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IRREDUCIBLE MAPPINGS AND HU-TERMINAL CONTINUA

Pavel Pyrih
Tomáš Bárta
David Opěla
Pavel Růžička
Robert Šámal

Abstract. We introduce the notion of an *irreducible* mapping between continua. We prove in the present paper that each atomic mapping is irreducible and each irreducible mapping is monotone. It is also shown that irreducible mappings between hereditarily unicoherent continua map an HU-terminal continuum of the domain onto an HU-terminal continuum of the range. Our examples demonstrate that neither monotone nor open mappings have this property in general. We show that the inverse image of an HU-terminal continuum of a nondegenerate range continuum under an atomic mapping is an HU-terminal continuum of the domain continuum. Some relevant counterexamples are indicated.

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1. Introduction and notation.

It is known that each atomic mapping between continua is monotone (see [8, 3.7, p. 13], [4, Theorem 1, p. 49]). It is natural to try to find some family of mappings which is somewhere between the families of all atomic and all monotone mappings. We introduce in the present paper the notion of an *irreducible* mapping between continua. In Section 2 we prove that each atomic mapping is irreducible, that

Department of Mathematical Analysis,Charles University, Sokolovská 83
CZ-186 75 Prague 8 Czech Republic
Research supported by the grant No. GAUK 186/96 of Charles University.
Email Address: pyrih@karlin.mff.cuni.cz, barta@kma14.karlin.mff.cuni.cz
opela@kma14.karlin.mff.cuni.cz, ruzicka@karlin.mff.cuni.cz samal@atrey.karlin.mff.cuni.cz

each irreducible mapping is monotone and that both these inclusions are proper (Corollary 2.9).

In Section 3 we will proceed with the study of HU-terminal continua (see [2, p. 380]). In his paper J. J. Charatonik posed a (still open) question ([2, Question 22, p. 382]):

1.1. Question. *What mappings among hereditarily unicoherent continua map HU-terminal continua of the domain onto HU-terminal continua of the range?*

Partially answering this question, we show that irreducible mappings between hereditarily unicoherent continua map HU-terminal continua of the domain onto HU-terminal continua of the range (Theorem 3.3). Thus we see that atomic mappings preserve the concept of an HU-terminal continuum, which is not in general preserved by monotone mappings (Corollary 3.5). We show that the inverse image of an HU-terminal continuum of a nondegenerate range continuum under an atomic mapping is an HU-terminal continuum of the domain continuum (Theorem 3.7).

In Section 4 we will look more closely at the open mappings. It is natural to ask if open mappings preserve HU-terminality. Our example demonstrates that open mappings do not in general preserve the concept of an HU-terminal continuum.

For the convenience of the reader we repeat the relevant material. All spaces considered in the paper are assumed to be metric and all mappings are continuous. A *continuum* means a compact connected space. Given a space X and its subset S, we denote by $\operatorname{cl} S$ the closure of S.

Let X be a continuum and $A \subset X$. Then X is said to be *irreducible about* A provided no proper subcontinuum of X contains A. A continuum X is said to be *irreducible* provided that X is irreducible about $\{p, q\}$ for some $p, q \in X$ (then X is said to be *irreducible between* p *and* q).

Let X be a continuum, and let A and B be nonempty compact subsets of X. We say that a subcontinuum C of X is *irreducible from* A *to* B, written $C = \operatorname{irr}(A, B)$, provided that $C \cap A \neq \emptyset$, $C \cap B \neq \emptyset$, and no proper subcontinuum of C intersects both A and B. If A or B is a singleton set, we omit the set notation; thus, e.g. we write $C = \operatorname{irr}(p, B)$, $C = \operatorname{irr}(p, q)$.

