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Solutions to Stationary Phase-Field Equations

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Abstract

In this paper we consider time-independent versions of the Phase-Field Equations introduced by O. Penrose and P. Fife. To this end we will investigate a system of semi-linear elliptic equations. We will show existence of solutions and give results on the multiplicity and the stability of solutions. This will be achieved by applying variational methods and bifurcation methods. We will also point out the physical significance of these results.

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1 Introduction

In recent years several authors investigated systems of non-linear parabolic equations known as phase field equations. These equations arise in studying temperature driven phase changes (cf., for example, [1, 3, 4, 9, 12, 16]) and usually consist of a non-linear parabolic equation of the form

$$\phi_t = \Delta\phi - s_\phi(x,\phi) + \lambda(x,\phi)u \tag{1}$$

for the phase parameter ϕ coupled with a linear or non-linear heat equation for the temperature u. The coupling term is of the form

$$\lambda(x,\phi)\phi_t, \tag{2}$$

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and represents the latent heat of the phase transition. The existence and uniqueness of solutions to such systems has been studied extensively (cf. [2, 4, 7, 8, 9, 11, 14, 16]), but to the best of our knowledge the steady state solutions to these equations have not been thoroughly investigated. In this paper we will consider stationary solutions to the system

$$\phi_t = \Delta\phi - s_\phi(x,\phi) + \lambda(x,\phi)u \tag{3}$$

$$\frac{u_t}{u^2} = \Delta u - \lambda(x,\phi)\phi_t \tag{4}$$

on a bounded domain $\Omega \subset \mathbf{R}^n$ with a sufficiently smooth boundary $\partial\Omega$. As in the papers on the time-dependent equations, we will restrict ourselves to the case that $n \leq 3$. However, most of the analysis can be carried out in higher dimensions as well.

We will use the same boundary conditions as in previous papers, namely

$$\frac{\partial\phi}{\partial\nu} = 0, \tag{5}$$

$$\frac{\partial u}{\partial\nu} = \left(u - u^2\theta_\Gamma\right), \tag{6}$$

however, one could also consider a linearized version of (6). In addition to these conditions we need to make the following assumptions:

- **(H 1)** The function $s(x,\cdot) \in C^3(A) \cap C(\overline{A})$ for an open interval $A \subseteq \mathbf{R}$, independent of $x \in \Omega$.
- **(H 2)** $s(x,\cdot)$ is a double-well potential for every $x \in \Omega$.
- **(H 3)** There exist positive constants α and β such that $\alpha \leq \theta_\Gamma \leq \beta$. Furthermore, $\theta_\Gamma \in H^1(\partial\Omega)$.
- **(H 4)** $\lambda(x,\phi) = a(x)\phi + b(x)$ with $a(x) \geq 0$ and $b(x) > 0$ smooth functions.

These are sufficient conditions for the existence of solutions to (3)–(6) with sufficiently smooth initial conditions. For the purpose of this paper the condition **(H 4)** could be eased to $\lambda_\phi > 0$ without significantly changing the arguments. For further analysis of the time-dependent problems we refer to the above mentioned papers as well as [6] for an overview of the problems. We would also like to mention [5] for a proof that such a model approximates a Stefan problem.

The function $-s_\phi(x,\phi)$ has three zeros (as a function of ϕ). Hence, the function $-s_\phi(x,\phi) + \lambda(x,\phi)u(x)$ will have at least one and at most three zeros (for fixed x). Zeros at which the derivative $-s_{\phi\phi}(x,\phi) + \lambda_\phi(x,\phi)u(x)$ is negative represent a single phase, zeros with positive derivative (with respect to ϕ) represent a point at which both phases are present.

In this paper we are interested in the time-independent stationary problem. In particular, we will investigate the system of non-linear elliptic equations

$$-\Delta\phi = -s_\phi(x,\phi) + \lambda(x,\phi)u, \quad (7)$$
$$-\Delta u = 0, \quad (8)$$

together with the above mentioned boundary conditions (5)– (6). This system is essentially decoupled, and (8) is just a homogeneous Laplace equation with non-linear boundary conditions. Aside from stating a maximum principle for u, we will concentrate on the study of solutions to (7) for a given function u satisfying (8).

In section 2 of the paper we will use variational techniques to prove a general existence result for (7) with arbitrary smooth $u(x)$. This is very similar to a result proved in [4] for a specific choice of the function s.

In section 3 we will give a more specific result for functions u which are close to a constant. This will also show that solutions to (7) are in general not unique. The results of this section rely on techniques from general bifurcation theory. The physical significance of these results will be summarized at the end of the section.

In section 4 we will analyze the linear stability of solutions to the time-independent system. In this section we will treat the entire system including the nonlinear heat equation. We will give a sufficient condition for stability and give an example of an unstable solution which consist entirely of "mushy regions", i. e. a solution for which at every point in the domain both phases exist.

Throughout the paper we will extensively use the Hilbert spaces $X = L^2(\Omega)$ and $Y = H^2(\Omega)$.

