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A Hopf bifurcation of interfaces in a Dirichlet problem

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Abstract

In this paper, we consider a free boundary problem which is reduced from the propagator-controller system satisfying the Dirichlet boundary condition as a layer parameter is equal to zero. We shall show that the equlibria $(v^*(x), s^*)$ *exist and there is a Hopf bifurcation at a critical point* τ *only for* $s^* \in (\frac{1}{3}, 1)$.

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1 Introduction

A free boundary problem is obtained from a singular limit of a reaction-diffusion system

$$(1) \qquad \begin{cases} \varepsilon\,\tau u_t = \varepsilon^2\, u_{xx} + f(u,v), \\ v_t = D\, v_{xx} + u - bv. \end{cases}$$

In this equations, u and v are called the activator and its inhibitor in morphogenesis ([7]), or the propagator and its controller in chemical waves of excitable media ([2]). The reaction term $f(u,v) = -u + H(u-a) - v$ ($a > 0$) is a McKean dynamics ([6]) and the parameter τ is a positive constant. We assume that the system (1) is bistable for a positive constant b which means that (1) has one unstable state and two stable states.

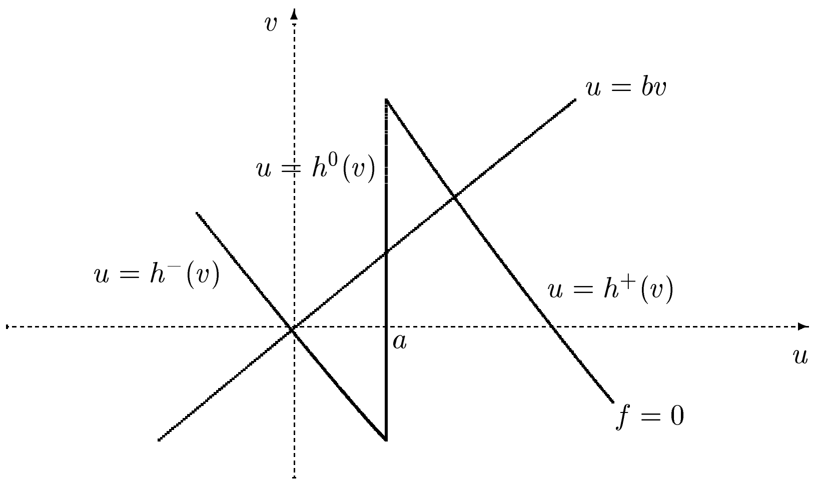

Figure 1: Nullclines $f(u) - v = 0$ and $g = 0$ for the bistable system (1).

When the diffusion constant ε is very small, the existence and uniqueness of solutions for the problem (1) are examined in [8]. The occurence of a Hopf bifurcation in (1) as τ varies was shown for a very small ε in [5] and [8]. When the diffusion constant ε is exactly zero in (1), a free boundary problem can be obtained in [3], and the well-posedness of this free boundary problem was shown in [5] and [9] using the fixed ponit theorem and using the semigroup theorey, respectively.

In [9], the authors proved the Hopf bifurcation near equlibria occurs at some τ for the Neumann boundary condition. In this paper, we consider the case for the Dirichlet boundary condition, and shall show that a Hopf bifurcation occurs for some equlibria for the following free boundary problem:

$$
\begin{cases}
v_t = D\, v_{xx} - c^2 v + H(x - s(t)) & \text{for } \ (x,t) \in \Omega^- \cup \Omega^+, \\
v(0,t) = 0 = v(1,t) & \text{for } \ t > 0, \\
v(x,0) = v_0(x) & \text{for } \ 0 \le x \le 1, \\
\tau \dfrac{ds}{dt} = C(v(s(t),t)) & \text{for } \ t > 0, \\
s(0) = s_0,
\end{cases} \tag{2}
$$

where $\Omega := \{(x,t) \in (0,1) \times (0,\infty)\}$, $\Omega^- = \{(x,t) \in \Omega : 0 < x < s(t)\}$ and $\Omega^+ = \{(x,t) \in \Omega : s(t) < x < 1\}$. In addition $v(x,t)$ and $v_x(x,t)$ are assumed to be continuous in Ω. The diffusion constant D is assumed to be finite (so, let $D = 1$) and $H(y)$ is the Heaviside function. The velocity function of a free boundary is

$$C : I \to \mathbf{R}; \ I = (-a, 1-a), \ 0 < a < 1;$$

and given by

$$C(r) := \frac{2r - 1 + 2a}{\sqrt{(a+r)(1-a-r)}}.$$

In section 2, we shall use a change of variables which regularizes problem (2) in such a way which was used in [9]. In this way, we can give an alternative proof of well-posedness and obtain enough regularity of the solution for an analysis of the bifurcation. In section 3, we show that a Hopf bifurcation occurs at a critical value of τ as τ decreases.