Let X and Y be continua. A mapping $f : X \to f(X) = Y$ is said to be:
– *atomic* provided that for each subcontinuum $K \subset X$ with nondegenerate image $f(K)$ the equality $f^{-1}(f(K)) = K$ holds;
– *irreducible* provided that for each nondegenerate irreducible subcontinuum $I \subset Y$ the inverse image $f^{-1}(I)$ is irreducible;
– *monotone* provided that for each subcontinuum Q of Y the inverse image $f^{-1}(Q)$ is connected;
– *interior at a point* $p \in X$ provided that for each open neighborhood U of p in X the point $f(p)$ is an interior point of the image $f(U)$ in Y;
– *open* provided that for each open subset U of X its image $f(U)$ is an open subset of Y.

Obviously a mapping is open if and only if it is interior at each point of its domain. It is known that each atomic mapping between continua is monotone (see [8, 3.7, p. 13], [4, Theorem 1, p. 49]).

Given continua X and Y, we will denote by

$$\mathsf{Atomic},\quad \mathsf{Irreducible},\quad \mathsf{Monotone}\quad \text{and}\quad \mathsf{Open}$$

the family of all atomic, irreducible, monotone and open mappings $f : X \to f(X) = Y$, respectively.

Let $\mathcal{M}$ be a class of mappings between continua. A mapping $f : X \to Y$ is said to be *hereditarily* $\mathcal{M}$ provided that for every subcontinuum $Z \subset X$ the partial mapping $f \restriction Z : Z \to f(Z) \subset Y$ is in $\mathcal{M}$.

In the next example we construct a mapping between continua which is irreducible but not hereditarily irreducible.

1.2. Example. Let $\varphi : \mathbb{R} \to \mathbb{R}^2$ be defined by $\varphi(t) = (\cos(t), \sin(t))$. Put $X = \{\varphi(t) \in \mathbb{R}^2 : t \in \mathbb{R}\}$. The function $f : X \to f(X) = X$ is defined by
(i) $f(\varphi(t)) = \varphi(2t)$ for $t \in [0, \pi]$;
(ii)$f(\varphi(t)) = (1, 0)$ for $t \in (\pi, 2\pi)$.
Then we observe

Claim 1. f is irreducible.

Proof. Indeed, any nondegenerate irreducible subcontinuum C of X is a proper subcontinuum and is homeomorphic to the closed unit interval. Then $f^{-1}(C)$ is again a proper subcontinuum of X due to the construction of f. It follows that $f^{-1}(C)$ is homeomorphic to the closed unit interval. Therefore $f^{-1}(C)$ is irreducible and Claim 1 is proved.

Claim 2. f is not hereditarily irreducible.

Proof. Fix a nondegenerate subcontinuum $C = \varphi([0, \pi]) \subsetneq X$ and put $g = f \restriction C$. Then $g(C) = X$ and $g : C \to X$ is not irreducible, for $g^{-1}(\varphi([0, 1]))$ is not connected (equals $(-1, 0) \cup \varphi([0, 2])$) in C. Claim 2 is proved. □

2. Irreducible mappings.

2.1. Proposition. *Let $f : X \to Y$ be an atomic mapping. Let $K \subset Y$ be a nondegenerate continuum irreducible about a set $L \subset K$. If $f(P) = L$ for some $P \subset X$, then $f^{-1}(K)$ is irreducible about P.*

Proof. Let $f : X \to Y$ be an atomic mapping. Let $K \subset Y$ be a nondegenerate continuum irreducible about a set $L \subset K$. Fix some set $P \subset X$ such that $f(P) = L$. We show that $f^{-1}(K)$ is irreducible about P.

Clearly L (and P as well) is not a singleton. It is known that every atomic mapping is monotone (see [8, 3.7, p. 13], [4, Theorem 1, p. 49]), hence $M = f^{-1}(K)$ is a continuum in X (we have a nondegenerate continuum K). Clearly $P \subset M$ because $L = f(P) \subset K$. Take some continuum N such that $P \subset N \subset M$.

We conclude that $f(N) = K$. (Indeed, obviously $f(N)$ is a continuum containing L, because $L = f(P)$. Hence $K \subset f(N)$, since K is irreducible about L. Conversely, $N \subset M$ implies $f(N) \subset f(M) = K$.)