2 Variational Approach — Existence of Solutions

The main result of this section is the generalization of a result that first appeared in [4]. We will use a variational approach to prove the general existence of a solution to (7). To begin we have to state a maximum principle for solutions of (8).

Lemma 2.1 *Let $u(x)$ be a smooth solution of (8) with boundary conditions (6). Then $u(x)$ takes its extrema on the boundary of Ω and*

$$\frac{1}{\beta} \le u(x) \le \frac{1}{\alpha}$$

for all $x \in \overline{\Omega}$.

We omit the simple proof of this lemma and continue with the investigation of (7). We will assume that $u(x)$ is a solution to (8) with boundary conditions (6).

We start by considering the equation

$$-s_\phi(x,\phi) + u(x)\lambda(x,\phi) = 0, \tag{9}$$

which can be solved for ϕ at any point $x \in \Omega$. The equation will have at least one and at most three solutions, which we denote by

$$\phi_1(x) \leq \phi_2(x) \leq \phi_3(x).$$

The functions $\phi_j(x)$ will be continuous in x, But their domains Ω_j might not be the entire domain Ω. However, for every $x \in \Omega$ at least one of the three functions exists and will be continuous. To continue we define

$$\underline{\phi}(x) = \min\left\{\phi_1(x), \phi_2(x), \phi_3(x)\right\}, \tag{10}$$

where the minimum is taken over the functions ϕ_j which exist at the point x. Similarily define

$$\overline{\phi}(x) = \max\left\{\phi_1(x), \phi_2(x), \phi_3(x)\right\}. \tag{11}$$

We are now ready to state a maximum principle for the solutions of (7).

Lemma 2.2 *Define*

$$c_1 = \inf_{x\in\Omega} \underline{\phi}(x),$$

and

$$c_2 = \sup_{x\in\Omega} \overline{\phi}(x),$$

where $\underline{\phi}(x)$ and $\overline{\phi}(x)$ are defined as above. Then for any smooth solution $\phi(x)$ of (7) we have:

$$c_1 \leq \phi(x) \leq c_2. \tag{12}$$

Proof: First we observe that since $u(x)$ is a bounded continuous function, the same holds for $\phi_j(x)$, hence c_1 and c_2 are finite numbers.

Now suppose $\phi(x)$ is a smooth solution to (7), and that ϕ has a maximum at $x_0 \in \Omega$. We have:

$$0 \leq -\Delta\phi(x_0) = -s_\phi(x_0, \phi(x_0)) + u(x_0)\lambda(x_0, \phi(x_0)). \tag{13}$$

Since the right hand side of this inequality becomes negative if $\phi(x_0)$ is larger than the largest root of the right-hand side it follows that:

$$\phi(x) \leq \phi(x_0) \leq \max\left\{\phi_1(x_0), \phi_2(x_0), \phi_3(x_0)\right\} \leq c_2.$$

Since we have $\partial_\nu \phi|_{\partial\Omega} = 0$, we have that ϕ must be concave down near a maximum at $x_0 \in \partial\Omega$ as well. For sufficiently smooth boundary and ϕ we have

$$\lim_{x \to x_0} \Delta\phi(x) \leq 0,$$

and (13) holds on the boundary as well. Analogously we prove the statement for the lower bound of $\phi(x)$.

$\square$

To continue we formulate our equation as a variational problem. (7) is the Euler-Lagrange equation for the following variational problem. We want to find a function $\phi \in H^1(\Omega)$ for which the functional

$$\mathcal{F}(\phi) = \int_\Omega \left(\frac{1}{2} |\nabla\phi|^2 + s(x,\phi) - u(x)\Lambda(x,\phi) \right) dx, \tag{14}$$

where $\Lambda_\phi(x,\phi) = \lambda(x,\phi)$, has a critical point. The functional $\mathcal{F}(\phi)$ represents of course the free energy of the system (cf. [12, 4], for example).

Any solution $\phi \in H^1(\Omega)$ to this variational problem is also a solution of the original equation. The additional regularity required for ϕ to be a classical solution follows from elliptic regularity theory.

We proceed by proving, that there exists a solution to the variational problem.

Theorem 2.1 *Let $C = \inf_{\phi \in H^1(\Omega)} \mathcal{F}(\phi)$. Then $|C| < \infty$ and there exists a $\phi^* \in H^1(\Omega)$ such that*

$$C = \mathcal{F}(\phi^*),$$

i. e. ϕ^* is a critical point of $\mathcal{F}(\phi)$.