2 The regularization

We rewrite the problem (2) as an abstract evolution equation:

$$\text{(F)} \qquad \begin{cases} \dfrac{d}{dt}(v,s) + \widetilde{A}(v,s) = F(v,s) \\ \qquad (v,s)(0) = (v_0(\cdot), s_0) \end{cases}$$

where

$$\widetilde{A} := \begin{pmatrix} -\dfrac{d^2}{dx^2} + c^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and the nonlinear operator F by

$$F(v,s) = \begin{pmatrix} F_1(v(\cdot,t), s(t)) \\ \\ F_2(v(\cdot,t), s(t)) \end{pmatrix} := \begin{pmatrix} H(\cdot - s(t)) \\ \dfrac{1}{\tau} C(v(s(t),t)) \end{pmatrix}.$$

We let the differential operator $A = -\dfrac{d^2}{dx^2} + c^2$ together with Dirichlet boundary conditions at $x = 0, 1$. The combination of the jump discontinuity of a Heaviside function in the first component of F and the nature of the dependence of v on s in the second component of F makes it impossible to find a function space of the form $X = L_p$, $1 \le p \le \infty$ such that F satisfies a Lipschitz condition on $\widetilde{X} \subset X \times \mathbf{R}$. As a first step we obtain more regularity for the solution by semigroup methods, considering A as a densely defined operator

$$\begin{cases} A : D(A) \subset_{\text{dense}} X \longrightarrow X \\ X := L_2((0,1)) \text{ with norm } \|\cdot\|_2 \\ D(A) := \{v \in H^{2,2}(0,1)\ :\ v(0) = 0 = v(1)\}\,. \end{cases}$$

We decompose v in (F) into a part u, which is a solution to a more regular problem, and a part g, which is worse, but explicitly known in terms of the Green's function G of the operator A.

Proposition 2.1 *Let $G : [0,1]^2 \to \mathbf{R}$ be a Green's function of the operator A. Define $g : [0,1]^2 \longrightarrow \mathbf{R}$*

$$g(x,s) := \int_s^1 G(x,y)\,dy = A^{-1}(H(\cdot - s))(x)$$

and $\gamma : [0,1] \longrightarrow \mathbf{R}$

$$\gamma(s) := g(s,s)\,.$$

Then $g(\cdot,s) \in D(A)$ for all s, $\dfrac{\partial g}{\partial s}(x,s) = -G(x,s)$ is in $H^{1,\infty}((0,1)\times(0,1))$, and $\gamma \in C^\infty([0,1])$.

The proof is followed from the fact of Green's function.

Using these preliminary observations, we decompose a solution (v,s) of (F) into two parts by defining

$$u(t)(x) := v(x,t) - g(x,s(t))\,.$$

We denote the space $X \times \mathbf{R}$ by $\widetilde{X}$ and define

$$\begin{cases} D(\widetilde{A}) := D(A) \times \mathbf{R}, \\ \widetilde{A} : D(\widetilde{A}) \subset_{\text{dense}} \widetilde{X} \longrightarrow \widetilde{X},\ \widetilde{A}(u,s) := (Au, 0) \end{cases}$$

The initial value problem for (u,s) can then be written as

$$\text{(R)} \qquad \begin{cases} \dfrac{d}{dt}(u,s) + \widetilde{A}(u,s) = \mu\, f(u,s) \\ (u,s)(0) = (u(0), s(0)) = (u_0, s_0)\,. \end{cases}$$

where $\mu = \dfrac{1}{\tau}$. The nonlinear reaction terms f is defined on a set

$$W := \{(u,s) \in C^1([0,1]) \times (0,1) \;:\; u(s) + \gamma(s) \in I\} \subset_{\text{open}} C^1([0,1]) \times \mathbf{R}$$

and given by

$$\begin{aligned} & f : W \to L_2(0,1) \times \mathbf{R}, \\ & f(u,s) := \begin{pmatrix} G(\cdot, s) \cdot C(u(s) + \gamma(s)) \\ C(u(s) + \gamma(s)) \end{pmatrix}. \end{aligned}$$

The advantage of (R) over (F) is, that the right hand side of (R) is one step more regular than that of (F), since it involves $G(x,s)$ instead of $H(x-s)$. We can now apply semigroup theory to (R) using domains of fractional powers $\alpha \in [0,1]$ of A and $\widetilde{A}$:

$$X^\alpha := D(A^\alpha)\,, \quad \widetilde{X}^\alpha := D(\widetilde{A}^\alpha)\,, \quad \widetilde{X}^\alpha = X^\alpha \times \mathbf{R}\,.$$

For this we need to find an $\alpha \in [0,1]$ such that $X^\alpha \subset C^1([0,1])$, because then $f : W \cap \widetilde{X}^\alpha \to \widetilde{X}$ is continuously differentiable. It was shown that it is true for $\alpha > 3/4$ in [4] and [11]. Hence an existence, uniqueness and dependence on initial conditions of solutions to (F) can be obtained as a similar arguments in [9].

