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SMALL AMPLITUDE CRITICAL PERIODS IN A CUBIC POLYNOMIAL SYSTEM

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Abstract. In this paper, we study the local bifurcations of critical periods in the neighbourhood of a nondegenerate centre of a parametrized family of plane cubic polynomial systems. We find that this family has no isochronous centres. We show that at most three local critical periods bifurcate from a centre at the origin. Moreover there exist perturbations which lead to the maximum number of critical periods. To get some of the results we reducing the question to that of transversal intersection of some algebraic surfaces, using computational algebra techniques such as the Theory of of Gröbner Bases.

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1. Introduction

Consider a family of plane polynomial autonomous vector fields $(\chi_\lambda)_{\lambda\in\mathbb{R}^N}$ parametrized by λ in the form

$$(\chi_\lambda) \qquad \chi_\lambda = (-y + P_\lambda(x,y))\frac{\partial}{\partial x} + (x + Q_\lambda(x,y))\frac{\partial}{\partial y},$$

P_λ and Q_λ polynomials in x and y, and with a nondegenerate center at the origin, i.e., a singular point surrounded by a continuous family of periodic trajectories with the system of differential equations normalized so that its linearization at the center has eigenvalues $\pm i$. We discuss the behaviour of the periods of such periodic solutions near the center. A period function $T(h,\lambda)$ associated to the center at the parameter value λ is defined by taking a semi-transversal Σ, Poincaré section,

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through the center with some parametrization h. $T(h,\lambda)$ is actually the time taken for the trajectory starting at h to next cross Σ, i.e., $T: \quad (0,L) \longrightarrow \mathbb{R}$ (perhaps $L=\infty$) assigns to each $h \in (0,L)$ the minimum period of the trajectory γ_h. $T(h,\lambda)$ is an even analytic function of the coordinate parametrization h, and has the asymptotic development

$$T(h,\lambda) = 2\pi + \sum_{k=1}^{\infty} p_{2k}(\lambda) h^{2k}, \tag{1-1}$$

for $|h|$ and $|\lambda - \lambda_*|$ sufficiently small, where λ_* corresponds to a center.

A number of results related to the period function has been obtained in the past few years, and specially on the monotonicity of the function, and the number of its critical points. See, for instance, [1,2,3] as well as for some historical details. A global study of the number of critical points of the period is a very difficult problem and is generally unsolved. Our interest lies in the local question of the bifurcation of critical points (*critical periods*) of the period function as the parameter λ varies in the neighbourhood of λ_* corresponding to a center. It is the *Small Amplitude Critical Periods problem.*

Our approach is based on the analysis on the Taylor expression of $T(h,\lambda)$ in (1-1) by determining the so-called *period coefficients* $p_{2k}(\lambda), k \geq 1$. These coefficients are polynomials in the components of the bifurcation parameter λ_*. By the Hilbert basis theorem, there exists $K \in \mathbb{N}$ such that the ideal of all coefficients, $\mathcal{I} =< p_2, \cdots, p_{2k}, \cdots >$ over the ring $\mathbb{R}[\lambda, \ldots, \lambda_N]$, that we call the *Bautin-like ideal*, has the finite generator $\mathcal{G} =< p_2, \cdots, p_{2K} >$. See [2] for more details. Determining K is at the core of the method. It also leads to uncover the isochronous centres (all periodic orbits have the same period), i.e., the ones for which all period coefficients $p_{2k}(\lambda), k \geq 1$ vanish. Using this approach the small amplitude critical periods problem has been completely solved for quadratic systems in [2], for cubic symmetric systems in [4], and for Kukles reduced system in [5]. Similarities were found: A greater number of critical periods is attained from the nonisochronous centres, with the maximum number on the symmetric stratum of the parameter values. (See below). To derive general laws such as the maximum number of local critical periods for a polynomial family in terms of the degree of the system, and tackle the global problem, more systems need to be investigated.

In particular, we consider the following quadratic-like family $(\mathcal{C}_\lambda)_{\lambda\in\mathbb{R}^5}$ of plane cubic polynomial systems

$$(\mathcal{C}_\lambda) \qquad \begin{aligned} \dot{x} &= -y + Ax^2 + (B+2D)xy + Cy^2 - y(x^2+y^2), \\ \dot{y} &= x + Dx^2 + (E-2A)xy - Dy^2 + x(x^2+y^2), \end{aligned}$$

parametrized by $\lambda = (A,B,C,D,E) \in \mathbb{R}^5$. It has a center at the origin for the parameter values $\lambda: \quad B=E=0$ (Hamiltonian Stratum) and $\lambda: \quad B=D=0$ (Symmetric Stratum). See [6]. We address and solve completely the *small amplitude critical periods* problem for such a family, and moreover we prove that the upper bound can be precisely attained.