Proof: As stated in the previous lemma, we have that

$$s_\phi(x,\phi) - u(x)\lambda(x,\phi) > 0 \quad \text{for} \quad \phi > c_2,$$

and

$$s_\phi(x,\phi) - u(x)\lambda(x,\phi) < 0 \quad \text{for} \quad \phi < c_1.$$

It follows that the function $s(x,\phi) - u(x)\Lambda(x,\phi)$ is bounded below, i. e. there is a $C_0 > 0$ such that

$$s(x,\phi) - u(x)\Lambda(x,\phi) > -C_0.$$

This implies immediately that

$$\begin{aligned} \mathcal{F}(\phi) = \int_\Omega \left(\frac{1}{2} |\nabla\phi|^2 + s(x,\phi) - u(x)\Lambda(\phi) \right) dx &\geq \int_\Omega (s(x,\phi) - u(x)\Lambda(\phi))\, dx \\ &\geq -\mu(\Omega) C_0 \\ &= -C_1 > -\infty. \end{aligned}$$

Hence, $\inf_{\phi \in H^1(\Omega)} \mathcal{F}(\phi) = C \geq -C_1 > -\infty$. Therefore, there exists a minimizing sequence $\phi_n \in H^1(\Omega)$ such that

$$\mathcal{F}(\phi_n) \to C \quad \text{as} \quad n \to \infty.$$

Without loss of generality we may assume that $\mathcal{F}(\phi_n) \leq C + 1$. To continue, we replace this minimizing sequence with one that is uniformly bounded as follows.

$$\hat{\phi}_n = \begin{cases} c_1 & \text{for} \quad \phi_n < c_1 \\ \phi_n & \text{for} \quad c_1 \leq \phi_n \leq c_2 \\ c_2 & \text{for} \quad \phi_n > c_2 \end{cases},$$

where c_1 and c_2 are the bounds of Lemma 2.2. It follows that $\hat{\phi}_n \in H^1(\Omega)$. Furthermore, the definitions of c_1 and c_2 imply that $s(x,\phi) - u(x)\Lambda(x,\phi)$ is an increasing function of ϕ for $\phi \geq c_2$ and a decreasing function of ϕ for $\phi \leq c_1$ (see Lemma 2.2). Thus we have

$$\mathcal{F}(\hat{\phi}_n) \leq \mathcal{F}(\phi_n),$$

i. e. $\left\{\hat{\phi}_n\right\}_{n=1}^{\infty}$ is also a minimizing sequence. Furthermore, since

$$\int_\Omega \frac{1}{2} \left|\nabla \hat{\phi}_n\right|^2 \, dx = \mathcal{F}(\hat{\phi}_n) - \int_\Omega \left(s(x,\hat{\phi}_n) - u(x)\Lambda(x,\hat{\phi}_n)\right) \, dx \leq C + 1 + C_2$$

for an appropriate constant C_2, the sequence is uniformly bounded in $H^1(\Omega)$. Since $H^1(\Omega)$ is a Hilbert space there exists a weakly convergent subsequence of this sequence, which we also denote by $\left\{\hat{\phi}_n\right\}_{n=1}^{\infty}$. Let $\tilde{\phi}$ denote the weak limit of the subsequence. It follows, that

$$\int_\Omega \frac{1}{2} \left|\nabla \tilde{\phi}\right|^2 \, dx \leq \liminf_{n\to\infty} \int_\Omega \frac{1}{2} \left|\nabla \hat{\phi}_n\right|^2 \, dx. \tag{15}$$

Since $c_1 \leq \hat{\phi}_n \leq c_2$, the same holds for $\tilde{\phi}$. We can use the mean-value theorem to get

$$\begin{aligned} &\left|\int_\Omega \left(s(x,\hat{\phi}_n) - u(x)\Lambda(x,\hat{\phi}_n) - s(x,\tilde{\phi}) + u(x)\Lambda(x,\tilde{\phi})\right) \, dx\right| \\ &\qquad\qquad \leq \int_\Omega \sup_{c_1 \leq \xi \leq c_2} \left|s_\phi(x,\xi) - u(x)\lambda(\xi)\right| \left|\hat{\phi}_n - \tilde{\phi}\right| \, dx \\ &\qquad\qquad \leq C_3 \left\|\hat{\phi}_n - \hat{\phi}\right\|_{L^2(\Omega)}, \end{aligned}$$

for some constant C_3. But by the Sobolev imbedding theorem, $\hat{\phi}_n$ converges strongly to $\hat{\phi}$ in $L^2(\Omega)$, which in turn implies

$$\int_\Omega \left(s(x,\hat{\phi}_n) - u(x)\Lambda(x,\hat{\phi}_n)\right) \, dx \to \int_\Omega \left(s(x,\tilde{\phi}) - u(x)\Lambda(x,\tilde{\phi})\right) \, dx$$

as $n \to \infty$. Together with (15) this implies

$$\mathcal{F}(\tilde{\phi}) \leq \liminf_{n\to\infty} \mathcal{F}(\hat{\phi}_n) = C,$$

i. e. $\tilde{\phi}$ is a solution to the variational problem.

$\square$

Remark:

This section shows that for any function u, there is a solution ϕ to the elliptic equation (7). For (ϕ, u) to be a solution of the system, u must also be a harmonic function which satisfies the boundary condition (6). For suitable boundary conditions, the inverse temperature u can vary over a quite large range of positive values. This implies that in some regions of Ω the function $-s_\phi(x,\phi) + \lambda(x,\phi)u(x)$ has only the largest root, whereas in other regions the function has only the smallest root. This insures the existence of solutions for which both phases are present in different areas of Ω. Lemma 2.2 gives a simple a priori bound for smooth solutions to the equation.