Since f is atomic and $f(N) = K$ is a nondegenerate continuum we obtain that $N = f^{-1}(f(N)) = f^{-1}(K) = M$ (we know that N is nondegenerate since $P \subset N$, f is atomic). Then $M = f^{-1}(K)$ is irreducible about P and the proof is complete. □

2.2. Theorem. *Every atomic mapping $f : X \to f(X) = Y$ from a continuum X onto a continuum Y is irreducible.*

Proof. Let $f : X \to f(X) = Y$ from a continuum X onto a continuum Y be an atomic mapping. Consider a nondegenerate subcontinuum $K \subset Y$ which is irreducible about $\{p, q\}$. Then $f^{-1}(K)$ is irreducible about any two points a, b in X such that $f(a) = p$ and $f(b) = q$ due to Proposition 2.1. Hence f is irreducible. □

We show that there exists an irreducible mapping which is neither open nor atomic in the following example.

2.3. Example. Let $X = [0, 2]$, $Y = [0, 1]$. The mapping $f : X \to f(X) = Y$ is defined for $x \in X$ by $f(x) = \min(x, 1)$. From this we have $f((1, 2)) = \{1\}$. Hence f is not open. We see that f is not atomic because $[1/2, 2] = f^{-1}f([1/2, 3/2]) \neq [1/2, 3/2]$. Since the inverse images of intervals in $[0, 1]$ are intervals in $[0, 2]$, it follows that f is an irreducible mapping, which is neither open nor atomic. □

We can formulate the previous results in

2.4. Corollary. Atomic$\subsetneq$ Irreducible.

Proof. See Theorem 2.2 and Example 3.2. □

We recall that a continuum X is said to be a *simple triod* if X is the union of three straight line segments pa, pb and pc any two of which have the point p in common only. Obviously, a simple triod is homeomorphic to the letter T).

We say that a topological space S is a *spiral* if it is homeomorphic to the real half line $[0, +\infty)$.

A continuous mapping $f : X \to f(X) = Y$ is called *strongly irreducible* if $f^{-1}(I)$ is irreducible in X for every irreducible $I \subset Y$ (i.e. singletons in Y are allowed). The following example shows that an atomic mapping need not to be strongly irreducible.

2.5. Example. Let X be the disjoint union of a simple triod T and a spiral S approximating T. The continuum X is illustrated in Figure 1.

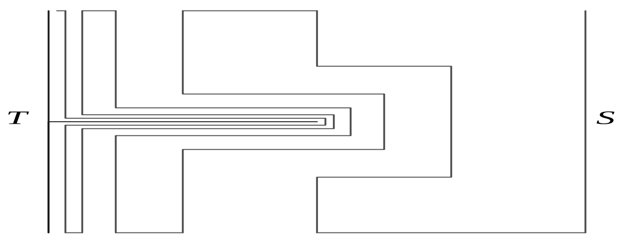

FIGURE 1 (EXAMPLE 2.5.)

Let $g : S \to [1, \infty)$ be a homeomorphism. Define $f : X \to [0, 1]$ by requiring $f(x) = 0$ for $x \in T$ and $f(x) = 1/g(x)$ for $x \in S$. Then f is not strongly irreducible, since $f^{-1}(0)$ is the simple triod T which is not an irreducible continuum. But f is atomic: Indeed, let C be a subcontinuum of X such that $f(C)$ is not a singleton. Clearly $C = S'$ or $C = S' \cup T$, for some $S' \subset S$. Since f is injective on S, $f^{-1}(f(S')) = S'$ and since $f^{-1}(f(T)) = T$, we have $f^{-1}(f(C)) = C$. We have shown that f is atomic.

The mapping f is an atomic mapping which is not strongly irreducible. □

Recall that a sequence of subsets of a metric space is called a *null-sequence* if the diameters of its members tend to zero.

For the convenience of the reader we include a useful statement (see [9, Corollary 5.5, p. 74]) which will be used in the following theorem.