After giving in section 2. some definitions and general results, we prove the followings in section 3.

(1) At most three critical periods can bifurcate from these centres. The family has no nonlinear isochronous centres.

(2) There are perturbations of the bifurcation parameter leading to the maximum number of critical periods. We prove this result by reducing the question to that of transversal intersections of some algebraic surfaces, using computational algebraic geometry techniques such as the Theory of Resultant, Sturm's algorithm ([7]), Gröbner Bases ([8],[9]), and the Transversality theory of algebraic curves ([10]). This is one of the originalities of the paper.

Another originality is the computer assisted method of proofs, such as the method of *reduction modulo a Gröbner basis* via the computer algebra system MapleV.

Actually system $(\mathcal{C}_\lambda)$ without the cubic terms is the pure quadratic system studied in [2], where the maximum number obtained is two, along with the existence of isochronous centres. On the isochronous parameter strata the order of degeneracy of the derivative of the period function is infinite. In other words, our results show that the addition of the cubic terms causes the system to have no center where the order of degeneracy is greater than three, and thus may be seen as a mechanism to destroy the isochronous strata, and to inscrease the number of critical periods.

Moreover, unlike other systems studied in [3,4,5], this maximum number of critical periods is attained on both Hamiltonian and symmetric strata.

2. Some General Results

Definition 2.1.

(1) *The origin is a weak centre of finite order k, if $p_1 = p_2 = \ldots = p_{2k} = p_{2k+1} = 0$ and $p_{2k+2} \neq 0$. It is of infinite order or an isochronous centre, if $p_k = 0$ for $k \geq 1$.*

(2) *We say that k local critical periods bifurcate from the weak centre corresponding to the parameter λ_* if, for every $\alpha > 0$, sufficiently small, there exists a neighbourhood W of λ_* such that for any $\lambda \in W$, $T(h, \lambda)$ has at most k critical points in $U = (0, \alpha)$. Moreover, any neighbourhood of λ_* contains a point λ^1 such that $T(h, \lambda)$ has exactly k critical points in $U = (0, \alpha)$.*

Definition 2.2. *Let $\{\chi_\lambda\}_{\lambda \in \mathbb{R}^N}$ be a family of systems with a centre at the origin and associated period coefficients $p_k(\lambda)$. The family is said to satisfy condition $(\mathcal{P})$ if for any $\lambda_* \in \mathbb{R}^N$ such that $p_2(\lambda_*) = \cdots = p_{2k}(\lambda_*) = 0$, $p_{2k+2}(\lambda_*) \neq 0$ and any neighbourhood $W \subset \mathbb{R}^N$ of λ_* in which $p_{2k+2} \neq 0$, there exists $\lambda_1 \in W$ such that*

$$p_{2k+2}(\lambda_1) \times p_{2k}(\lambda_1) < 0, \quad \text{with } p_2(\lambda_1) = \cdots = p_{2k-2}(\lambda_1) = 0. \tag{2-2}$$

The system χ_{λ_*} is said to satisfy condition $(\mathcal{P}_k)$.

We proved the following in [5]

Proposition 2.3. *Let $\{\chi_\lambda\}_{\lambda \in \mathbb{R}^N}$ be a family of systems with a centre at the origin, satisfying condition $(\mathcal{P})$. And let $\chi_{\lambda*}$ satisfy condition $(\mathcal{P}_k)$. Then for any sufficiently small $\epsilon > 0$, any neighbourhood V of λ_*, and for any $0 \leq n \leq k$, there exists $\lambda_{**} \in V$ such that $\chi_{\lambda**}$ has exactly n periodic orbits with critical periods passing through points $(x, 0)$ for $x \in (0, \epsilon)$.*

Straightforward from the Weierstrass-Malgrange Preparation Theorem, we derive the following theorem applied throughout the study.

Finite Order Bifurcation Theorem. *From weak centres of finite order at the parameter value* λ_* *, no more than* k *local critical periods can bifurcate. Moreover there are perturbations with exactly* $n \leq k$ *local critical periods, if the coefficients* $p_2, \dots, p_{2k+2}$ *satisfy the condition* $(\mathcal{P}_k)$ *at* λ_*.

3. Critical periods of $(\mathcal{C}_\lambda)_{\lambda \in \mathbb{R}^5}$

Blows and Rousseau in [6] obtained the following conditions of center.