3 Bifurcation Theory and Single Phase Solutions

In this section we study the solutions to (7) more closely. We will restrict ourselves to the generic case when the function s does not explicitely depend on x. Furthermore, we will assume that

$$\lambda(\phi) = a\phi + b,$$

for constants $a \geq 0$ and $b > 0$.

We will apply a result from generic bifurcation theory to show the multiplicity of specific solutions, which are nearly constant. We start by stating the obvious result for constant temperatures $u(x) = \overline{u}$.

Lemma 3.1 *Let $\hat{\phi}$ be a solution of the equation*

$$-s_\phi(\hat{\phi}) + \overline{u}\lambda(\hat{\phi}) = 0. \tag{16}$$

Then $\phi(x) \equiv \hat{\phi}$ is clearly a solution of (7).

The proof of this is obvious. This result also shows, that we cannot expect uniqueness for solutions of this equation, since the algebraic equation (16) may have up to three distinct solutions.

We continue by showing that there exist solutions for $u \in X = L^2(\Omega)$ provided that u is close enough to a constant $\overline{u}$. To do this we let Σ denote the spectrum of the negative Laplacian on Ω with Neumann boundary conditions. It is well known that Σ is a pure point spectrum consisting of countably many non-negative numbers. Furthermore, Σ is nowhere dense in $\mathbf{R}^+$. Using this one can state:

Theorem 3.1 *Let $\overline{u}$ be constant and $\hat{\phi}$ be one of the solutions to*

$$-s_\phi(\hat{\phi}) + \overline{u}\lambda(\hat{\phi}) = -s_\phi(\hat{\phi}) + \overline{u}\left(a\hat{\phi} + b\right) = 0.$$

Furthermore, assume that

$$\overline{u}a - s_{\phi\phi}(\hat{\phi}) \notin \Sigma.$$

Then there exist positive numbers ξ_0 and η_0, such that for every $u \in X$ with $\|u - \overline{u}\|_X < \eta_0$ there exists a unique $\phi \in Y = H^2(\Omega)$ with $\left\|\phi - \hat{\phi}\right\|_Y < \xi_0$ that solves (7).

Proof: We will apply a result from genral bifurcation theory, namely Theorem 8.C of [15]. Observe that

$$\begin{aligned} -\Delta\phi &= -s_\phi(\phi) + au\phi + bu \\ -\Delta\hat{\phi} &= -s_\phi(\hat{\phi}) + a\overline{u}\hat{\phi} + b\overline{u} \end{aligned}$$

implies

$$\begin{aligned} -\Delta\left(\phi - \hat{\phi}\right) &= -s_\phi(\phi) + s_\phi(\hat{\phi}) + au\left(\phi - \hat{\phi}\right) + a\hat{\phi}\,(u - \overline{u}) + b\,(u - \overline{u}) \\ &= -s_\phi(\phi) + s_\phi(\hat{\phi}) + s_{\phi\phi}(\hat{\phi})\left(\phi - \hat{\phi}\right) + a\,(u - \overline{u})\left(\phi - \hat{\phi}\right) \\ &\quad - \left(s_{\phi\phi}(\hat{\phi}) - a\overline{u}\right)\left(\phi - \hat{\phi}\right) + \left(a\hat{\phi} + b\right)(u - \overline{u}) \end{aligned}$$

Let $\psi = \phi - \hat{\phi}$ and $\theta = u - \overline{u}$. After bringing $\left(s_{\phi\phi}(\hat{\phi}) - a\overline{u}\right)\psi$ to the left the last equation becomes

$$-\Delta\psi - \left(a\overline{u} - s_{\phi\phi}(\hat{\phi})\right)\psi = G(\psi, \theta). \tag{17}$$

Since $a\overline{u} - s_{\phi\phi}(\hat{\phi}) \notin \Sigma$, the linear operator $L = -\Delta - (a\overline{u} - s_{\phi\phi}(\hat{\phi}))$ on the left has an inverse operator $L^{-1} : X \to Y$. Furthermore, the operator

$$G : Y \times X \to X$$

mapping

$$G : (\psi, \theta) \mapsto G(\psi, \theta)$$

has the following properties:

- $G(0,0) = 0$
- If $\|\psi\|_Y < \xi$ and $\|\theta\|_X < \eta$ for positive numbers ξ and η, then

$$\|G(\psi,\theta)\|_X \leq g(\xi,\eta), \tag{18}$$

where $g(\xi,\eta) = O(\xi^2) + O(\eta)$ as $\xi \to 0$ and $\eta \to 0$. Indeed, by applying the mean-value theorem twice we get

$$\begin{aligned} |G(\psi,\theta)| &\leq \left|-s_\phi(\phi) + s_\phi(\hat{\phi}) + s_{\phi\phi}(\hat{\phi})\psi\right| + |a\theta\psi| + \left|\left(a\hat{\phi}+b\right)\theta\right| \\ &\leq \left|s_{\phi\phi}(\hat{\phi})\psi - s_{\phi\phi}(\tau)\psi\right| + |a\theta\psi| + \left|\left(a\hat{\phi}+b\right)\theta\right| \\ &\leq C_1 |\psi|^2 + |a\theta\psi| + \left|\left(a\hat{\phi}+b\right)\theta\right| \end{aligned}$$