2.6. Statement. *Let X be a nondegenerate continuum. Then, X contains a nondegenerate proper subcontinuum. Furthermore: If A is a proper subcontinuum of X and U is an open subset of X such that $A \subset U$, then there is a subcontinuum B of X such that*

$$A \subset B \neq A \quad \text{and} \quad B \subset U.$$

2.7. Theorem. Irreducible $\subset$ Monotone.

Proof. Let $f : X \to f(X) = Y$ be an irreducible mapping. We prove that f is monotone. We first prove a reduced form of the definition :

Claim. For each nondegenerate subcontinuum Q of Y the inverse image $f^{-1}(Q)$ is connected.

Proof. To deduce this, take a nondegenerate subcontinuum Q of Y. Suppose, contrary to this, that $f^{-1}(Q)$ is not connected.

Fix a, b in different components of $f^{-1}(Q)$ such that $f(a) \neq f(b)$. To fix such two points we proceed as follows :

(i) if for each component C of $f^{-1}(Q)$ the image $f(C)$ is a singleton in Q, then we have at least two disjoint components C_a and C_b such that $f(C_a) \neq f(C_b)$; then we choose $a \in C_a$ and $b \in C_b$.

(ii) if for some component C_a of $f^{-1}(Q)$ the image $f(C_a)$ is not a singleton in Q, then we find a component C_b of $f^{-1}(Q)$ disjoint with C_a; we choose any $b \in C_b$ and find $a \in C_a$ such that $f(a) \neq f(b)$.

Then $I = \mathrm{irr}(f(a), f(b))$ is a nondegenerate irreducible subcontinuum of Y contained in Q. Hence $f^{-1}(I)$ is an irreducible continuum of X contained in $f^{-1}(Q)$. Consequently, a and b are in the same component of $f^{-1}(Q)$, contrary to the choice of a, b. This contradicts our assumption. Claim is proved.

We are now in a position to show that f is monotone. Let $Q = \{p\}$ be a degenerate subcontinuum of Y. We show that the inverse image $f^{-1}(Q)$ is connected.

We first construct in Y a null sequence of nondegenerate subcontinua $K_1 \supset K_2 \supset \cdots \supset K_n \cdots \ni p$. Set $K_1 = Y$. We now proceed by induction. We construct K_{n+1} as a nondegenerate proper subcontinuum of K_n of diameter less then $1/n$ containing p (see Statement 2.6).

Then $f^{-1}(K_n)$ is a monotone sequence of nondegenerate subcontinua of X according to Claim. Hence $A = \cap f^{-1}(K_n)$ is a subcontinuum of X as well due to [9, Theorem 1.8, p. 6]. We observe that $A = f^{-1}(Q)$. Hence $f^{-1}(Q)$ is connected. □

The following example was presented by J. J. Charatonik in [2, p. 381].

2.8. Example. Let a continuum X be a simple triod, i.e. the union of three straight line segments pa, pb and pc any two of which have the point p in common only. The mapping $f : X \to f(X) \subset X$ is defined by requiring f to be the identity on $pa \cup pb$, $f(pc) = \{p\}$.

Clearly f is monotone, but f is not irreducible, since its range $f(T)$ is a space homeomorphic to the closed unit interval, which is irreducible, but its domain (the simple triod T) is not. $\square$

We have proved the following

2.9. Corollary. Atomic $\subsetneq$ Irreducible $\subsetneq$ Monotone.

Proof. See Corollary 2.4, Theorem 2.7 and Example 2.8. $\square$

3. HU-terminality.

A continuum X is said to be *hereditarily unicoherent* provided that the intersection of any two its subcontinua is connected.

J. J. Charatonik in [2, p. 380-381] compared the concept of a terminal continuum (see [1, Definition 4.1. p. 34], [1, Example 1.2(a), p. 7], [2, p. 377], [5] and Nadler's book [9, 1.54, p. 107]) with the concept of a terminal continuum as defined by G. R. Gordh in [6]. Since G. R. Gordh restricts his considerations to subcontinua of hereditarily unicoherent continua only, J. J. Charatonik renamed the concept of G. R. Gordh as HU-terminal (see [6, p. 458], [2, p. 380-381]).

