z_2,z_3) \qquad (2.3)
\end{aligned}
$$

and

$$(D+c-1)H(a,b,b^{'};c;z_1,z_2,z_3) = (c-1)H(a,b,b^{'};c-1;z_1,z_2,z_3). \qquad (2.4)$$

From (2.1), (2.2), (2.3) and (2.4), it follows at once that

$$
\begin{aligned}
(a-b)H(a,b,b^{'};c;z_1,z_2,z_3) &= aH(a+1,b,b^{'};c;z_1,z_2,z_3) \\
&- bH(a,b+1,b^{'};c;z_1,z_2,z_3) + (d_2-d_3)H(a,b,b^{'};c;z_1,z_2,z_3),
\end{aligned} \qquad (2.5)
$$

$$
\begin{aligned}
(a-b^{'})H(a,b,b^{'};c;z_1,z_2,z_3) &= aH(a+1,b,b^{'};c;z_1,z_2,z_3) \\
&- b^{'}H(a,b,b^{'}+1;c;z_1,z_2,z_3) + (d_2-d_1)H(a,b,b^{'};c;z_1,z_2,z_3),
\end{aligned} \qquad (2.6)
$$

$$
\begin{aligned}
(b-b^{'})H(a,b,b^{'};c;z_1,z_2,z_3) &= bH(a,b+1,b^{'};c;z_1,z_2,z_3) \\
&- b^{'}H(a,b,b^{'}+1;c;z_1,z_2,z_3) + (d_3-d_1)H(a,b,b^{'};c;z_1,z_2,z_3)
\end{aligned} \qquad (2.7)
$$

and

$$
\begin{aligned}
(a-c+1)H(a,b,b^{'};c;z_1,z_2,z_3) &= aH(a+1,b,b^{'};c;z_1,z_2,z_3) \\
&- (c-1)H(a,b,b^{'};c-1;z_1,z_2,z_3) + d_2H(a,b,b^{'};c;z_1,z_2,z_3).
\end{aligned} \qquad (2.8)
$$

From these relations the other contiguous function relations can be deduced.

3. The hypergeometric Differential Equation

The operator D, that is used in the derivation of the contiguous function relations, is also in deriving a differential equation satisfied by (1.1) From (1.1) we obtain

$$
\begin{aligned}
&D(D+c-1)H(a,b,b^{'};c;z_1,z_2,z_3)\\
&\quad=\sum_{m,n,p}\frac{(m+n+p)(m+n+p-1)(a)_{m+p}(b)_{m+n}(b^{'})_{n+p}}{(c)_{m+n+p}}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\frac{z_3^p}{p!}\\
&\quad=\sum_{m,n,p}\frac{(m+n+p)(a)_{m+p}(b)_{m+n}(b^{'})_{n+p}}{(c)_{m+n+p-1}}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\frac{z_3^p}{p!}\\
&\quad=z_1\sum_{m,n,p}\frac{(a)_{m+1+p}(b)_{m+1+n}(b^{'})_{n+p}}{(c)_{m+n+p}}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\frac{z_3^p}{p!}\\
&\qquad+z_2\sum_{m,n,p}\frac{(a)_{m+p}(b)_{m+n+1}(b^{'})_{n+1+p}}{(c)_{m+n+p}}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\frac{z_3^p}{p!}\\
&\qquad+z_3\sum_{m,n,p}\frac{(a)_{m+p+1}(b)_{m+n}(b^{'})_{n+p+1}}{(c)_{m+n+p}}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\frac{z_3^p}{p!}\\
&\quad=\{z_1(a+D-d_2)(b+D-d_3)+z_2(b+D-d_3)(b^{'}+D-d_1)\\
&\qquad+z_3(a+D-d_2)(b^{'}+D-d_1)\}H(a,b,b^{'};c;z_1,z_2,z_3).
\end{aligned}
$$

It is easy to see that the hypergeometric function $H(a,b,b^{'};c;z_1,z_2,z_3)$ should be a solution of a partial differential equation given by

$$
\begin{aligned}
\{D(D+c-1)&-[(a+D)-d_2][(b+D)-d_3]z_1\\
&-[(b+D)-d_3][(b^{'}+D)-d_1]z_2\\
&-[(a+D)-d_2][(b^{'}+D)-d_1]z_3\}H(a,b,b^{'};c;z_1,z_2,z_3)=0
\end{aligned}
\tag{3.1}
$$

4. $D^{(N)}$-operator of hypergeometric function of three complex variables

The $D^{(N)}$-operator which is defined and studied in [1,3] in the case of two complex variables and refined here in the case of three complex variables in the form

$$
D^{(N)}=\begin{cases}(d_1+d_2+d_3)^N, & m,n,p\geq 1\\ 1, & otherwise\end{cases}
\tag{4.1}
$$

where N is a finite positive integer. In terms of the $D^{(N)}$-operator, equation (1.5) can be written in the form

$$\begin{aligned} DH(a,b,b^{'};c;z_1,z_2,z_3) =&(c-1)[H(a,b,b^{'};c-1;z_1,z_2,z_3) \\ &- H(a,b,b^{'};c;z_1,z_2,z_3)]. \end{aligned} \tag{4.2}$$

Now, we employ the differential operator

$$\begin{aligned} D^2 =&z_1^2\frac{\partial^2}{\partial z_1^2}+z_2^2\frac{\partial^2}{\partial z_2^2}+z_3^2\frac{\partial^2}{\partial z_3^2} \\ &+2z_1z_2\frac{\partial^2}{\partial z_1\partial z_2}+2z_1z_3\frac{\partial^2}{\partial z_1\partial z_3}+2z_2z_3\frac{\partial^2}{\partial z_2\partial z_3} \end{aligned} \tag{4.3}$$

on the hypergeometric function $H(a,b,b^{'};c;z_1,z_2,z_3)$ to get

$$\begin{aligned} D^2H(a,b,b^{'};c;z_1,z_2,z_3) = &(c-1)(c-2)H(a,b,b^{'};c-2;z_1,z_2,z_3) \\ &-2(c-1)^2H(a,b,b^{'};c-1;z_1,z_2,z_3) \\ &+c(c-1)H(a,b,b^{'};c;z_1,z_2,z_3). \end{aligned} \tag{4.4}$$

Then, the following relation can be derived by induction

$$\begin{aligned} D^NH(a,b,b^{'};c;z_1,z_2,z_3) =&(c-1)\sum_{k=0}^{N}(-1)^k\binom{N}{k}\prod_{j=0}^{N-2}(c-N+j+k) \\ &H(a,b,b^{'};c-N+k;z_1,z_2,z_3) \end{aligned} \tag{4.5}$$

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A Hopf bifurcation of interfaces in a Dirichlet problem

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Abstract

In this paper, we consider a free boundary problem which is reduced from the propagator-controller system satisfying the Dirichlet boundary condition as a layer parameter is equal to zero. We shall show that the equlibria $(v^*(x), s^*)$ *exist and there is a Hopf bifurcation at a critical point* τ *only for* $s^* \in (\frac{1}{3}, 1)$.