After squaring this last inequality, integrating the result over Ω and applying the Sobolev imbedding theorem (18) follows immediately. Furthermore, one sees that one can choose

$$g(\xi,\theta) = C_1\xi^2 + C_2\eta$$

- If $\|\psi\|_Y \leq \xi$ and $\|\theta\|_X \leq \eta$ for positive ξ and η, then

$$\|G(\psi_1,\theta) - G(\psi_2,\theta)\|_X \leq h(\xi,\eta)\,\|\psi_1 - \psi_2\|_Y\,, \tag{19}$$

where $h(\xi,\eta) \to 0$ as $\xi \to 0$ and $\eta \to 0$. To prove this we observe that the mean value theorem implies

$$\begin{aligned} |G(\psi_1,\theta) - G(\psi_2,\theta)| &\leq \left|-s_\phi(\phi_1) + s_\phi(\phi_2) + s_{\phi\phi}(\hat{\phi})\,(\psi_1-\psi_2)\right| + |a\theta| \quad |\psi_1-\psi_2| \\ &\leq \left|s_{\phi\phi}(\phi^*) - s_{\phi\phi}(\hat{\phi})\right| \quad |\psi_1-\psi_2| + |a\theta| \quad |\psi_1-\psi_2| \\ &\leq \left(\left|s'_{\phi\phi}(\phi^{**})\left(\phi^* - \hat{\phi}\right)\right| + |a\theta|\right)|\psi_1-\psi_2|\,. \end{aligned}$$

Squaring and integrating this inequality gives the desired result with

$$h(\xi,\eta) = C_3\xi + C_4\eta.$$

Therefore, all the hypotheses of Theorem 8.C of [15] are satisfied, and the result follows by applying this theorem.

□

This theorem has the following immediate consequences.

Remarks:

- If
$$-s_\phi(\phi) + \overline{u}\lambda(\phi) < 0,$$
then it is automatically not a part of the spectrum Σ and the theorem applies. Physically this case corresponds with single phase solutions.

- Clearly the theorem cannot be applied if $\hat{\phi}$ is a multiple root of
$$-s_\phi(\phi) + \overline{u}\lambda(\phi) = 0,$$
since in this case
$$\overline{u} - s_{\phi\phi}(\hat{\phi}) = 0 \in \Sigma.$$
However, this is not a generic case and will not be studied here.

In order to more closely investigate this result we need the following classical lemma which gives an estimate of $\|L^{-1}\|$.

Lemma 3.2 *Let $L = -\Delta - \nu$ on Ω together with Neumann boundary conditions. Then the operator $L^{-1} : X \to Y$ satisfies*

$$\|L^{-1}\| \leq C_0 \left(1 + \frac{|\nu|}{\inf_{\lambda \in \Sigma} |\nu - \lambda|}\right),$$

where C_0 is a constant which does not depend on ν. Moreover, for negative values of ν $\|L^{-1}\|$ is uniformly bounded.

Proof: The proof of this lemma is a straight forward application of some results from Fourier analysis and can be found in most standard texts on Spectral Theory. We will only briefly outline it. For an arbitrary $f \in X$ let v be the solution of
$$Lv = f.$$
We can expand both v and f in terms of the eigenfunctions φ_j of the negative Laplacian with Neumann boundary conditions. Using $v = \sum_{j=0}^{\infty} v_j \varphi_j$ and $f = \sum_{j=0}^{\infty} f_j \varphi_j$ we see that
$$(\lambda_j - \nu)\, v_j = f_j,$$
where λ_j is the j-th eigenvalue of the negative Laplacian. Plancherel's theorem implies that

$$\|v\|_X^2 = \sum_{j=0}^{\infty} v_j^2 = \sum_{j=0}^{\infty} \frac{f_j^2}{(\lambda_j - \nu)^2} \leq \frac{1}{\inf_{\lambda \in \Sigma} |\nu - \lambda|^2} \|f\|_X^2 .$$

In order to get an estimate on the $Y-$norm of the solution we observe that

$$-\Delta v = \nu v + f,$$

which immediately implies

$$\|v\|_{H^2(\Omega)} \leq C_0 \left(|\nu| \, \|v\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right),$$

which immediately implies the first part of the Lemma. Since all eigenvalues of the negative Laplacian are non-negative it follows that for

$$\inf_{\lambda \in \Sigma} |\nu - \lambda| = |\nu|,$$

for negative values of ν, i.e. L^{-1} is uniformly bounded for such values of ν.