Theorem 3.1. *System* $((\mathcal{C}_\lambda)_{\lambda \in \mathbb{R}^5})$ *has a center at the origin for a parameter value* λ_* *if and only if* λ_* *is on either of the following stratas:*

$$\begin{cases} B = E = 0, & \textit{(Hamiltonian stratum } (\mathcal{H})) \\ B = D = 0, & \textit{(Symmetric stratum } (\mathcal{S})) \end{cases}$$

Throughout the paper, $(\mathcal{H})$ and $(\mathcal{S})$ indicate respectively the Hamiltonian and the symmetric stratum. We consider a point $\lambda_* \in (\mathcal{I}) = (\mathcal{H}) \cup (\mathcal{S})$, and study the bifurcations of critical periods for perturbations of λ_* which remain in $(\mathcal{I})$.

Under condition $(\mathcal{H})$, $(\mathcal{C}_\lambda)_{\lambda \in \mathbb{R}^5}$ is a Hamiltonian system with a Hamiltonian in the form

$$H(x,y) = \frac{x^2+y^2}{2} + \frac{Dx^3}{3} - Ax^2y - Dxy^2 - \frac{Cy^3}{3} + \frac{(x^2+y^2)^2}{4}. \tag{3-1}$$

Under condition $(\mathcal{S})$, $(\mathcal{C}_\lambda)_{\lambda \in \mathbb{R}^5}$ is symmetric with respect to the y-axis.

Assuming the parameter value λ_* in $(\mathcal{I})$, changing to polar coordinates (r, θ), and eliminating t bring system $(\mathcal{C}_\lambda)$ into the form

$$\frac{dr}{d\theta} = \frac{r^2 f_0(\theta, \lambda_*)}{1 + r g_0(\theta, \lambda_*) + r^2}, \tag{3-2}$$

where

$$\begin{aligned} f_0(\theta, \lambda_*) &= A\cos^3\theta + (A+3D)\cos^2\theta\sin\theta + (E+C-2A)\cos\theta\sin^2\theta - D\sin^3\theta \\ g_0(\theta, \lambda_*) &= D\cos^3\theta + (D-3A)\cos^2\theta\sin\theta + (3D+B)\cos\theta\sin^2\theta - C\sin^3\theta. \end{aligned}$$

In the neighbourhood of the centre at the origin, the period function $T(h, \lambda)$ is computed as follows:

$$T(h,\lambda) = \int_0^{2\pi} \frac{d\theta}{\dot\theta} = \int_0^{2\pi} \frac{d\theta}{1 + r g_0(\theta, \lambda_*) + r^2} = 2\pi + \sum_{k \geq 1} p_{2k}(\lambda) h^{2k}, \tag{3-3}$$

for $|h|$ and $|\lambda - \lambda_*|$ sufficiently small. The solution $\theta \mapsto r(\theta, h, \lambda)$ of the differential equation in (3-2) with initial condition $r(0, \lambda) = h$ is well-known to be locally represented as the convergent power series in h

$$r(\theta, h, \lambda) = \sum_{k \geq 1} u_k(\theta, \lambda) h^k. \tag{3-4}$$

Substituting back into (3-2) yields the following recurrent relation to determine the coefficients $u_k(\theta, \lambda)$:

$$\begin{aligned} \dot{u}_1 &= 0, \quad \text{(from the initial condition)}, \\ \dot{u}_k &= \sum_{1 \le i \le (k-1)} (f_0 u_i u_{k-i} - (g_0 u_{k-i} + v_{k-i})\dot{u}_i), \end{aligned} \tag{3-5}$$

with $v_1 = 0, \quad v_k = \sum_{i=1}^{k-1} u_i u_{k-i}$. Using the integral representation of T in (3-3), we derive the period coefficients p_k by simple integration

$$p_k(\lambda) = \int_0^{2\pi} A_k(\theta, \lambda) d\theta, \tag{3-6}$$

with the terms A_k recursively defined by

$$A_0(\theta, \lambda) \equiv 1, \quad A_k(\theta, \lambda) = -\sum_{1 \le i \le (k-1)} (g_0 u_{k-i} + v_{k-i}) A_i. \tag{3-7}$$

$p_k(\lambda)$ is clearly a polynomial of degree k in (A, B, C, D, E). For $k = 2$, we get

$$p_2(\lambda) = -24 + 18A^2 + B^2 + 12AC + 10C^2 + 8BD + 16D^2 - 9AE - CE + E^2. \tag{3-8}$$

We next study the bifurcations of the critical periods from centres corresponding to a parameter value λ_* on each one of the strata $(\mathcal{H})$ and $(\mathcal{S})$.