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1 Introduction

A free boundary problem is obtained from a singular limit of a reaction-diffusion system

$$(1)\qquad \begin{cases} \varepsilon\,\tau u_t = \varepsilon^2\,u_{xx} + f(u,v), \\ v_t = D\,v_{xx} + u - bv. \end{cases}$$

In this equations, u and v are called the activator and its inhibitor in morphogenesis ([7]), or the propagator and its controller in chemical waves of excitable media ([2]). The reaction term $f(u,v) = -u+H(u-a)-v$ ($a>0$) is a McKean dynamics ([6]) and the parameter τ is a positive constant. We assume that the system (1) is bistable for a positive constant b which means that (1) has one unstable state and two stable states.

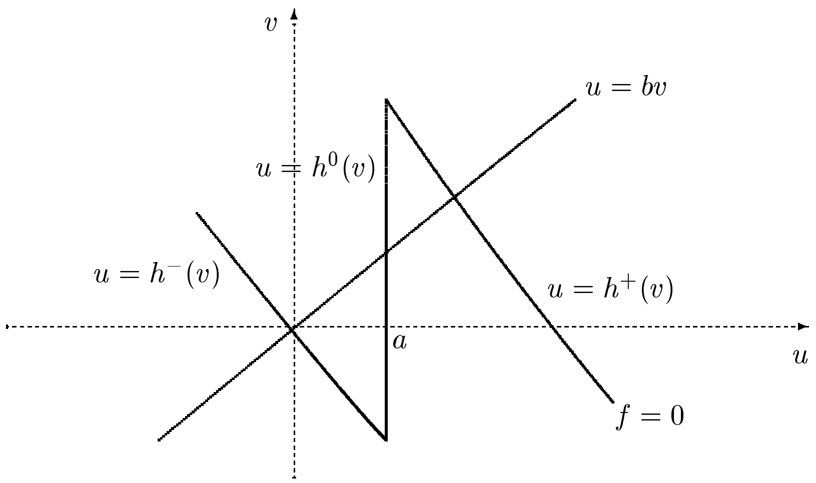

Figure 1: Nullclines $f(u) - v = 0$ and $g = 0$ for the bistable system (1).

When the diffusion constant ε is very small, the existence and uniqueness of solutions for the problem (1) are examined in [8]. The occurence of a Hopf bifurcation in (1) as τ varies was shown for a very small ε in [5] and [8]. When the diffusion constant ε is exactly zero in (1), a free boundary problem can be obtained in [3], and the well-posedness of this free boundary problem was shown in [5] and [9] using the fixed ponit theorem and using the semigroup theorey, respectively.

In [9], the authors proved the Hopf bifurcation near equlibria occurs at some τ for the Neumann boundary condition. In this paper, we consider the case for the Dirichlet boundary condition, and shall show that a Hopf bifurcation occurs for some equlibria for the following free boundary problem:

$$\begin{cases} v_t = D\, v_{xx} - c^2 v + H(x - s(t)) & \text{for } \ (x,t) \in \Omega^- \cup \Omega^+, \\ v(0,t) = 0 = v(1,t) & \text{for } \ t > 0, \\ v(x,0) = v_0(x) & \text{for } \ 0 \le x \le 1, \\ \tau \dfrac{ds}{dt} = C(v(s(t),t)) & \text{for } \ t > 0, \\ s(0) = s_0, \end{cases} \tag{2}$$

where $\Omega := \{(x,t) \in (0,1) \times (0,\infty)\}$, $\Omega^- = \{(x,t) \in \Omega : 0 < x < s(t)\}$ and $\Omega^+ = \{(x,t) \in \Omega : s(t) < x < 1\}$. In addition $v(x,t)$ and $v_x(x,t)$ are assumed to be continuous in Ω. The diffusion constant D is assumed to be finite (so, let $D = 1$) and $H(y)$ is the Heaviside function. The velocity function of a free boundary is

$$C : I \to \mathbf{R};\ I = (-a, 1-a),\ 0 < a < 1;$$

and given by

$$C(r) := \frac{2r - 1 + 2a}{\sqrt{(a+r)(1-a-r)}}\,.$$

In section 2, we shall use a change of variables which regularizes problem (2) in such a way which was used in [9]. In this way, we can give an alternative proof of well-posedness and obtain enough regularity of the solution for an analysis of the bifurcation. In section 3, we show that a Hopf bifurcation occurs at a critical value of τ as τ decreases.

2 The regularization

We rewrite the problem (2) as an abstract evolution equation:

$$\text{(F)} \qquad \begin{cases} \dfrac{d}{dt}(v,s) + \widetilde{A}(v,s) = F(v,s) \\ \qquad\qquad (v,s)(0) = (v_0(\cdot), s_0) \end{cases}$$

where

$$\widetilde{A} := \begin{pmatrix} -\dfrac{d^2}{dx^2} + c^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and the nonlinear operator F by

$$F(v,s) = \begin{pmatrix} F_1(v(\cdot,t), s(t)) \\ \\ F_2(v(\cdot,t), s(t)) \end{pmatrix} := \begin{pmatrix} H(\cdot - s(t)) \\ \\ \dfrac{1}{\tau} C(v(s(t),t)) \end{pmatrix}.$$

We let the differential operator $A = -\dfrac{d^2}{dx^2} + c^2$ together with Dirichlet boundary conditions at $x = 0, 1$. The combination of the jump discontinuity of a Heaviside function in the first component of F and the nature of the dependence of v on s in the second component of F makes it impossible to find a function space of the form $X = L_p$, $1 \leq p \leq \infty$ such that F satisfies a Lipschitz condition on $\widetilde{X} \subset X \times \mathbf{R}$. As a first step we obtain more regularity for the solution by semigroup methods, considering A as a densely defined operator

$$\begin{cases} A : D(A) \subset_{\text{dense}} X \longrightarrow X \\ X := L_2((0,1)) \text{ with norm } \|\cdot\|_2 \\ D(A) := \{v \in H^{2,2}(0,1) \,:\, v(0) = 0 = v(1)\}\,. \end{cases}$$

We decompose v in (F) into a part u, which is a solution to a more regular problem, and a part g, which is worse, but explicitly known in terms of the Green's function G of the operator A.