A subcontinuum K of a hereditarily unicoherent continuum X is said to be an HU-*terminal continuum* of X provided that K is contained in an irreducible subcontinuum of X and for every irreducible subcontinuum I of X containing K there is a point $x \in X$ such that I is irreducible about the union $K \cup \{x\}$ (see [6, p. 458], [2, p. 380-381]).

Given hereditarily unicoherent continua X and Y, we will denote by

$$\mathsf{HU\text{-}terminal}$$

the family of all mappings $f : X \to f(X) = Y$ which map each HU-terminal continuum of X onto an HU-terminal continuum of Y.

3.1. Proposition. *Let $f : X \to f(X) = Y$ be a monotone mapping. Let J be a subcontinuum of Y and $P \subset J$. Let $f^{-1}(J)$ be a continuum irreducible about Q and $f(Q) = P$. Then J is irreducible about P.*

Proof. Let $f : X \to f(X) = Y$ be a monotone mapping. Let J be a subcontinuum of Y and $P \subset J$. Let $f^{-1}(J)$ be a continuum irreducible about Q and $f(Q) = P$. We show that J is irreducible about P.

To prove this let $I \subset J$ be a continuum containing P. Then $f^{-1}(I)$ is a subcontinuum of $f^{-1}(J)$ (inverse images under a monotone mapping) containing Q, which implies $f^{-1}(I) = f^{-1}(J)$, since $f^{-1}(J)$ be a continuum irreducible about Q. But then $I = J$ and J is irreducible about P. $\square$

3.2 Proposition. *Let $f : X \to f(X) = Y$ be an irreducible mapping. If I is a subcontinuum of X irreducible about $\{p, q\}$, then $f(I)$ is irreducible about $\{f(p), f(q)\}$.*

Proof. Let $f : X \to f(X) = Y$ be an irreducible mapping. Let I be a subcontinuum of X irreducible about $\{p, q\}$. We prove that $f(I)$ is irreducible about $\{f(p), f(q)\}$.

The case $p = q$ is simple. We may assume that $p \neq q$. Take any continuum $J \subset f(I)$ containing $f(p), f(q)$. Then $f^{-1}(J)$ is a continuum in X (irreducible mappings are monotone due to Theorem 2.7) containing $\{p, q\}$. Hence $I \subset f^{-1}(J)$. But the inclusion cannot be proper since $J \subset f(I)$. We conclude that $f^{-1}(J) = I$, which implies $J = f(I)$. We have showed that $f(I)$ is irreducible about $\{f(p), f(q)\}$. □

3.3. Theorem. *Let X and Y be hereditarily unicoherent continua. Let $f : X \to f(X) = Y$ be irreducible. If $K \subset X$ is an HU-terminal continuum of X, then $f(K) \subset Y$ is an HU-terminal continuum of Y.*

Proof. Let X and Y be hereditarily unicoherent continua. Let $f : X \to f(X) = Y$ be irreducible. Let $K \subset X$ be an HU-terminal continuum of X. We show that $f(K) \subset Y$ is an HU-terminal continuum of Y.

(i) We know that K is contained in some irreducible continuum I of X. Hence $f(K)$ is contained in the irreducible continuum $f(I)$ due to Proposition 3.2.

(ii) If $f(K)$ is contained in a nondegenerate irreducible continuum J of Y, then $f^{-1}(J)$ is a nondegenerate irreducible subcontinuum of X (the mapping f is irreducible) and contains K. Since K is an HU-terminal continuum of X, the inverse image $f^{-1}(J)$ is irreducible about $K \cup \{x\}$ for some $x \in f^{-1}(J)$. So by Proposition 3.1 the continuum J is irreducible about $f(K) \cup \{f(x)\}$ (the mapping f is irreducible, hence monotone due to Theorem 2.7).