$\square$

In the setting of Theorem 3.1, negative values of ν correspond to the smallest and largest simple root of $-s_\phi(\phi) + \overline{u}\lambda(\phi)$. Physically these roots correspond to single phase steady states. We can now formulate the following theorem.

Theorem 3.2 *Let $0 < \alpha \leq \theta_\Gamma \leq \beta$ be the bounds on the boundary conditions for the inverse temperature u. Furthermore, let C_i, $i = 1, 2, 3, 4$ be the constants in the proof of Theorem 3.1. Then there exists at least one solution in the sense of Theorem 3.1 if*

$$\min\{ \left(\frac{1}{2\|L^{-1}\|C_1} \right)^4, \left(\frac{\left(1 - \frac{C_3}{2C_1}\right)}{\|L^{-1}\|C_4} \right)^2 \} > \mu(\Omega) \left(\frac{1}{\alpha^2} - \frac{1}{\beta^2} \right), \tag{20}$$

where $\mu(\Omega)$ denotes the usual Lebesgue measure of Ω.

Proof: Let $u(x)$ be an arbitrary positive solution to equation (8) with boundary values as in the statement of the theorem. Define

$$\overline{u} = \frac{1}{\mu(\Omega)} \int_\Omega u(x) \, dx = \frac{1}{\mu(\Omega)} \|u\|_{L^1(\Omega)}.$$

It follows that

$$\|u - \overline{u}\|_X^2 = \int_\Omega |u(x) - \overline{u}|^2 \, dx = \|u\|_X^2 - \frac{1}{\mu(\Omega)} \|u\|_{L^1(\Omega)}.$$

Using the bounds on u from Lemma 2.1 we obtain

$$\|u - \overline{u}\|_X^2 \leq \mu(\Omega) \left(\frac{1}{\alpha^2} - \frac{1}{\beta^2} \right). \tag{21}$$

So we can ensure existence of a solution if η_0 in the statement of Theorem 3.1 satisfies:

$$\eta_0^2 \geq \mu(\Omega) \left(\frac{1}{\alpha^2} - \frac{1}{\beta^2} \right). \tag{22}$$

¿From the proof of Theorem 8.C of [15] we have that ξ_0 and η_0 are chosen to satisfy

$$\left\|L^{-1}\right\| \left(C_1\xi_0^2 + C_2\eta_0\right) \le \xi_0, \tag{23}$$

and

$$\left\|L^{-1}\right\| \left(C_3\xi_0 + C_4\eta_0\right) < 1. \tag{24}$$

¿From the first equation we get that

$$\eta_0 \le \frac{\xi_0}{C_2}\left(\frac{1}{\|L^{-1}\|} - C_1\xi_0\right) \le \frac{1}{\left(2\|L^{-1}\|C_1\right)^2}.$$

At the ξ_0 value corresponding to this maximum the second condition gives

$$\eta_0 < \frac{1 - \frac{C_3}{C_1}}{\|L^{-1}\|C_4}.$$

Combining these results we get the sufficient condition from the theorem.

□

Remarks:

- The graph of $-s_\phi(\phi)+\overline{u}\lambda(\phi)$ has at least one root. Theorem 3.2 states that for temperature fields which are nearly constant, there will always be at least one solution. If there is only one solution, this solution corresponds to a single phase.
- The other possibility is that $-s_\phi(\phi)+\overline{u}\lambda(\phi)$ has three solutions for which inequality (20) is satisfied. Then equation (7) has also three solutions, one of which corresponds to a root of $-s_\phi(\phi)+\overline{u}\lambda(\phi)$, for which both phases are present at every point. At this root the derivative of $-s_\phi(\phi)+\overline{u}\lambda(\phi)$ with respect to ϕ is positive. This solution makes no physical sense. Areas in which both phases are present are commonly called mushy regions.
- The conditions (20) in the statement of Theorem 3.2 are not optimal. Indeed, the conditions (22)–(24) in the proof imply that (ξ_0, η_0) lie in a region in the $\xi\eta$-plane bounded by a parabola (23), a straight decreasing line (24), and a horizontal line (22). One can easily see that this area is not empty if and only if

  $$\frac{1}{\|L^{-1}\|C_4} - \frac{C_3}{C_4}\hat{\xi} > \sqrt{\mu(\Omega)\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)},$$

  where $\hat{\xi}$ is the smallest root of

  $$\frac{\xi}{C_2}\left(\frac{1}{\|L^{-1}\|} - C_1\xi\right) - \sqrt{\mu(\Omega)\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)}.$$

The last theorem also has some implications on the stability of solutions. Solutions (ϕ, u) for which the quantity $u(x) - s_{\phi\phi}(\phi)$ is negative correspond to a single phase. Theorem 8.C of [15] can always be applied (even if u is not a constant) and shows that the map

$$u(x) \mapsto \phi(x),$$

which assigns a single phase solution ϕ to a function $u(x)$ is continuous, i. e. single phase solutions are stable in the sense that small changes in temperature field $u(x)$ will imply small changes in the phase field ϕ. However, this is not the case for solutions for which $u(x) - s_{\phi\phi}(\phi)$ is negative. In this case continuity cannot be guaranteed, and a rather small change of $u(x)$ could cause a relatively large change of ϕ. These latter solutions correspond to values of the phase parameter at which both phases are present at the same location. Regions in which this happens are so called mushy regions. Solutions which consist of entirely mushy regions are inconsistent with the experimental experience. It is therefore to be expected that such solutions should be unstable.