3.1 Hamiltonian stratum.

The parameter value λ_* satisfies the condition $B = E = 0$. We first compute sucessively the period coefficients $p_2(\lambda_*)$, $p_4(\lambda_*)$, $p_6(\lambda_*)$, $p_8(\lambda_*)$. Substituting $B = E = 0$ in formula (3-8) gives

$$p_2(\lambda_*) = b_1^2 + 4C^2 + 8D^2 - 12 \quad \text{with } b_1 = 3A + C. \tag{3-9}$$

Thus

$$p_2(\lambda_*) = 0 \quad \text{if and only if } D^2 = b_4 = \frac{12 - b_1^2 - 4C^2}{8} \ge 0, \tag{3-10}$$

that is, for A and C in $[-\sqrt{3}, \sqrt{3}]$. Therefore, out of the ellipse $b_1^2 + 4C^2 + 8D^2 - 12 = 0$, we have necessarily $p_2 \neq 0$. Hence no critical period can bifurcate from the corresponding centre.

From formulas (3-6) and (3-7), and modulo $p_2(\lambda_*)$ we obtain the period coefficient $p_4(\lambda_*)$ as

$$\begin{aligned} p_4(\lambda_*) = &- 64 + 120A^2 - 75A^4 + 240AC - 260A^3C + 120C^2 - 330A^2C^2 \\ &- 180AC^3 - 35C^4. \end{aligned} \tag{3-11}$$

Through the linear transformation $(\mathcal{T}_1)$: $\quad (A, C) = (\frac{-7u+v}{8}, \frac{15u-v}{8})$ for u and v such that $|u| \le 2\sqrt{3}$ and $|v| \le 22\sqrt{3}$ (using condition (3-10)), the coefficient $p_4(\lambda_*)$ is simplified to

$$\tilde{p}_4(u, v) = -5u^3v + 120u^2 - 64. \tag{3-12}$$

First let us assume $u = 0$, i.e., $A = -C$. Let λ_*^1 be the corresponding parameter value; then $p_4(\lambda_*^1) = -64 < 0$. The centre associated with λ_*^1 is therefore of order 1. Moreover, $p_2(\lambda_*^1)$ is reduced to

$$p_2(\lambda_*^1) = 2C^2 + 2D^2 - 3. \tag{3-13}$$

It suffices to take any value $\tilde{\lambda}_*^1$ next to the exterior of the ellipse $p_2(\lambda_*^1) = 0$, to obtain a perturbation leading to $p_2(\tilde{\lambda}_*^1) > 0$ and $p_4(\tilde{\lambda}_*^1) < 0$. Therefore, condition $(\mathcal{P}_k)$, $k = 1$ is fulfilled. Hence the following lemma.

Lemma 3.2. *No more than one critical period can bifurcate from the centre corresponding to the parameter value* λ_*^1.

We denote by $(\mathcal{C}_{\lambda\in\mathcal{H}})$ the corresponding system and prove the following theorem.

Theorem 3.3.

(1) *System* $(\mathcal{C}_{\lambda\in\mathcal{H}})$ *has a weak centre of finite order at most three at the origin, i.e., at most three critical periods can bifurcate from this centre.*
(2) *Moreover, there exists a perturbation of the centre of finite order three leading to the maximum number of critical periods.*
(3) *System* $(\mathcal{C}_{\lambda\in\mathcal{H}})$ *has no non-linear isochronous centre.*

Proof. For $u \neq 0$ in (3-12) and λ_*^2 the corresponding parameter value, we have

$$p_4(\lambda_*^2) = 0 \quad \text{if and only if } v = \frac{120u^2 - 64}{5u^3}. \tag{3-14}$$

Modulo p_2, and through the above transformation $(\mathcal{T}_1)$, the period coefficient $p_6(\lambda_*^2)$ is simplified to

$$p_6(u,v) = 15u^3(63u^2 - 116)v - 850u^6 + 15120u^4 + 1440u^2 - 3328. \tag{3-15}$$

Thus the analysis of the system of equations

$$p_2 = p_4 = p_6 = 0 \quad \text{with } A + C = 0 \tag{3-16}$$

is easily reduced to that of

$$\begin{aligned} v = f_4(u) &= \frac{120u^2 - 64}{5u^3} \\ f_6(u) &= 18944 - 52416u^2 + 37800u^4 - 8505u^6 = 0. \end{aligned} \tag{3-17}$$