If $f(K)$ is contained in a degenerate irreducible continuum J of Y, then J is irreducible about $f(K) \cup \{f(x)\}$ for some $x \in K$.

We have showed that $f(K) \subset Y$ is an HU-terminal continuum of Y. □

The following example establishes a relation between the classes $\mathsf{Monotone}$ and $\mathsf{HU\text{-}terminal}$.

3.4. Example. Let a continuum X be the union of three straight line segments pa, pb and pc any two of which have the point p in common only, where $a = (-1, 0)$, $b = (1, 0)$, $c = (0, 1)$ and $p = (0, 0)$. The continuum X is a simple triod. Let $Y = pa \cup pb$.

The mapping $g : X \to g(X) = Y \subset X$ is defined by $g((x, y)) = (x + y, 0)$ (i.e. g is the identity on $pa \cup pb$, linear on pc, $g(c) = b$ and g is sending pc to pb.)

The mapping g is not monotone, because the inverse image of b is not connected (equals $\{b, c\}$). Note that a subcontinuum K of Y is an HU-terminal continuum of Y if and only if $K \cap \{a, b\} \neq \emptyset$. Note also that each HU-terminal continuum of X contains at least one point of $\{a, b, c\}$. Since the mapping g maps $\{a, b, c\}$ onto $\{a, b\}$ we see that g belongs to the class $\mathsf{HU\text{-}terminal}$.

J. J. Charatonik in [2, p. 381] noticed that the mapping f defined in Example 2.8 (defined for $x \in X$ by $f((x, y)) = (x, 0)$, i.e. f is the identity on $pa \cup pb$, $f(pc) = \{p\}$) is monotone and that f maps the HU-terminal continuum $\{c\}$ of X to p, which is not an HU-terminal continuum of Y.

Consequently, $f \in \mathsf{Monotone} \setminus \mathsf{HU\text{-}terminal}$ and $g \in \mathsf{HU\text{-}terminal} \setminus \mathsf{Monotone}$. □

We can summarize the properties of HU-terminal in

3.5. Corollary. *Let X and Y be hereditarily unicoherent continua. Then*

(i) Irreducible$\subsetneq$ HU-terminal;
(ii) Atomic$\subsetneq$ HU-terminal;
(iii) HU-terminal$\not\subset$ Monotone;
(iv) Monotone$\not\subset$ HU-terminal.

Proof. See Theorem 2.2, Theorem 3.3 and Example 3.4. $\square$

For irreducible mappings $f : X \to f(X) = Y$, the inverse image of an HU-terminal continuum of Y is not necessarily an HU-terminal continuum of X. We construct an example with this property.

3.6 Example. Let $a = (0,1)$, $a' = (0,-1)$, $a'' = (-1,-1)$, $A = aa'$, $L_A = a'a''$; $b = (1,1)$, $b' = (1,-1)$, $b'' = (2,-1)$, $B = bb'$, $L_B = b'b''$ and

$$S = \left\{ (x,y) : \quad 0 < x < 1, \quad y = \sin \frac{1}{x(1-x)} \right\}.$$

Put $X = A \cup L_A \cup B \cup L_B \cup S$. The continuum X is illustrated in Figure 2.

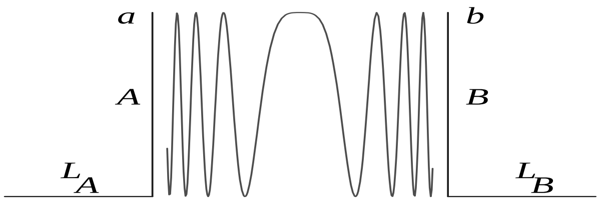

FIGURE 2 (EXAMPLE 3.6)

Consider $f : X \to f(X) = Y$, which maps L_A to a', L_B to b' and is an identity elsewhere. Clearly a is an HU-terminal continuum of $f(X) = A \cup S \cup B$, but $a = f^{-1}(a)$ is not an HU-terminal continuum of X. Moreover, f is clearly irreducible. $\square$

For irreducible mappings, the inverse image of $\mathrm{irr}(a,b)$ is not necessarily irreducible about $f^{-1}(a) \cup f^{-1}(b)$. Indeed, consider X, f, a and b from the previous example. Then $f(X) = \mathrm{irr}(a,b)$, but X is not irreducible about $f^{-1}(a) \cup f^{-1}(b)$.