4 Linearization and Asymptotic Stability

This section is devoted to the investigation of the asymptotic stability of the solutions to the stationary system (7) and (8). Whereas in the last section we considered the map

$$u(x) \mapsto \phi(x),$$

we now assume that the pair (u_0, ϕ_0) is a solution to the stationary phase field system and we want to analyze its stability. Furthermore, we allow λ tobe explicitely dependent on x.

We start by linearizing the parabolic system (3)-(4) about a solution $(\phi_0(x), u_0(x))$ to the elliptic system. Substituting

$$\begin{aligned} \phi(x,t) &= \phi_0(x) + \varepsilon\phi_1(x,t), \\ u(x,t) &= u_0(x) + \varepsilon u_1(x,t), \end{aligned}$$

in the parabolic equations and the boundary conditions one gets – after using the properties of $(\phi_0(x), u_0(x))$ – a linear parabolic system

$$\phi_{1t} = \Delta\phi_1 - (s_{\phi\phi}(x,\phi_0) - \lambda_\phi(x,\phi_0)u_0)\,\phi_1 + \lambda(x,\phi_0)u_1, \qquad (25)$$

$$\frac{u_{1t}}{u_0^2} = \Delta u_1 - \lambda(x,\phi_0)\phi_{1t}, \qquad (26)$$

with the following boundary conditions

$$\frac{\partial\phi_1}{\partial\nu} = 0, \qquad (27)$$

$$\frac{\partial u_1}{\partial\nu} = u_1\,(1 - 2u_0\theta_\Gamma)\,. \qquad (28)$$

In addition we assume that the solutions $(\phi_1(x,t), u_1(x,t))$ satisfy the initial conditions

$$\begin{aligned} \phi_1(x,0) &= \phi_1^0(x), && (29)\\ u_1(x,0) &= u_1^0(x). && (30) \end{aligned}$$

Using the terminology of [13], we say that the stationary solution (ϕ_0, u_0) is **asymptotically stable**, if the solutions (ϕ_1, u_1) of the linearized system satisfy

$$\lim_{t\to\infty} (\|\phi_1(t)\|_X + \|u_1(t)\|_X) = 0,$$

for all initial values with $\left\|\phi_1^0\right\|_X \leq 1$ and $\left\|u_1^0(x)\right|_X \leq 1$. Since the non-linear parabolic system is not semi-linear, we cannot directly apply the usual methods (see, for example, Chapter 11 of [13]). However, only slight modifications are necessary.

In the following theorem we give sufficient conditions for asymptotic stability. This will show that single phase solutions are asymptotically stable.

Theorem 4.1 *Suppose that $(\phi_0, u_0)$is a solution to the stationary problem (7)-(8) with*

1. $s_{\phi\phi}(x,\phi_0) - \lambda_\phi(x,\phi_0) > 0$ *for all* $x \in \overline{\Omega}$.
2. $u_0 > \frac{1}{2\theta_\Gamma}$ *for all* $x \in \partial\Gamma$.

Then the solution is asymptotically stable.

Proof: We only sketch the proof of this theorem and omit some of the details. The theory of systems of linear parabolic equations implies that (25)-(26) has a unique classical solution.

To continue we multiply (25) by ${\phi_1}_t$ and (26) by u_1. Adding the results and integrating over Ω yields:

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\frac{u_1(t)}{u_0}\right\|_X^2 + \|\nabla\phi_1(t)\|_X^2 + \int_\Omega (s_{\phi\phi}(x,\phi_0) - \lambda_\phi(x,\phi_0))\,\phi_1^2(x,t)\,dx\right) = -C(t), \tag{31}$$

where

$$\begin{aligned} C(t) &= \|{\phi_1}_t(t)\|_X^2 + \|\nabla u_1(t)\|_X^2 - \int_{\partial\Omega} u_1(t)\frac{\partial u_1(t)}{\partial\nu}\,d\gamma \\ &= \|{\phi_1}_t(t)\|_X^2 + \|\nabla u_1(t)\|_X^2 - \int_{\partial\Omega} u_1^2(t)\,(1 - 2u_0\theta_\Gamma)\,dx \\ &\geq 0. \end{aligned}$$

In the next step we integrate (31) to get

$$-\int_0^t C(s)\,ds = \frac{1}{2}\left(\left\|\frac{u_1}{u_0}\right\|_X^2 + \|\nabla\phi_1\|_X^2 + \int_\Omega (s_{\phi\phi}(x,\phi_0) - \lambda_\phi(x,\phi_0))\,\phi_1^2(x)\,dx\right)\Bigg|_{s=0}^t.$$