Proposition 2.1 *Let $G : [0,1]^2 \to \mathbf{R}$ be a Green's function of the operator A. Define $g : [0,1]^2 \longrightarrow \mathbf{R}$*

$$g(x,s) := \int_s^1 G(x,y)\,dy = A^{-1}(H(\cdot - s))(x)$$

and $\gamma : [0,1] \longrightarrow \mathbf{R}$

$$\gamma(s) := g(s,s)\,.$$

Then $g(\cdot, s) \in D(A)$ for all s, $\dfrac{\partial g}{\partial s}(x,s) = -G(x,s)$ is in $H^{1,\infty}((0,1)\times(0,1))$, and $\gamma \in C^\infty([0,1])$.

The proof is followed from the fact of Green's function.

Using these preliminary observations, we decompose a solution (v,s) of (F) into two parts by defining

$$u(t)(x) := v(x,t) - g(x,s(t))\,.$$

We denote the space $X \times \mathbf{R}$ by $\widetilde{X}$ and define

$$\begin{cases} D(\widetilde{A}) := D(A) \times \mathbf{R}, \\ \widetilde{A} : D(\widetilde{A}) \subset_{\text{dense}} \widetilde{X} \longrightarrow \widetilde{X},\ \widetilde{A}(u,s) := (Au, 0) \end{cases}$$

The initial value problem for (u, s) can then be written as

$$\text{(R)} \qquad \begin{cases} \dfrac{d}{dt}(u, s) + \widetilde{A}(u, s) = \mu\, f(u, s) \\ (u, s)(0) = (u(0), s(0)) = (u_0, s_0)\,. \end{cases}$$

where $\mu = \dfrac{1}{\tau}$. The nonlinear reaction terms f is defined on a set

$$W := \{(u, s) \in C^1([0,1]) \times (0,1) \;:\; u(s) + \gamma(s) \in I\} \subset_{\text{open}} C^1([0,1]) \times \mathbf{R}$$

and given by

$$\begin{aligned} & f : W \to L_2(0,1) \times \mathbf{R}, \\ & f(u, s) := \begin{pmatrix} G(\cdot, s) \cdot C(u(s) + \gamma(s)) \\ C(u(s) + \gamma(s)) \end{pmatrix}. \end{aligned}$$

The advantage of (R) over (F) is, that the right hand side of (R) is one step more regular than that of (F), since it involves $G(x, s)$ instead of $H(x - s)$. We can now apply semigroup theory to (R) using domains of fractional powers $\alpha \in [0,1]$ of A and $\widetilde{A}$:

$$X^\alpha := D(A^\alpha)\,, \;\; \widetilde{X}^\alpha := D(\widetilde{A}^\alpha)\,, \;\; \widetilde{X}^\alpha = X^\alpha \times \mathbf{R}\,.$$

For this we need to find an $\alpha \in [0,1]$ such that $X^\alpha \subset C^1([0,1])$, because then $f : W \cap \widetilde{X}^\alpha \to \widetilde{X}$ is continuously differentiable. It was shown that it is true for $\alpha > 3/4$ in [4] and [11]. Hence an existence, uniqueness and dependence on initial conditions of solutions to (F) can be obtained as a similar arguments in [9].

3 A Hopf bifurcation

In order to examine a Hopf bifurcation for (R), we need to investigate a linearized eigenvalue problem for (R). Hence we shall find stationary solutions for (R) which is represented by

$$\begin{aligned} &Au^* = \mu\, G(x,s^*)\, C(u^*(s^*) + \gamma(s^*)),\ \ u^*(0) = 0 = u^*(1) \\ &0 = \mu\, C(u^*(s^*) + \gamma(s^*)) \end{aligned}$$

for $(u^*, s^*) \in D(\widetilde{A}) \cap W$. The function $\gamma(s) = \int_s^1 G(s,y)\, dy$ then becomes

$$\gamma(s) = \frac{1}{c^2 \sinh c}\, \sinh cs\, (\cosh c(1-s) - 1)\ .$$

For $\mu \neq 0$, this system is equivalent to the pair of equations

$$(3) \qquad u^* = 0\,, \quad C(\gamma(s^*)) = 0\,.$$

We thus obtain

Proposition 3.1 *If* $0 < \frac{1}{2} - a < \dfrac{2\sinh^3 \frac{c}{3}}{c^2 \sinh c}$, *then for all* $\mu \neq 0$, (R) *has a unique stationary solution* (u^*, s^*) *with* $u^*(x) = 0$ *and* $s^* \in (0, \frac{1}{3}) \cup (\frac{1}{3}, 1)$. *The linearization of* f *at* $(0, s^*)$ *is*

$$Df(0,s^*)(\hat{u},\hat{s}) = \Big(\hat{u}(s^*) + \gamma'(s^*)\hat{s}\Big) \cdot \Big(G(\cdot, s^*), 1\Big)\ .$$

The pair $(0, s^*)$ *corresponds to a unique steady state* (v^*, s^*) *of* (F) *for* $\mu \neq 0$ *with*

$$v^*(x) = g(x, s^*)\,.$$

Proof. We note $C(r) = 0$ if and only if $r = \frac{1}{2} - a$. Define $\Gamma(s) = \gamma(s) - (\frac{1}{2} - a)$. Then $\Gamma'(s) > 0$ if $0 < s < \frac{1}{3}$ and $\Gamma'(s) < 0$ if $\frac{1}{3} < s < 1$. In order to have solutions, $\Gamma(s) = 0$ must satisfy either $\Gamma(0) < 0 < \Gamma(\frac{1}{3})$ or $\Gamma(1) < 0 < \Gamma(\frac{1}{3})$. That means $0 < \frac{1}{2} - a < \gamma(\frac{1}{3})$ with $\gamma(\frac{1}{3}) = \dfrac{2\sinh^3 \frac{c}{3}}{c^2 \sinh c}$. Thus, there exists a unique solution s^* in $(0, \frac{1}{3}) \cup (\frac{1}{3}, 1)$. The formula for $Df(0, s^*)$ follows from the relation $C'(1/2 - a) = 1$. $\square$

It suffices to show the stationary solution $(u^*(x), s^*, \mu^*)$ is a Hopf point, and since the Hopf bifurcation theorem in [1, 9, 10] gurantees that there is a periodic solution near the Hopf point point. Hence we now consider the eigenvalue problem for (R)

$$-\widetilde{A}(u, s) + \mu B(u, s) = \lambda(u, s)$$

which is equivalent to

$$\begin{aligned} (A + \lambda)u &= \mu \cdot (\gamma'(s^*)s + u(s^*)) \cdot G(\cdot, s^*) \\ \lambda s &= \mu \cdot (\gamma'(s^*)s + u(s^*)) \,. \end{aligned} \tag{4}$$

We now state the stationary solution is a Hopf point.