As we stated in Proposition 2.1, atomic mappings have the mentioned property. That allows us to prove that the inverse image under an atomic mapping of an HU-terminal continuum is an HU-terminal continuum.

3.7. Theorem. *Let $f : X \to f(X) = Y$ be an atomic mapping between two nondegenerate hereditarily unicoherent continua X and Y. Let $K \subset Y$ be an HU-terminal continuum of Y. Then $f^{-1}(K)$ is an HU-terminal continuum of X.*

Proof. Let $f : X \to f(X) = Y$ be an atomic mapping between two nondegenerate hereditarily unicoherent continua X and Y. Let $K \subset Y$ be an HU-terminal continuum of Y. We prove that $f^{-1}(K)$ is an HU-terminal continuum of X.

(i) Since K is an HU-terminal continuum of a nondegenerate continuum Y, K is contained in some nondegenerate irreducible subcontinuum L of Y. Then $f^{-1}(K)$ is a continuum, because atomic mappings are monotone (see [8, 3.7, p. 13], [4, Theorem 1, p. 49]). Clearly $f^{-1}(K)$ is contained in $f^{-1}(L)$, which is irreducible (atomic mappings are irreducible - see Theorem 2.2).

(ii) Now let the continuum $f^{-1}(K)$ is contained in some subcontinuum M of X which is irreducible about $\{p, q\}$. Then $f(M)$ is irreducible about $\{f(p), f(q)\}$ by Proposition 3.2. The continuum K is an HU-terminal continuum of Y, $K \subset f(M)$, therefore there is a point $a \in Y$, such that $f(M)$ is irreducible about $K \cup \{a\}$.

If $f(M)$ is a singleton, then $K = f(M)$ and $f^{-1}(K) = M$. Hence $f^{-1}(K)$ is irreducible about $M \cup \{b\}$ for any $b \in M$.

If $f(M)$ is not a singleton, by Proposition 2.1 we see that $f^{-1}(f(M)) = M$ is irreducible about $f^{-1}(K) \cup \{b\}$, where b is such that $f(b) = a$.

Therefore we have proved that $f^{-1}(K)$ is an HU-terminal continuum of X. □

The atomic mapping sending a simple triod to a singleton shows that Theorem 3.7 does not hold if Y is a degenerate continuum, because a simple triod is not irreducible.

3.8. Corollary. *Let $f : X \to f(X) = Y$ be an atomic mapping between two nondegenerate hereditarily unicoherent continua X and Y. Then $K \subset X$ is an HU-terminal continuum of X if and only if $f(K)$ is an HU-terminal continuum of Y.*

Proof. See Corollary 3.5 and Theorem 3.7. □

Notice that the mapping f in Example 2.3 is not atomic, but $K \subset X$ is an HU-terminal continuum of X if and only if $f(K)$ is an HU-terminal continuum of Y.

4. Open mappings.

We have seen in Example 2.3 that there exists an irreducible mapping which is not open. In the next example we show how to find an open mapping which is not in HU-terminal. This construction is adapted from [3, Example 7.1, p. 29].