System (3-17) has at least one real root, denoted (u_0, v_0), admissible for p_2. Indeed, by Sturm's algorithm, we verify that $f_6(u)$ has a unique real root u_0 in the interval $\left]\frac{74654}{10000}, \frac{74655}{10000}\right)$, with the corresponding v_0 in the interval $\left]\frac{13837}{10000}, \frac{138461}{10000}\right)$. The root (u_0, v_0) is not a zero of the next period coefficient p_8; precisely, modulo p_2, and, through the transformation $(\mathcal{T}_1)$ p_8 is determined, simplified via (3-17) to

$$\begin{aligned} f_8(u) = &- 3559140352 - 446772960u^2 + 15477215400u^4 \\ &- 12057953625u^6 + 266716800u^8. \end{aligned} \tag{3-18}$$

A direct calculation of the resultant of the polynomials $f_6(u)$ and $f_8(u)$ yields a nonzero constant. Therefore, from the theory of Resultant, these polynomials have no common root. Hence there exists a parameter value $\tilde{\lambda}_*^2$ satisfying

$$p_2(\tilde{\lambda}_*^2) = p_4(\tilde{\lambda}_*^2) = p_6(\tilde{\lambda}_*^2) = 0, \quad \text{with } p_8(\tilde{\lambda}_*^2) \neq 0. \tag{3-19}$$

The corresponding weak centre is therefore of finite order $k = 3$; and at most three critical periods can bifurcate from this centre by the Finite Order Bifurcation Theorem.

To show that there exists a perturbation leading to the maximum number of critical periods we consider the algebraic curves

$$\begin{cases} \tilde{p}_4(u,v) = 0, & \text{(equation of curve } (\mathcal{C}_4)) \\ \tilde{p}_6(u,v) = 0, & \text{(equation of curve } (\mathcal{C}_6)) \end{cases} \tag{3-20}$$

$(\mathcal{C}_4)$ and $(\mathcal{C}_6)$ intersect transversally at their common root; indeed, the assumption $(\mathcal{A}_u): \quad u \neq 0 \quad$ and $63u^2 - 116 \neq 0$, allows the equations

(3-21)

$$v = g_4(u) = \frac{120u^2 - 64}{5u^3} = g_6(u) = -\frac{-8505u^6 + 15120u^4 + 1440u^2 - 3328}{15u^3(63u^2 - 116)}.$$

$63u^2 - 116 = 0$ yields the period coefficients

$$p_4 = \pm\sqrt{\frac{116}{63}} \times \frac{568}{63}v + \frac{3296}{21}, \quad \text{whereas } p_6 = -\frac{368576}{147} < 0. \tag{3-22}$$

This implies a corresponding weak centre of order at most two, and therefore leads to a bifurcation of at most two critical periods.

Now assuming $(\mathcal{A}_u)$, the curves g_4 and g_6 in (3-21) cannot be tangent at their common root; this results from the fact that the system of equations

$$F(u) = g_4(u) - g_6(u) = 0, \quad T(u) = \frac{dg_4(u)}{du} - \frac{dg_6(u)}{du} = 0 \tag{3-23}$$

has no real solution. Indeed, direct calculation of the resultant of $F(u)$ and $T(u)$ gives a nonzero constant. Consequently, the curves $(\mathcal{C}_4)$ and $(\mathcal{C}_6)$ have a transversal intersection that guarantees, in the neighbourhood of the common root (u_0, v_0), the existence of a perturbation λ_*^3 such that

$$p_4(\lambda_*^3) \times p_6(\lambda_*^3) < 0, \quad \text{with } p_8(\lambda_*^3) \neq 0. \tag{3-24}$$

Finally, in the neighbourhood of such a perturbation we may choose $\tilde{\lambda}_*^3$, to realize condition $(\mathcal{P}_k)$ with $k = 3$, that is, the birth of three critical periods.

Hence system $p_2 = p_4 = p_6 = p_8 = 0$ has no real solution, yielding an obstruction to the existence of an isochrone subclass in the family $(\mathcal{C}_{\lambda\in\mathcal{H}})$. $\square$

3.2 Symmetric stratum.

The parameter value satisfies the condition $B = D = 0$. The corresponding system is denoted $(\mathcal{C}_{\lambda\in\mathcal{S}})$. From (3-8) the first period coefficient p_2 is given by

$$p_2(A, C, E) = 18A^2 + 12AC + 10C^2 + E^2 - CE - 9AE - 24. \tag{3-25}$$

The next three period coefficients are computed and simplified by the method of reduction modulo a Gröbner basis. We get the following.