Theorem 3.2 *Assume that for $\mu^* \in \mathbf{R} \setminus \{0\}$, the operator $-\widetilde{A} + \mu^* B$ has a unique pair $\{\pm i\beta\}$ of purely imaginary eigenvalues. Then $(0, s^*, \mu^*)$ with $\frac{1}{3} < s^* < 1$ is a Hopf point for* (R).

Proof. Without loss of generality, let $\beta > 0$, and let ϕ^* be the (normalized) eigenfunction of $-\widetilde{A} + \mu^* B$ with eigenvalue $i\beta$. We have to show that

$(\phi^*, i\beta)$ can be extended to a C^1-curve $\mu \mapsto (\phi(\mu), \lambda(\mu))$ of eigendata for $-\widetilde{A} + \mu B$ with $\lambda'(\mu^*) \neq 0$.

For this let $\phi^* = (\psi_0, s_0) \in D(A) \times \mathbf{R}$. First, we see that $s_0 \neq 0$, for otherwise, by (4), $(A + i\beta)\psi_0 = i\beta s_0 G(\cdot, s^*) = 0$, which is impossible because A is symmetric. So without loss of generality, let $s_0 = 1$. Then by (4), $E(\psi_0, i\beta, \mu^*) = 0$, where

$$E : D(A)_{\mathbf{C}} \times \mathbf{C} \times \mathbf{R} \longrightarrow X_{\mathbf{C}} \times \mathbf{C}\,,$$

$$E(u, \lambda, \mu) := \begin{pmatrix} (A + \lambda)u - \mu \cdot ((\gamma'(s^*) + u(s^*))G(\cdot, s^*) \\ \lambda - \mu \cdot (\gamma'(s^*) + u(s^*)) \end{pmatrix}.$$

The equation $E(u, \lambda, \mu) = 0$ is equivalent to λ being and eigenvalue of $-\widetilde{A} + \mu B$ with eigenfunction $(u, 1)$. The proofs of E is C^1 and the mapping

$$D_{(u,\lambda)}E(\psi_0, i\beta, \mu^*)(\hat{u}, \hat{\lambda}) = \Big((A + i\beta)\hat{u} - \mu^*\hat{u}(s^*)G(\cdot, s^*) + \hat{\lambda}\psi_0, -\mu^*\hat{u}(s^*) + \hat{\lambda}\Big)$$

is a compact perturbation of the mapping

$$(\hat{u}, \hat{\lambda}) \longmapsto \Big((A + i\beta)\hat{u}, \hat{\lambda}\Big)$$

which is invertible, are similar to in [9]. Therefore, we only need to show that the transversality condition $\operatorname{Re}\lambda'(\mu^*) \neq 0$ holds. Let $\phi(\mu) = (\psi(\mu), 1)$. Since $E(\psi_0, i\beta, \mu^*) = 0$, we have

$$(A + i\beta)\psi_1 = -\delta_{s^*} \tag{5}$$

and

$$i\beta = \mu^*(\gamma'(s^*) + \psi_1(s^*) + G(s^*, s^*)\,). \tag{6}$$

Implicit differentiation of $E(\psi(\mu),\lambda(\mu),\mu)=0$ implies that

$$D_{(u,\lambda)}E(\psi_0,i\beta,\mu^*)(\psi'(\mu^*),\lambda'(\mu^*))=\begin{pmatrix}(\gamma'(s^*)+\psi'(\mu^*)(s^*))\cdot G(\cdot,s^*)\\ \gamma'(s^*)+\psi'(\mu^*)(s^*)\end{pmatrix}$$

This means that the function $\hat{u}:=\psi'(\mu^*)$ and $\hat{\lambda}:=\lambda'(\mu^*)$ satisfy the equations

$$(A+i\beta)\hat{u}-\mu^*\,\hat{u}(s^*)\,G(\cdot,s^*)+\hat{\lambda}\psi_0=(\gamma'(s^*)+\psi_0(s^*))\,G(\cdot,s^*) \tag{7}$$

and

$$-\mu^*\hat{u}(s^*)+\hat{\lambda}=\gamma'(s^*)+\psi_0(s^*)\,. \tag{8}$$

Putting (8) into (7) and using $\psi_1:=\psi_0-G(\cdot,s^*)$, as before, we obtain

$$(A+i\beta)\hat{u}+\hat{\lambda}\psi_1=0\,. \tag{9}$$

$$\hat{u}(s^*)=\frac{\hat{\lambda}}{\mu^*}-\frac{i\beta}{\mu^{*2}}\,. \tag{10}$$

Multiplying $\overline{\hat{u}}$ by (5) and (9) and integrating then we obtain

$$2i\,\beta\int_0^1\overline{\hat{u}}\psi_1=\frac{\overline{\hat{\lambda}}}{\mu^*}-\overline{\hat{u}(s^*)} \tag{11}$$

and

$$\textstyle\int_0^1|A^{1/2}\hat{u}|^2+i\beta\int_0^1|\hat{u}|^2+\hat{\lambda}\int_0^1\psi_1\,\overline{\hat{u}}=0\,. \tag{12}$$

Compare with (11) and (12) and, then substitute (10), then we get

$$2i\beta\int_0^1|A^{1/2}\hat{u}|^2-2\beta^2\int_0^1|\hat{u}|^2-i\beta\frac{\hat{\lambda}}{(\mu^*)^2}=0.$$

Hence, the transversality condition

$$\mathrm{Re}\hat{\lambda}=2(\mu^*)^2\int_0^1|A^{1/2}\hat{u}|^2>0$$

is satisfied. ∎

We now show that, whenever (R) admits a stationary solution, there is a unique $\mu^* > 0$ such that $(0, s^*, \mu^*)$ is a Hopf point, thus μ^* is the origin of a branch of nontrivial periodic orbits.