3 A Hopf bifurcation

In order to examine a Hopf bifurcation for (R), we need to investigate a linearized eigenvalue problem for (R). Hence we shall find stationary solutions for (R) which is represented by

$$\begin{aligned} &Au^* = \mu\, G(x, s^*)\, C(u^*(s^*) + \gamma(s^*)),\ \ u^*(0) = 0 = u^*(1) \\ &0 = \mu\, C(u^*(s^*) + \gamma(s^*)) \end{aligned}$$

for $(u^*, s^*) \in D(\widetilde{A}) \cap W$. The function $\gamma(s) = \displaystyle\int_s^1 G(s,y)\,dy$ then becomes

$$\gamma(s) = \frac{1}{c^2 \sinh c} \sinh cs\, (\cosh c(1-s) - 1)\ .$$

For $\mu \neq 0$, this system is equivalent to the pair of equations

$$u^* = 0\,, \quad C(\gamma(s^*)) = 0\,. \tag{3}$$

We thus obtain

Proposition 3.1 *If* $0 < \frac{1}{2} - a < \dfrac{2\sinh^3 \frac{c}{3}}{c^2 \sinh c}$, *then for all* $\mu \neq 0$, (R) *has a unique stationary solution* (u^*, s^*) *with* $u^*(x) = 0$ *and* $s^* \in (0, \frac{1}{3}) \cup (\frac{1}{3}, 1)$. *The linearization of* f *at* $(0, s^*)$ *is*

$$Df(0,s^*)(\hat{u}, \hat{s}) = \Big(\hat{u}(s^*) + \gamma'(s^*)\hat{s}\Big) \cdot \Big(G(\cdot, s^*), 1\Big)\ .$$

The pair $(0, s^*)$ *corresponds to a unique steady state* (v^*, s^*) *of* (F) *for* $\mu \neq 0$ *with*

$$v^*(x) = g(x, s^*)\,.$$

Proof. We note $C(r) = 0$ if and only if $r = \frac{1}{2} - a$. Define $\Gamma(s) = \gamma(s) - (\frac{1}{2} - a)$. Then $\Gamma'(s) > 0$ if $0 < s < \frac{1}{3}$ and $\Gamma'(s) < 0$ if $\frac{1}{3} < s < 1$. In order to have solutions, $\Gamma(s) = 0$ must satisfy either $\Gamma(0) < 0 < \Gamma(\frac{1}{3})$ or $\Gamma(1) < 0 < \Gamma(\frac{1}{3})$. That means $0 < \frac{1}{2} - a < \gamma(\frac{1}{3})$ with $\gamma(\frac{1}{3}) = \dfrac{2\sinh^3\frac{c}{3}}{c^2 \sinh c}$. Thus, there exists a unique solution s^* in $(0, \frac{1}{3}) \cup (\frac{1}{3}, 1)$. The formula for $Df(0, s^*)$ follows from the relation $C'(1/2 - a) = 1$. $\Box$

It suffices to show the stationary solution $(u^*(x), s^*, \mu^*)$ is a Hopf point, and since the Hopf bifurcation theorem in [1, 9, 10] gurantees that there is a periodic solution near the Hopf point point. Hence we now consider the eigenvalue problem for (R)

$$-\widetilde{A}(u, s) + \mu B(u, s) = \lambda(u, s)$$

which is equivalent to

$$\begin{array}{rcl} (A + \lambda)u & = & \mu \cdot (\gamma'(s^*)s + u(s^*)) \cdot G(\cdot, s^*) \\ \lambda s & = & \mu \cdot (\gamma'(s^*)s + u(s^*)) \,. \end{array} \tag{4}$$

We now state the stationary solution is a Hopf point.

Theorem 3.2 *Assume that for $\mu^* \in \mathbf{R} \setminus \{0\}$, the operator $-\widetilde{A} + \mu^* B$ has a unique pair $\{\pm i\beta\}$ of purely imaginary eigenvalues. Then $(0, s^*, \mu^*)$ with $\frac{1}{3} < s^* < 1$ is a Hopf point for* (R).