Lemma 3.4. *From formulas* (3-6) *and* (3-7)*, and using transformation* $(\mathcal{T}_1)$*, the period coefficients* p_4, p_6, *and* p_8, *reduced modulo the Gröbner basis of the preceding, are respectively*

$$p_4(A, C, E) = Ag_4(C, E) + f_4(C, E), \quad \textit{[Gröbner basis of } [p_2]] \tag{3-26}$$

with

$$g_4(C,E) = -120C - 54E + 32C^2 + 3CE^2 + 80C^3,$$
$$f_4(C,E) = -19CE - 600C^2 + 6E^2 + 320C^4 + 120C^3E + 6C^2E^2 - CE^3 - 252.$$

(3-27) $$p_6(A,C,E) = Ag_6(C,E) + f_6(C,E), \quad \textit{[Gröbner basis of } [p_2,p_4]\textit{]},$$

with

$$\begin{aligned} g_6(C,E) &= 20784C^2E^3 + 1647E^4C + 3472320C^2E - 2432496CE^2 \\ &- 11815E^3 - 75479040C - 71743968E, \\ f_6(C,E) &= -1387210CE^3 + 12810720CE - 21529280C^3E - 12514400C^2E^2 \\ &- 441360C^4E^2 + 167858E^4 - 45130C^2E^4 - 2399C^5E - 22440240 \\ &+ 29352960C^2 + 12012396E^2 + 6988800C^4 - 238280C^3E^3; \end{aligned}$$

(3-28) $$p_8(A,C,E) = Ag_8(C,E) + f_8(C,E), \quad \textit{[Gröbner basis of } [p_2,p_4,p_6]\textit{]},$$

where g_8 and f_8 are polynomials of respective degree 5 and 6 in the variables C and E.

As in the Hamiltonian case, the results are derived from the analysis of the system of equations:

(3-29) $$p_2(A,C,E) = p_4(A,C,E) = p_6(A,C,E) = p_8(A,C,E) = 0.$$

We first prove

Theorem 3.5. *The hamiltonian stratum intersects the symmetric stratum to generate one local critical period.*

Proof. Using the above reduced period coefficients, we get at $(C,E) = (0,0)$

(3-30) $$p_2(A,0,0) = 3A^2 - 4, \quad \text{with } p_4(A,0,0) = -252 < 0.$$

Let λ_*^3 be the corresponding parameter value with components $C = E = B = D = 0$, that is, at the intersection of the Hamiltonian and symmetric strata. Therefore the corresponding centre is of order at most one. Let us consider such a centre of order one; we construct a perturbation $\tilde{\lambda}_*^3$ of λ_*^3 with exactly one critical period. That is:

(3-31) $$p_4(\tilde{\lambda}_*^1) \times p_2(\tilde{\lambda}_*^1) < 0,, \quad \text{condition } (\mathcal{P}_1).$$

To realize this conditon, we simply perturb the component A into $\tilde{A} = A + \epsilon_1$ with $A\epsilon_1 > 0$. From (3-31), it suffices to show that $p_2(\tilde{\lambda}_*^3) > 0$, that is,

(3-32) $$3(A+\epsilon_1)^2 - 4 > 0, \quad \text{with } 3A^2 - 4 = 0.$$

This follows from

(3-33) $$2A\epsilon_1 + \epsilon_1 > 0, \quad \text{for } A\epsilon_1 > 0.$$

We may always choose ϵ_1 to meet the above requirements. □

A more general analysis leads to the following theorem:

Theorem 3.6.

(1) *System* $(\mathcal{C}_{\lambda\in\mathcal{S}})$ *has no non-linear isochrone centre at the origin.*
(2) *The origin is a weak centre of finite order of at most three, i.e., at most three critical periods can bifurcate from a such a centre.*

Proof. We note from lemma (3.4) that $g_4(C,E)$ and $f_4(C,E)$ are in $\mathbb{R}[C][E]$, i.e., polynomials in the variable E with coefficients in $\mathbb{R}[C]$. Thus, for $C \neq 0$, we easily derive a nonzero resultant of g_4 and f_4 using Sturm's algorithm. In other words, system

$$g_4(C,E) = f_4(C,E) = 0 \tag{3-34}$$

has no real solution.