Theorem 3.3 *There exists a unique, purely imaginary eigenvalue $\lambda = i\beta$ of (4) with $\beta > 0$ for a unique critical point $\mu^* > 0$ in order for $(0, s^*, \mu^*)$ to be a Hopf point only for $\frac{1}{3} < s^* < 1$.*

Proof. To do this, we only have to show that the function $(u, \beta, \mu) \mapsto E(u, i\beta, \mu)$ has a unique zero with $\beta > 0$ and $\mu > 0$. This means solving the system

$$\begin{aligned} (A + i\beta)u &= \mu \cdot \Big(\gamma'(s^*) + u(s^*)\Big) \cdot G(\cdot, s^*) \\ i\beta &= \mu \cdot \Big(\gamma'(s^*) + u(s^*)\Big). \end{aligned}$$

As before, with $v := u - G(\cdot, s^*)$, this system is equivalent to the weak system of equations

$$\begin{aligned} (A + i\beta)w &= -\delta_{s^*} \\ i\beta &= \mu \cdot \Big(\gamma'(s^*) + G(s^*, s^*) + v(s^*)\Big). \end{aligned} \tag{13}$$

Now the first equation in (13) has, for fixed $\beta \geq 0$, the unique solution $v = -G_\beta(\cdot, s^*)$, where G_β is the Green's function for the operator $A + i\beta$. We are thus left with having to solve the complex valued equation

$$i\beta = \mu \cdot (\gamma'(s^*) + G(s^*, s^*) - G_\beta(s^*, s^*))$$

which is equivalent to the real valued system

$$\gamma'(s^*) + G(s^*, s^*) - \operatorname{Re} G_\beta(s^*, s^*) = 0 \tag{14}$$

$$\mu \operatorname{Im} G_\beta(s^*, s^*) + \beta = 0\,. \tag{15}$$

By the Lemma 12 in [9], the expression $\operatorname{Re} G_\beta(s^*, s^*)$ is strictly decreasing in $\beta \in \mathbf{R}^+$, $\operatorname{Re} G_0(s^*, s^*) = G(s^*, s^*)$ and $\lim_{\beta\to\infty} \operatorname{Re} G_\beta(s^*, s^*) = 0$ and, $\operatorname{Im} G_\beta(s^*, s^*) < 0$ for any $\beta > 0$. Since $\gamma'(s^*) + G(s^*, s^*)$ is real valued, there is a unique solution (β, μ^*) of (14) and (15) with $\beta > 0$ and $\mu^* > 0$. □

The following theorem summarizes what we have proved for the free boundary problem with the Dirichlet boundary condition:

Theorem 3.4 *Assume that* $0 < \frac{1}{2} - a < \dfrac{2\sinh^3 \frac{c}{3}}{c \sinh c}$, *so that* (R), *respectively* (F), *has a unique stationary solution* (u^*, s^*) *where* $u^* = 0$ *and* $s^* \in (0, \frac{1}{3}) \cup (\frac{1}{3}, 1)$ *respectively* (v^*, s^*), *for all* $\mu > 0$. *Then there exists a unique* $\mu^* > 0$ *such that the linearization* $-\widetilde{A} + \mu^* B$ *has a purely imaginary pair of eigenvalues. The point* $(0, s^*, \mu^*)$ *with* $0 < s^* < \frac{1}{3}$ *is then a Hopf point for* (R) *and there exists a* C^0*-curve of nontrivial periodic orbits for* (R), (F), *respectively, bifurcating from* $(0, s^*, \mu^*)$, (v^*, s^*, μ^*), *respectively.*

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A Hopf bifurcation of interfaces in a Dirichlet problem

YoonMee Ham & Sang-Gu Lee

Abstract

In this paper, we consider a free boundary problem which is reduced from the propagator-controller system satisfying the Dirichlet boundary condition as a layer parameter is equal to zero. We shall show that the equlibria $(v^*(x), s^*)$ *exist and there is a Hopf bifurcation at a critical point* τ *only for* $s^* \in (\frac{1}{3}, 1)$.

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1 Introduction

A free boundary problem is obtained from a singular limit of a reaction-diffusion system

$$(1) \qquad \begin{cases} \varepsilon\,\tau u_t = \varepsilon^2\, u_{xx} + f(u,v), \\ v_t = D\, v_{xx} + u - bv. \end{cases}$$

In this equations, u and v are called the activator and its inhibitor in morphogenesis ([7]), or the propagator and its controller in chemical waves of excitable media ([2]). The reaction term $f(u,v) = -u+H(u-a)-v$ ($a > 0$) is a McKean dynamics ([6]) and the parameter τ is a positive constant. We assume that the system (1) is bistable for a positive constant b which means that (1) has one unstable state and two stable states.

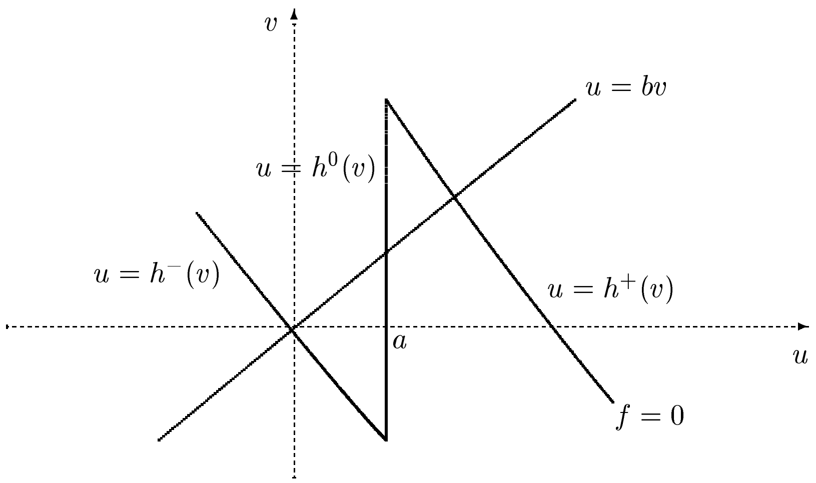

Figure 1: Nullclines $f(u) - v = 0$ and $g = 0$ for the bistable system (1).

When the diffusion constant ε is very small, the existence and uniqueness of solutions for the problem (1) are examined in [8]. The occurence of a Hopf bifurcation in (1) as τ varies was shown for a very small ε in [5] and [8]. When the diffusion constant ε is exactly zero in (1), a free boundary problem can be obtained in [3], and the well-posedness of this free boundary problem was shown in [5] and [9] using the fixed ponit theorem and using the semigroup theorey, respectively.