If there exists a $\delta > 0$ such that $C(t) \geq \delta$ for all $t > 0$, there is a number $\tau > 0$ such that

$$\int_0^\tau C(s)\,ds = \frac{1}{2}\left(\left\|\frac{u_1^0}{u_0}\right\|_X^2 + \left\|\nabla\phi_1^0\right\|_X^2 + \int_\Omega (s_{\phi\phi}(x,\phi_0) - \lambda_\phi(x,\phi_0))\,(\phi_1^0)^2(x)\,dx\right),$$

i.e.

$$\frac{1}{2}\left(\left\|\frac{u_1(\tau)}{u_0}\right\|_X^2 + \|\nabla\phi_1(\tau)\|_X^2 + \int_\Omega (s_{\phi\phi}(x,\phi_0) - \lambda_\phi(x,\phi_0))\,\phi_1^2(x,\tau)\,dx\right) = 0,$$

and hence we have $\phi_1(x,\tau) = 0$ and $u_1(x,\tau) = 0$.

In the other case observe that the right-hand-side of (4) is bounded below by

$$-\frac{1}{2}\left(\left\|\frac{u_1^0}{u_0}\right\|_X^2 + \left\|\nabla\phi_1^0\right\|_X^2 + \int_\Omega (s_{\phi\phi}(x,\phi_0) - \lambda_\phi(x,\phi_0))\,(\phi_1^0)^2(x)\,dx\right),$$

which immedately implies that $\int_0^\infty C(s)\,ds$ exists. Thus, we have $\lim_{t\to\infty} C(t) = 0$. This implies that $\phi_1(x,t) \to \hat{\phi}(x)$ and $u_1(x,t) \to \hat{u}(x)$ as $t \to \infty$ for some functions $\hat{\phi}$ and $\hat{u}$ which satisfy

$$\begin{aligned}
\Delta\hat{\phi} &= (-s_{\phi\phi}(x,\phi_0) + \lambda_\phi(x,\phi_0))\,\hat{\phi} + \lambda(x,\phi_0)\hat{u} \\
\Delta\hat{u} &= 0,
\end{aligned}$$

and the boundary conditions (27) and (28). The result now follows from immediately from the following lemma.

□

Lemma 4.1 *Suppose* $(\hat{\phi},\hat{u})$ *is a smooth solution to*

$$\begin{aligned}
-\Delta\hat{\phi} &= (-s_{\phi\phi}(x,\phi_0) + \lambda_\phi(x,\phi_0)u_0)\,\hat{\phi} + \lambda(x,\phi_0)\hat{u}, \\
-\Delta\hat{u} &= 0,
\end{aligned}$$

with the same boundary conditions as above. Furthermore, assume that the hypotheses of the theorem hold.

Then $(\hat{\phi},\hat{u}) = (0,0)$

Proof: Suppose $\hat{u}$ has a positive maximum, then this maximum is on the boundary and

$$\frac{\partial\hat{u}}{\partial\nu} \geq 0.$$

But we have

$$\frac{\partial\hat{u}}{\partial\nu} = \hat{u}\,(1 - 2u_0\theta_\Gamma) < 0,$$

and therefore we have

$$\hat{u} \leq 0, \qquad \forall x \in \overline{\Omega}.$$

The analogous argument for negative minima shows that $\hat{u} \geq 0$ and therefore

$$\hat{u} = 0, \qquad \forall x \in \overline{\Omega}.$$

Using this in the first equation we get

$$\left\| \nabla \hat{\phi} \right\|_X^2 = \int_\Omega \left(-s_{\phi\phi}(x, \phi_0) + \lambda_\phi(x, \phi_0) u_0 \right) \hat{\phi}^2 \, dx \leq 0,$$

which immediately implies $\hat{\phi} = 0$.

□

We finish this section by giving a specific example of an unstable solution. For this let Ω be a region with smooth boundary in $\mathbf{R}^n$ and let

$$s_\phi - \lambda u = \phi^3 - \phi + 1 - u$$

and

$$\theta_\Gamma = 1.$$

It follows that $u_0 = 1$ satisfies the stationary equation for u and $\phi_0 = 0$ satisfies the stationary equation for ϕ. The linearized system becomes:

$$\begin{aligned} \phi_{1t} &= \Delta\phi_1 + \phi_1 - u_1 \\ u_{1t} &= \Delta u_1 - \phi_{1t} \end{aligned}$$

with boundary conditions

$$\begin{aligned} \frac{\partial \phi_1}{\partial \nu} &= 0, \\ \frac{\partial u_1}{\partial \nu} &= -u_1 \end{aligned}$$

For the initial values $u_1^0(x) = 0$ and $\phi_1^0(x) = 1$ the unique solution to this system is given by $u_1(x,t) = 0$ and $\phi_1(x,t) = 1$, i.e. $\phi(x,t) \not\to 0$ as $t \to \infty$.

This implies that ϕ_0 is an unstable stationary solution. This solution does however make no physical sense, since it would imply the existence of both phases at every point in the domain.

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