Now assume $g_4(C,E)$ is nonzero. This yields

$$p_4(A,C,E) = 0 \quad \text{if and only if } A = q_4(C,E) = -\frac{f_4(C,E)}{g_4(C,E)}, \tag{3-35}$$

and allows us to reduce $p_2(A,C,E)$, $p_6(A,C,E)$, and $p_8(A,C,E)$ respectively to $q_2(C,E)$, $q_6(C,E)$, $q_8(C,E)$ polynomials in the variables C and E. The corresponding system of equations

$$\begin{gathered} A = q_4(C,E), \\ q_2(C,E) = q_6(C,E) = q_8(C,E) = 0 \end{gathered} \tag{3-36}$$

has no real solution by calculating the resultants of the pairs (q_2,q_6), (q_6,q_8), and (q_2,q_8). Thus we cannot get simultaneously $q_2 = q_6 = 0$ and $q_2 = q_8 = 0$. Hence the claim.

The origin is therefore a weak centre of finite order at most $k = 3$, i.e., by the Finite Order Bifurcation Theorem, no more than three critical periods can bifurcate from such a centre. In fact, we have shown that, if there exists a parameter value λ_* such that

$$p_2(\lambda_*) = p_4(\lambda_*) = p_6(\lambda_*) = 0, \tag{3-37}$$

then necessarily, $p_8(\lambda_*) \neq 0$. Moreover, as in the Hamiltonian case, this implies the non-existence of an isochrone system in the sub-family $(\mathcal{C}_{\lambda\in\mathcal{S}})$. □

Here also, the maximum number of three critical periods can be attained by perturbation as below.

Theorem 3.7. *For a centre of order three as in theorem* (3-6)*, we may construct a perturbation with the maximum number three of critical periods. Moreover, condition* $(\mathcal{P}_k)$ *ascertains that, for every* $n \leq 3$, *there exists a perturbation with exactly* n *periods.*

Proof. Consider a weak centre of order three. Keeping the same notations, we first simplify the period coefficients $p_{2k}(A,C,E)$, for $k = 1,2,3,4$ via the following transformation $(\mathcal{T}_2)$

$$(A,C,E) = \left(\frac{2b_3 + 5b_5}{3}, b_5, b_1 + 3b_3 + 8b_5\right). \tag{3-38}$$

We obtain successively $p_2(b_1, b_3, b_5) = b_1^2 - b_3^2 + 16b_5^2 - 24$, and

(3-39) $\quad p_4(b_1, b_3, b_5) = b_1 g_4(b_3, b_5) + f_4(b_3, b_5), \quad$ modulo Gröbner basis of $[p_2]$,

with

$$\begin{aligned} g_4(b_3, b_5) &= -36b_3 - 198b_5 + 220b_5^3 + 71b_3b_5^2 + 4b_3^2b_5 \\ f_4(b_3, b_5) &= -1146b_3b_5 + 1188b_3b_5^3 + 12b_3^3b_5 + 225b_3^2b_5^2 - 132b_3^2 - 54. \end{aligned} \tag{3-40}$$

(3-41) $\; p_6(b_1, b_3, b_5) = b_1 g_6(b_3, b_5) + f_6(b_3, b_5), \quad$ modulo Gröbner basis of $[p_2, p_4]$,

with g_6 and f_6 polynomials of degree 6 in b_3 and b_5, and so on for p_8 modulo the Gröbner basis of $[p_2, p_4, p_6]$. Having done this, we reduce the analysis of the system $p_2 = p_4 = p_6 = 0$ to a question of transversal intersection of some algebraic surfaces. More precisely, we consider the following surface equations:
(3-42)
$$p_2(b_1, b_3, b_5) = 0, \quad (\mathcal{S}_2), \quad p_4(b_1, b_3, b_5) = 0, \quad (\mathcal{S}_4), \quad p_6(b_1, b_3, b_5) = 0, \quad (\mathcal{S}_6).$$

From the equation of $(\mathcal{S}_4)$ we derive $b_1 = -\frac{f_4(b_3,b_5)}{g_4(b_3,b_5)}$. Substitution in $(\mathcal{S}_2)$ and $(\mathcal{S}_6)$ yields the following curves

$$\begin{cases} p_2(b_1, b_3, b_5) = \bar{p}_2(b_3, b_5) = 0, & (\tilde{\mathcal{C}}_2) \\ p_6(b_1, b_3, b_5) = \bar{p}_6(b_3, b_5) = 0, & (\tilde{\mathcal{C}}_6). \end{cases} \tag{3-43}$$

The algebraic curves $(\tilde{C}_j), j = 2, 6$, are non-singular and intersect transversally at a point $M_* = (b_3^*, b_5^*)$ in the rectangle

$$R_* = R_*^1 \times R_*^2 = \left[\frac{72}{100}, \frac{78}{100}\right] \times \left[-\frac{4}{100}, -\frac{2}{100}\right]. \tag{3-44}$$