In [9], the authors proved the Hopf bifurcation near equlibria occurs at some τ for the Neumann boundary condition. In this paper, we consider the case for the Dirichlet boundary condition, and shall show that a Hopf bifurcation occurs for some equlibria for the following free boundary problem:

$$
(2) \qquad \begin{cases} v_t = D\, v_{xx} - c^2 v + H(x - s(t)) & \text{for } (x,t) \in \Omega^- \cup \Omega^+, \\ v(0,t) = 0 = v(1,t) & \text{for } t > 0, \\ v(x,0) = v_0(x) & \text{for } 0 \le x \le 1, \\ \tau \dfrac{ds}{dt} = C(v(s(t),t)) & \text{for } t > 0, \\ s(0) = s_0, \end{cases}
$$

where $\Omega := \{(x,t) \in (0,1) \times (0,\infty)\}$, $\Omega^- = \{(x,t) \in \Omega : 0 < x < s(t)\}$ and $\Omega^+ = \{(x,t) \in \Omega : s(t) < x < 1\}$. In addition $v(x,t)$ and $v_x(x,t)$ are assumed to be continuous in Ω. The diffusion constant D is assumed to be finite (so, let $D = 1$) and $H(y)$ is the Heaviside function. The velocity function of a free boundary is

$$
C : I \to \mathbf{R};\ I = (-a, 1-a),\ 0 < a < 1;
$$

and given by

$$
C(r) := \frac{2r - 1 + 2a}{\sqrt{(a+r)(1-a-r)}}.
$$

In section 2, we shall use a change of variables which regularizes problem (2) in such a way which was used in [9]. In this way, we can give an alternative proof of well-posedness and obtain enough regularity of the solution for an analysis of the bifurcation. In section 3, we show that a Hopf bifurcation occurs at a critical value of τ as τ decreases.

2 The regularization

We rewrite the problem (2) as an abstract evolution equation:

$$\text{(F)} \qquad \begin{cases} \dfrac{d}{dt}(v,s) + \widetilde{A}(v,s) = F(v,s) \\ \qquad (v,s)(0) = (v_0(\cdot), s_0) \end{cases}$$

where

$$\widetilde{A} := \begin{pmatrix} -\dfrac{d^2}{dx^2} + c^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and the nonlinear operator F by

$$F(v,s) = \begin{pmatrix} F_1(v(\cdot,t), s(t)) \\ F_2(v(\cdot,t), s(t)) \end{pmatrix} := \begin{pmatrix} H(\cdot - s(t)) \\ \dfrac{1}{\tau} C(v(s(t),t)) \end{pmatrix}.$$

We let the differential operator $A = -\dfrac{d^2}{dx^2} + c^2$ together with Dirichlet boundary conditions at $x = 0, 1$. The combination of the jump discontinuity of a Heaviside function in the first component of F and the nature of the dependence of v on s in the second component of F makes it impossible to find a function space of the form $X = L_p$, $1 \le p \le \infty$ such that F satisfies a Lipschitz condition on $\widetilde{X} \subset X \times \mathbf{R}$. As a first step we obtain more regularity for the solution by semigroup methods, considering A as a densely defined operator

$$\begin{cases} A : D(A) \subset_{\text{dense}} X \longrightarrow X \\ X := L_2((0,1)) \text{ with norm } \|\cdot\|_2 \\ D(A) := \{v \in H^{2,2}(0,1) \,:\, v(0) = 0 = v(1)\}\,. \end{cases}$$

We decompose v in (F) into a part u, which is a solution to a more regular problem, and a part g, which is worse, but explicitly known in terms of the Green's function G of the operator A.

Proposition 2.1 *Let $G : [0,1]^2 \to \mathbf{R}$ be a Green's function of the operator A. Define $g : [0,1]^2 \longrightarrow \mathbf{R}$*

$$g(x,s) := \int_s^1 G(x,y)\,dy = A^{-1}(H(\cdot - s))(x)$$

and $\gamma : [0,1] \longrightarrow \mathbf{R}$

$$\gamma(s) := g(s,s)\,.$$

Then $g(\cdot, s) \in D(A)$ for all s, $\dfrac{\partial g}{\partial s}(x,s) = -G(x,s)$ is in $H^{1,\infty}((0,1)\times(0,1))$, and $\gamma \in C^\infty([0,1])$.

The proof is followed from the fact of Green's function.

Using these preliminary observations, we decompose a solution (v,s) of (F) into two parts by defining

$$u(t)(x) := v(x,t) - g(x,s(t))\,.$$

We denote the space $X \times \mathbf{R}$ by $\widetilde{X}$ and define

$$\begin{cases} D(\widetilde{A}) := D(A) \times \mathbf{R}, \\ \widetilde{A} : D(\widetilde{A}) \subset_{\text{dense}} \widetilde{X} \longrightarrow \widetilde{X},\ \widetilde{A}(u,s) := (Au, 0) \end{cases}$$

The initial value problem for (u,s) can then be written as

$$\text{(R)} \qquad \begin{cases} \dfrac{d}{dt}(u,s) + \widetilde{A}(u,s) = \mu\, f(u,s) \\ (u,s)(0) = (u(0), s(0)) = (u_0, s_0)\,. \end{cases}$$

where $\mu = \dfrac{1}{\tau}$. The nonlinear reaction terms f is defined on a set

$$W := \{(u,s) \in C^1([0,1]) \times (0,1) \,:\, u(s) + \gamma(s) \in I\} \subset_{\text{open}} C^1([0,1]) \times \mathbf{R}$$

and given by

$$\begin{aligned} &f : W \to L_2(0,1) \times \mathbf{R}, \\ &f(u,s) := \begin{pmatrix} G(\cdot, s) \cdot C(u(s) + \gamma(s)) \\ C(u(s) + \gamma(s)) \end{pmatrix}. \end{aligned}$$

The advantage of (R) over (F) is, that the right hand side of (R) is one step more regular than that of (F), since it involves $G(x,s)$ instead of $H(x-s)$. We can now apply semigroup theory to (R) using domains of fractional powers $\alpha \in [0,1]$ of A and $\widetilde{A}$:

$$X^\alpha := D(A^\alpha)\,, \;\; \widetilde{X}^\alpha := D(\widetilde{A}^\alpha)\,, \;\; \widetilde{X}^\alpha = X^\alpha \times \mathbf{R}\,.$$

For this we need to find an $\alpha \in [0,1]$ such that $X^\alpha \subset C^1([0,1])$, because then $f : W \cap \widetilde{X}^\alpha \to \widetilde{X}$ is continuously differentiable. It was shown that it is true for $\alpha > 3/4$ in [4] and [11]. Hence an existence, uniqueness and dependence on initial conditions of solutions to (F) can be obtained as a similar arguments in [9].