Proof. Without loss of generality, let $\beta > 0$, and let ϕ^* be the (normalized) eigenfunction of $-\widetilde{A} + \mu^* B$ with eigenvalue $i\beta$. We have to show that

$(\phi^*, i\beta)$ can be extended to a C^1-curve $\mu \mapsto (\phi(\mu), \lambda(\mu))$ of eigendata for $-\widetilde{A} + \mu B$ with $\lambda'(\mu^*) \neq 0$.

For this let $\phi^* = (\psi_0, s_0) \in D(A) \times \mathbf{R}$. First, we see that $s_0 \neq 0$, for otherwise, by (4), $(A + i\beta)\psi_0 = i\beta s_0 G(\cdot, s^*) = 0$, which is impossible because A is symmetric. So without loss of generality, let $s_0 = 1$. Then by (4), $E(\psi_0, i\beta, \mu^*) = 0$, where

$$E : D(A)_{\mathbf{C}} \times \mathbf{C} \times \mathbf{R} \longrightarrow X_{\mathbf{C}} \times \mathbf{C}\,,$$
$$E(u, \lambda, \mu) := \begin{pmatrix} (A + \lambda)u - \mu \cdot ((\gamma'(s^*) + u(s^*))G(\cdot, s^*) \\ \lambda - \mu \cdot (\gamma'(s^*) + u(s^*)) \end{pmatrix}.$$

The equation $E(u, \lambda, \mu) = 0$ is equivalent to λ being and eigenvalue of $-\widetilde{A} + \mu B$ with eigenfunction $(u, 1)$. The proofs of E is C^1 and the mapping

$$D_{(u,\lambda)}E(\psi_0, i\beta, \mu^*)(\hat{u}, \hat{\lambda}) = \Big((A + i\beta)\hat{u} - \mu^*\hat{u}(s^*)G(\cdot, s^*) + \hat{\lambda}\psi_0, -\mu^*\hat{u}(s^*) + \hat{\lambda}\Big)$$

is a compact perturbation of the mapping

$$(\hat{u}, \hat{\lambda}) \longmapsto \Big((A + i\beta)\hat{u}, \hat{\lambda}\Big)$$

which is invertible, are similar to in [9]. Therefore, we only need to show that the transversality condition $\operatorname{Re}\lambda'(\mu^*) \neq 0$ holds. Let $\phi(\mu) = (\psi(\mu), 1)$. Since $E(\psi_0, i\beta, \mu^*) = 0$, we have

$$(A + i\beta)\psi_1 = -\delta_{s^*} \tag{5}$$

and

$$i\beta = \mu^*(\gamma'(s^*) + \psi_1(s^*) + G(s^*, s^*)\,). \tag{6}$$

Implicit differentiation of $E(\psi(\mu), \lambda(\mu), \mu) = 0$ implies that

$$D_{(u,\lambda)}E(\psi_0, i\beta, \mu^*)(\psi'(\mu^*), \lambda'(\mu^*)) = \begin{pmatrix} (\gamma'(s^*) + \psi'(\mu^*)(s^*)) \cdot G(\cdot, s^*) \\ \gamma'(s^*) + \psi'(\mu^*)(s^*) \end{pmatrix}$$

This means that the function $\hat{u} := \psi'(\mu^*)$ and $\hat{\lambda} := \lambda'(\mu^*)$ satisfy the equations

$$(A + i\beta)\hat{u} - \mu^* \, \hat{u}(s^*) \, G(\cdot, s^*) + \hat{\lambda}\psi_0 = (\gamma'(s^*) + \psi_0(s^*)) \, G(\cdot, s^*) \tag{7}$$

and

$$-\mu^* \hat{u}(s^*) + \hat{\lambda} = \gamma'(s^*) + \psi_0(s^*) \,. \tag{8}$$

Putting (8) into (7) and using $\psi_1 := \psi_0 - G(\cdot, s^*)$, as before, we obtain

$$(A + i\beta)\hat{u} + \hat{\lambda}\psi_1 = 0 \,. \tag{9}$$

$$\hat{u}(s^*) = \frac{\hat{\lambda}}{\mu^*} - \frac{i\beta}{\mu^{*2}} \,. \tag{10}$$

Multiplying $\overline{\hat{u}}$ by (5) and (9) and integrating then we obtain

$$2i\,\beta \int_0^1 \overline{\hat{u}}\psi_1 = \frac{\overline{\hat{\lambda}}}{\mu^*} - \overline{\hat{u}(s^*)} \tag{11}$$

and

$$\textstyle\int_0^1 |A^{1/2}\hat{u}|^2 + i\beta \int_0^1 |\hat{u}|^2 + \hat{\lambda} \int_0^1 \psi_1 \, \overline{\hat{u}} = 0 \,. \tag{12}$$