Indeed, letting $J_b^1 = J_b^1(b_1, b_3, b_5)$ be the determinant of the Jacobian matrix $\left(\frac{\partial(p_2,p_4,p_6)}{\partial(b_1,b_3,b_5)}\right)$, it follows, using resultants, that system

$$p_2 = p_4 = p_6 = J_b^1 = 0 \tag{3-45}$$

has no real solution. Therefore, the surfaces (S_j), $j = 2, 4, 6$ are transversal. By Sturm's algorithm, each one of the polynomials

$$f_2^1(b_5) = \bar{p}_2\left(\frac{72}{100}, b_5\right), \quad f_2^2(b_5) = \bar{p}_2\left(\frac{78}{100}, b_5\right) \tag{3-46}$$

has a single real root in the interval $It^1 = (-\frac{4}{100}, -\frac{2}{100}]$, whereas polynomials

$$f_6^1(b_5) = \bar{p}_2\left(\frac{72}{100}, b_5\right), \quad f_6^2(b_5) = \bar{p}_2\left(\frac{78}{100}, b_5\right) \tag{3-47}$$

do not vanish over this interval. On the other hand, the interval $It^2 = (\frac{72}{100}, \frac{78}{100}]$ contains a single common real root of the polynomials

$$g_6^1(b_5) = \bar{p}_6\left(b_3, \frac{-2}{100}\right), \quad g_6^2(b_5) = \bar{p}_6\left(b_3, \frac{4}{100}\right), \tag{3-48}$$

and none for

$$g_2^1(b_5) = \bar{p}_2\left(b_3, \frac{-2}{100}\right); \quad g_2^2(b_5) = \bar{p}_2\left(b_3, \frac{4}{100}\right). \tag{3-49}$$

Moreover, the curves $\bar{p}_2 = 0$ and $\bar{p}_6 = 0$ are non-singular, for the polynomials $\bar{p}_2$ and $\bar{p}_6$ have nonvanishing discriminants respectively in the intervals It^2 and It^1. Finally, setting $J_b^2(b_3, b_5)$ as the determinant of the Jacobian matrix $\left(\frac{\partial(\bar{p}_2,\bar{p}_4)}{\partial(b_3,b_5)}\right)$, and again using Sturm's algorithm , one shows that this determinant does not vanish on the sides on the rectangle R_*, on the one hand; on the other hand, from discriminant(J_b^2, b_3) and discriminant(J_b^2, b_5), respectively polynomials in the variables b_3 and b_5, it results that the curve $J_b^2(b_3, b_5) = 0$ has no component in R_*. Thus the curves $(\tilde{C}_j), j = 2, 6$ are non-singular with a transversal intersection in the surface (S_4).

Substituting $b_1 = -\frac{f_4(b_3,b_5)}{g_4(b_3,b_5)}$ in the expression $J_b^1(b_1, b_3, b_5)$ gives a polynomial $H(b_3, b_5)$ in b_3 and b_5. As above, computing the resultants of the pairs $(\bar{p}_2, \bar{p}_6)$, $(\bar{p}_2, H)$ and $(\bar{p}_6, H)$ leads us to conclude that, on the algebraic surfaces, we do not get simultaneously

$$\bar{p}_2(b_3, b_5) = \bar{p}_6(b_3, b_5) = 0, \quad \text{and } \bar{p}_6(b_3, b_5) = H(b_3, b_5) = 0. \tag{3-50}$$

Therefore, the three algebraic surfaces (S_j), $j = 2, 4, 6$ are in general position, that is, transversal in the space $\{(b_1, b_3, b_5), b_j \in \mathbb{R}^3, j = 1, 3, 5\}$. This transversality argument guarantees the existence of a perturbation of the weak centre of order three to produce exactly three local critical periods. □

4. Concluding Remarks

Comparing our results with [2], we may conclude that the cubic terms $(-y(x^2 + y^2), x(x^2 + y^2))$, provide a mechanism to destroy the isochrone stratas found in the case of quadratic systems in Bautin's normal form studied therein, on one hand, and, on the other hand, to increase by one the maximum number of critical periods. Such a mechanism might be generalized.

Our approach, efficiently integrating techniques from computational algebraic geometry, ideal theory, and transversality theory of algebraic curves, might be applied with much success to the global study, and as well as the more difficult case of the general return time function, i.e., period function of a polycycle using its Dulac asymptotic expansion.

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