3 A Hopf bifurcation

In order to examine a Hopf bifurcation for (R), we need to investigate a linearized eigenvalue problem for (R). Hence we shall find stationary solutions for (R) which is represented by

$$\begin{aligned} &Au^* = \mu\, G(x,s^*)\, C(u^*(s^*) + \gamma(s^*)),\ \ u^*(0) = 0 = u^*(1) \\ &0 = \mu\, C(u^*(s^*) + \gamma(s^*)) \end{aligned}$$

for $(u^*, s^*) \in D(\widetilde{A}) \cap W$. The function $\gamma(s) = \int_s^1 G(s,y)\, dy$ then becomes

$$\gamma(s) = \frac{1}{c^2 \sinh c} \sinh cs\, (\cosh c(1-s) - 1)\ .$$

For $\mu \neq 0$, this system is equivalent to the pair of equations

$$u^* = 0\,, \quad C(\gamma(s^*)) = 0\,. \tag{3}$$

We thus obtain

Proposition 3.1 *If* $0 < \frac{1}{2} - a < \dfrac{2\sinh^3 \frac{c}{3}}{c^2 \sinh c}$, *then for all* $\mu \neq 0$, (R) *has a unique stationary solution* (u^*, s^*) *with* $u^*(x) = 0$ *and* $s^* \in (0, \frac{1}{3}) \cup (\frac{1}{3}, 1)$. *The linearization of* f *at* $(0, s^*)$ *is*

$$Df(0,s^*)(\hat{u},\hat{s}) = \Big(\hat{u}(s^*) + \gamma'(s^*)\hat{s}\Big) \cdot \Big(G(\cdot, s^*), 1\Big)\ .$$

The pair $(0, s^*)$ *corresponds to a unique steady state* (v^*, s^*) *of* (F) *for* $\mu \neq 0$ *with*

$$v^*(x) = g(x, s^*)\,.$$

Proof. We note $C(r) = 0$ if and only if $r = \frac{1}{2} - a$. Define $\Gamma(s) = \gamma(s) - (\frac{1}{2} - a)$. Then $\Gamma'(s) > 0$ if $0 < s < \frac{1}{3}$ and $\Gamma'(s) < 0$ if $\frac{1}{3} < s < 1$. In order to have solutions, $\Gamma(s) = 0$ must satisfy either $\Gamma(0) < 0 < \Gamma(\frac{1}{3})$ or $\Gamma(1) < 0 < \Gamma(\frac{1}{3})$. That means $0 < \frac{1}{2} - a < \gamma(\frac{1}{3})$ with $\gamma(\frac{1}{3}) = \dfrac{2\sinh^3 \frac{c}{3}}{c^2 \sinh c}$. Thus, there exists a unique solution s^* in $(0, \frac{1}{3}) \cup (\frac{1}{3}, 1)$. The formula for $Df(0, s^*)$ follows from the relation $C'(1/2 - a) = 1$. □

It suffices to show the stationary solution $(u^*(x), s^*, \mu^*)$ is a Hopf point, and since the Hopf bifurcation theorem in [1, 9, 10] gurantees that there is a periodic solution near the Hopf point point. Hence we now consider the eigenvalue problem for (R)

$$-\widetilde{A}(u, s) + \mu B(u, s) = \lambda(u, s)$$

which is equivalent to

$$\begin{array}{rcl} (A + \lambda)u & = & \mu \cdot (\gamma'(s^*)s + u(s^*)) \cdot G(\cdot, s^*) \\ \lambda s & = & \mu \cdot (\gamma'(s^*)s + u(s^*)) \, . \end{array} \tag{4}$$

We now state the stationary solution is a Hopf point.

Theorem 3.2 *Assume that for* $\mu^* \in \mathbf{R} \setminus \{0\}$, *the operator* $-\widetilde{A} + \mu^* B$ *has a unique pair* $\{\pm i\beta\}$ *of purely imaginary eigenvalues. Then* $(0, s^*, \mu^*)$ *with* $\frac{1}{3} < s^* < 1$ *is a Hopf point for* (R).

Proof. Without loss of generality, let $\beta > 0$, and let ϕ^* be the (normalized) eigenfunction of $-\widetilde{A} + \mu^* B$ with eigenvalue $i\beta$. We have to show that

$(\phi^*, i\beta)$ can be extended to a C^1-curve $\mu \mapsto (\phi(\mu), \lambda(\mu))$ of eigendata for $-\widetilde{A} + \mu B$ with $\lambda'(\mu^*) \neq 0$.

For this let $\phi^* = (\psi_0, s_0) \in D(A) \times \mathbf{R}$. First, we see that $s_0 \neq 0$, for otherwise, by (4), $(A + i\beta)\psi_0 = i\beta s_0 G(\cdot, s^*) = 0$, which is impossible because A is symmetric. So without loss of generality, let $s_0 = 1$. Then by (4), $E(\psi_0, i\beta, \mu^*) = 0$, where

$$E : D(A)_{\mathbf{C}} \times \mathbf{C} \times \mathbf{R} \longrightarrow X_{\mathbf{C}} \times \mathbf{C}\,,$$
$$E(u, \lambda, \mu) := \begin{pmatrix} (A + \lambda)u - \mu \cdot ((\gamma'(s^*) + u(s^*))G(\cdot, s^*) \\ \lambda - \mu \cdot (\gamma'(s^*) + u(s^*)) \end{pmatrix}.$$

The equation $E(u, \lambda, \mu) = 0$ is equivalent to λ being and eigenvalue of $-\widetilde{A} + \mu B$ with eigenfunction $(u, 1)$. The proofs of E is C^1 and the mapping

$$D_{(u,\lambda)}E(\psi_0, i\beta, \mu^*)(\hat{u}, \hat{\lambda}) = \Big((A + i\beta)\hat{u} - \mu^*\hat{u}(s^*)G(\cdot, s^*) + \hat{\lambda}\psi_0, -\mu^*\hat{u}(s^*) + \hat{\lambda}\Big)$$

is a compact perturbation of the mapping

$$(\hat{u}, \hat{\lambda}) \longmapsto \Big((A + i\beta)\hat{u}, \hat{\lambda}\Big)$$

which is invertible, are similar to in [9]. Therefore, we only need to show that the transversality condition $\operatorname{Re}\lambda'(\mu^*) \neq 0$ holds. Let $\phi(\mu) = (\psi(\mu), 1)$. Since $E(\psi_0, i\beta, \mu^*) = 0$, we have

$$(A + i\beta)\psi_1 = -\delta_{s^*} \tag{5}$$

and

$$i\beta = \mu^*(\gamma'(s^*) + \psi_1(s^*) + G(s^*, s^*)\,). \tag{6}$$

Implicit differentiation of $E(\psi(\mu), \lambda(\mu), \mu) = 0$ implies that

$$D_{(u,\lambda)}E(\psi_0, i\beta, \mu^*)(\psi'(\mu^*), \lambda'(\mu^*)) = \begin{pmatrix} (\gamma'(s^*) + \psi'(\mu^*)(s^*)) \cdot G(\cdot, s^*) \\ \gamma'(s^*) + \psi'(\mu^*)(s^*) \end{pmatrix}$$