Compare with (11) and (12) and, then substitute (10), then we get

$$2i\beta \int_0^1 |A^{1/2}\hat{u}|^2 - 2\beta^2 \int_0^1 |\hat{u}|^2 - i\beta \frac{\hat{\lambda}}{(\mu^*)^2} = 0.$$

Hence, the transversality condition

$$\mathrm{Re}\hat{\lambda} = 2(\mu^*)^2 \int_0^1 |A^{1/2}\hat{u}|^2 > 0$$

is satisfied. ∎

We now show that, whenever (R) admits a stationary solution, there is a unique $\mu^* > 0$ such that $(0, s^*, \mu^*)$ is a Hopf point, thus μ^* is the origin of a branch of nontrivial periodic orbits.

Theorem 3.3 *There exists a unique, purely imaginary eigenvalue $\lambda = i\beta$ of (4) with $\beta > 0$ for a unique critical point $\mu^* > 0$ in order for $(0, s^*, \mu^*)$ to be a Hopf point only for $\frac{1}{3} < s^* < 1$.*

Proof. To do this, we only have to show that the function $(u, \beta, \mu) \mapsto E(u, i\beta, \mu)$ has a unique zero with $\beta > 0$ and $\mu > 0$. This means solving the system

$$\begin{aligned} (A + i\beta)u &= \mu \cdot \Big(\gamma'(s^*) + u(s^*)\Big) \cdot G(\cdot, s^*) \\ i\beta &= \mu \cdot \Big(\gamma'(s^*) + u(s^*)\Big) . \end{aligned}$$

As before, with $v := u - G(\cdot, s^*)$, this system is equivalent to the weak system of equations

$$\begin{aligned} (A + i\beta)w &= -\delta_{s^*} \\ i\beta &= \mu \cdot \Big(\gamma'(s^*) + G(s^*, s^*) + v(s^*)\Big) . \end{aligned} \tag{13}$$

Now the first equation in (13) has, for fixed $\beta \geq 0$, the unique solution $v = -G_\beta(\cdot, s^*)$, where G_β is the Green's function for the operator $A + i\beta$. We are thus left with having to solve the complex valued equation

$$i\beta = \mu \cdot (\gamma'(s^*) + G(s^*, s^*) - G_\beta(s^*, s^*))$$

which is equivalent to the real valued system

$$\gamma'(s^*) + G(s^*, s^*) - \operatorname{Re} G_\beta(s^*, s^*) = 0 \tag{14}$$

$$\mu \operatorname{Im} G_\beta(s^*, s^*) + \beta = 0\,. \tag{15}$$

By the Lemma 12 in [9], the expression $\operatorname{Re} G_\beta(s^*, s^*)$ is strictly decreasing in $\beta \in \mathbf{R}^+$, $\operatorname{Re} G_0(s^*, s^*) = G(s^*, s^*)$ and $\lim_{\beta\to\infty} \operatorname{Re} G_\beta(s^*, s^*) = 0$ and, $\operatorname{Im} G_\beta(s^*, s^*) < 0$ for any $\beta > 0$. Since $\gamma'(s^*) + G(s^*, s^*)$ is real valued, there is a unique solution (β, μ^*) of (14) and (15) with $\beta > 0$ and $\mu^* > 0$. ∎

The following theorem summarizes what we have proved for the free boundary problem with the Dirichlet boundary condition:

Theorem 3.4 *Assume that* $0 < \frac{1}{2} - a < \dfrac{2\sinh^3 \frac{c}{3}}{c \sinh c}$, *so that* (R), *respectively* (F), *has a unique stationary solution* (u^*, s^*) *where* $u^* = 0$ *and* $s^* \in (0, \frac{1}{3}) \cup (\frac{1}{3}, 1)$ *respectively* (v^*, s^*), *for all* $\mu > 0$. *Then there exists a unique* $\mu^* > 0$ *such that the linearization* $-\widetilde{A} + \mu^* B$ *has a purely imaginary pair of eigenvalues. The point* $(0, s^*, \mu^*)$ *with* $0 < s^* < \frac{1}{3}$ *is then a Hopf point for* (R) *and there exists a* C^0*-curve of nontrivial periodic orbits for* (R), (F), *respectively, bifurcating from* $(0, s^*, \mu^*)$, (v^*, s^*, μ^*), *respectively.*

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