4.1. Example. Let $I = \{(0, y) \in \mathbb{R} : y \in [-1, 1]\}$ and

$$T = \{(0, y) \in \mathbb{R} : y \in [-1/3, 1/3]\} \cup \{(x, 0) \in \mathbb{R} : x \in [0, 1]\} \quad .$$

Set $v = (0, 0)$, $s = (0, -1)$, $t = (0, 1)$ and for each $n \in \mathbb{N}$ put $a_n = (1/n, -1/5n)$, $b_n = (1 + 1/n, -1/5n)$, $c_n = (1 + 1/n, 1/5n)$, $d_n = (1/n, 1/5n)$, $e_n = (1/n, 1)$, and finally

$$L_n = sa_n \cup a_n b_n \cup b_n c_n \cup c_n d_n \cup d_n e_n \quad .$$

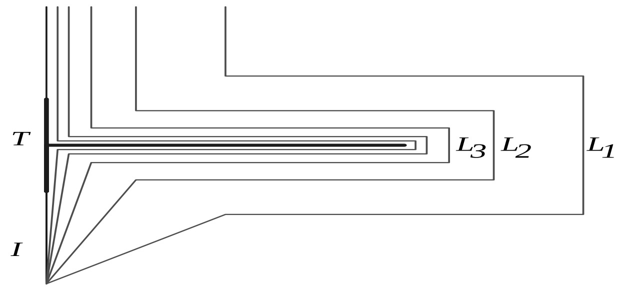

FIGURE 3 (EXAMPLE 4.1)

We will denote by X the space

$$I \cup T \cup \bigcup_{n \in \mathbb{N}} L_n \quad .$$

The continuum X is illustrated in Figure 3.

In the sequel, p_ε stands for the line $\{(x, \varepsilon) : x \in \mathbb{R}\}$ and P stands for the strip $\{(x, y) : x \in \mathbb{R}, y \in (-1/3, 1/3)\}$. We find a sequence $\{r_n\} \subset (X \cap P)$ such that $\mathrm{cl}\,(\{r_n\}) = T$ and $r_n \in L_n$ for $n \in \mathbb{N}$. This is easy. It is sufficient to use the separability of T.

Let us denote by $f : X \to I$ the mapping satisfying

(i) $f(T) = \{v\}$, $f(s) = \{s\}$;
(ii) $f(X \cap p_1) = \{t\}$;
(iii) f is injective on $I \setminus T$;

and for each $n \in \mathbb{N}$

(iv) f is injective on L_n;
(v) $f(L_n \cap p_{1/3}) = (0, 1/3n)$;
(vi) $f(L_n \cap p_{-1/3}) = (0, -1/3n)$;
(vii) $f(r_n) = v$.

CLAIM 1. $f \in \mathsf{Open}$.

Proof. We prove this as follows :

- f is interior at each point of $L_n \setminus I$ due to (iv), $n \in \mathbb{N}$;
- f is interior at each point of $I \setminus T$ due to (i) - (iii);
- f is interior at each point of T due to the construction of $\{r_n\}$. (Indeed, each open neighborhood U of each point $x \in T$ contains some r_n. Since f is open at $L_n \setminus I$, we see that $f(U)$ is open.)

CLAIM 2. $f \notin \mathsf{HU\text{-}terminal}$.

Proof. It is a simple matter to check that the point $r = (1, 0)$ is an HU-terminal continuum of X. We examine the irreducible subcontinua of X containing r. It is immediate that any such subcontinuum is irreducible about $\{r, q\}$ for some $q \in X$.

We now observe that $f(X) = I$. But $p = f(r)$ is not an HU-terminal continuum of I since I is irreducible about $\{s, t\}$ and for any $x \in I$ the continuum I is not irreducible about any $\{p, x\}$.

CLAIM 3. $f \notin \mathsf{Irreducible}$.

Proof. It is a simple matter to check that the inverse image of $I \cap T$ is not connected in X. $\square$

We may summarize the previous by the following corollary.

4.2. Corollary. *Let X and Y be hereditarily unicoherent continua. Then*

(i) $\mathsf{HU\text{-}terminal} \not\subset \mathsf{Open}$;
(ii) $\mathsf{Open} \not\subset \mathsf{HU\text{-}terminal}$;
(iii) $\mathsf{Irreducible} \not\subset \mathsf{Open}$;
(iv) $\mathsf{Open} \not\subset \mathsf{Irreducible}$.

Proof. See Example 2.3, Theorem 3.3 and Example 4.1. $\square$

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