This means that the function $\hat{u} := \psi'(\mu^*)$ and $\hat{\lambda} := \lambda'(\mu^*)$ satisfy the equations

$$(A + i\beta)\hat{u} - \mu^* \, \hat{u}(s^*) \, G(\cdot, s^*) + \hat{\lambda}\psi_0 = (\gamma'(s^*) + \psi_0(s^*)) \, G(\cdot, s^*) \tag{7}$$

and

$$-\mu^* \hat{u}(s^*) + \hat{\lambda} = \gamma'(s^*) + \psi_0(s^*) \, . \tag{8}$$

Putting (8) into (7) and using $\psi_1 := \psi_0 - G(\cdot, s^*)$, as before, we obtain

$$(A + i\beta)\hat{u} + \hat{\lambda}\psi_1 = 0 \, . \tag{9}$$

$$\hat{u}(s^*) = \frac{\hat{\lambda}}{\mu^*} - \frac{i\beta}{\mu^{*2}} \, . \tag{10}$$

Multiplying $\overline{\hat{u}}$ by (5) and (9) and integrating then we obtain

$$2i \, \beta \int_0^1 \overline{\hat{u}}\psi_1 = \frac{\overline{\hat{\lambda}}}{\mu^*} - \overline{\hat{u}(s^*)} \tag{11}$$

and

$$\textstyle\int_0^1 |A^{1/2}\hat{u}|^2 + i\beta \int_0^1 |\hat{u}|^2 + \hat{\lambda} \int_0^1 \psi_1 \, \overline{\hat{u}} = 0 \, . \tag{12}$$

Compare with (11) and (12) and, then substitute (10), then we get

$$2i\beta \int_0^1 |A^{1/2}\hat{u}|^2 - 2\beta^2 \int_0^1 |\hat{u}|^2 - i\beta \frac{\hat{\lambda}}{(\mu^*)^2} = 0.$$

Hence, the transversality condition

$$\mathrm{Re}\hat{\lambda} = 2(\mu^*)^2 \int_0^1 |A^{1/2}\hat{u}|^2 > 0$$

is satisfied. □

We now show that, whenever (R) admits a stationary solution, there is a unique $\mu^* > 0$ such that $(0, s^*, \mu^*)$ is a Hopf point, thus μ^* is the origin of a branch of nontrivial periodic orbits.

Theorem 3.3 *There exists a unique, purely imaginary eigenvalue* $\lambda = i\beta$ *of (4) with* $\beta > 0$ *for a unique critical point* $\mu^* > 0$ *in order for* $(0, s^*, \mu^*)$ *to be a Hopf point only for* $\frac{1}{3} < s^* < 1$.

Proof. To do this, we only have to show that the function $(u, \beta, \mu) \mapsto E(u, i\beta, \mu)$ has a unique zero with $\beta > 0$ and $\mu > 0$. This means solving the system

$$\begin{aligned}(A + i\beta)u &= \mu \cdot \Big(\gamma'(s^*) + u(s^*)\Big) \cdot G(\cdot, s^*) \\ i\beta &= \mu \cdot \Big(\gamma'(s^*) + u(s^*)\Big).\end{aligned}$$

As before, with $v := u - G(\cdot, s^*)$, this system is equivalent to the weak system of equations

$$\begin{aligned}(A + i\beta)w &= -\delta_{s^*} \\ i\beta &= \mu \cdot \Big(\gamma'(s^*) + G(s^*, s^*) + v(s^*)\Big).\end{aligned} \tag{13}$$

Now the first equation in (13) has, for fixed $\beta \geq 0$, the unique solution $v = -G_\beta(\cdot, s^*)$, where G_β is the Green's function for the operator $A + i\beta$. We are thus left with having to solve the complex valued equation

$$i\beta = \mu \cdot (\gamma'(s^*) + G(s^*, s^*) - G_\beta(s^*, s^*))$$

which is equivalent to the real valued system

$$\gamma'(s^*) + G(s^*, s^*) - \operatorname{Re} G_\beta(s^*, s^*) = 0 \tag{14}$$

$$\mu \operatorname{Im} G_\beta(s^*, s^*) + \beta = 0\,. \tag{15}$$

By the Lemma 12 in [9], the expression $\operatorname{Re} G_\beta(s^*, s^*)$ is strictly decreasing in $\beta \in \mathbf{R}^+$, $\operatorname{Re} G_0(s^*, s^*) = G(s^*, s^*)$ and $\lim_{\beta\to\infty} \operatorname{Re} G_\beta(s^*, s^*) = 0$ and, $\operatorname{Im} G_\beta(s^*, s^*) < 0$ for any $\beta > 0$. Since $\gamma'(s^*) + G(s^*, s^*)$ is real valued, there is a unique solution (β, μ^*) of (14) and (15) with $\beta > 0$ and $\mu^* > 0$. $\square$

The following theorem summarizes what we have proved for the free boundary problem with the Dirichlet boundary condition:

Theorem 3.4 *Assume that* $0 < \frac{1}{2} - a < \dfrac{2\sinh^3\frac{c}{3}}{c\sinh c}$, *so that* (R), *respectively* (F), *has a unique stationary solution* (u^*, s^*) *where* $u^* = 0$ *and* $s^* \in (0, \frac{1}{3}) \cup (\frac{1}{3}, 1)$ *respectively* (v^*, s^*), *for all* $\mu > 0$. *Then there exists a unique* $\mu^* > 0$ *such that the linearization* $-\widetilde{A} + \mu^* B$ *has a purely imaginary pair of eigenvalues. The point* $(0, s^*, \mu^*)$ *with* $0 < s^* < \frac{1}{3}$ *is then a Hopf point for* (R) *and there exists a* C^0*-curve of nontrivial periodic orbits for* (R), (F), *respectively, bifurcating from* $(0, s^*, \mu^*)$, (v^*, s^*, μ^*), *respectively.*

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