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EIGENVALUE DISTRIBUTION OF INTEGRAL OPERATORS DEFINED BY ORLICZ KERNELS

E. Elshobaky and M. Faragallah

Abstract

In this paper the new interpolation method (see [GP], [FG]) has been applied to drive some theorems, e.g., the interpolation of L_p spaces, l_p spaces, and b^{σ}_{pu} (see [EF]). Using the interpolation results in [EF] with the eigenvalue results we give a good estemate for the eigenvalues of integral operators defined by kernels belong to vector valued Orlicz spaces $L_p(L_\Phi)$, $l_p(l_\Phi)$ and vector valued Besov - Orlicz sequence spaces $b^s_{pu}(b^t_{\Phi v})$.

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1. Interpolation spaces

In this section we summarize some well known definitions and theorems and we will also drive some interpolation results.

Theorem 1.1 (FG). *Let* (X_0, X_1) *and* (Y_0, Y_1) *be Banach couples,* $\rho \in P^{+-}$ *and* $1 \leq p, q < \infty$. *Then* $\langle X_0, X_1\rangle_{\rho,p} = \langle X_0, X_1\rangle_{\rho,q}$, *with norm equivalence.*

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Theorem 1.2 (FG). *Let (X_0, X_1) be a Banach couple. Then for all $p, 1 \leq p < \infty$, we have*

$$\langle L_p(X_0), L_p(X_1)\rangle_{\rho,p} = L_p(\langle X_0, X_1\rangle_{\rho,p}), \tag{1}$$

with equivalent norms.

Remark 1.3. *Using the fact that l_q^s-spaces are special L_q-spaces and the previous theorem, we have*

$$\langle l_q^s(X_0), l_q^s(X_1)\rangle_{\rho,p} = l_q^s(\langle X_0, X_1\rangle_{\rho,p}), \tag{2}$$

with the same assumption as in Theorem (1.2).

By an Orlicz function we mean a convex and continuous function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$.

Theorem 1.4 (FG). *Let Φ be an Orlicz function and let $L_\Phi(E)$ be the corresponding E-valued Orlicz space on a σ-finite measure space, where E is a Banach space. Suppose that Φ and the conjugate Φ^* satisfy the Δ_2-condition. Then there exist p_0 and p_1, with $1 < p_1 < p_0 < \infty$,*

$$\langle L_{p_0}(E), L_{p_1}(E)\rangle_{\rho,p} = L_\Phi(E), \qquad \text{where} \tag{3}$$

$$\rho(t) = \begin{cases} t^{\frac{p_1}{p_1-p_0}} \quad \Phi^{-1}(t^{\frac{p_0p_1}{p_0-p_1}}), & t > 0 \\ 0 & t = 0 \end{cases} \tag{4}$$

and $1 < p < \infty$.

Theorem 1.5. *Let Φ be an Orlicz function and let $l_\Phi(E)$ be the corresponding E-valued Orlicz space on a σ-finite measure space, where E is a Banach space. Suppose that Φ and the conjugate Φ^* satisfy the Δ_2-condition. Then there exist p_0 and p_1, with $1 < p_1 < p_0 < \infty$,*

$$\langle l_{p_0}(E), l_{p_1}(E)\rangle_{\rho,p} = l_\Phi(E) \tag{5}$$

where ρ is given by (4).

Theorem 1.6. *Let Φ be an Orlicz function which satisfies, together with its conjugate Φ^*, the Δ_2-condition. Then there exist p_0 and p_1, with $1 < p_0, p_1 < \infty$, such that*

$$\langle b_{p_0q}^s(X), b_{p_1q}^s(X)\rangle_{\rho,p} = b_{\Phi q}^s(X), \tag{6}$$

where ρ is given by (4).

Proof: Using Remark (1.3) and Theorem (1.5) we have

$$\begin{aligned}\langle b_{p_0q}^s(X), b_{p_1q}^s(X)\rangle_{\rho,p} &= \langle l_q^s(l_{p_0}(X)), l_q^s(l_{p_1}(X))\rangle_{\rho,p} \\ &= l_q^s\langle l_{p_0}(X), l_{p_1}(X)\rangle_{\rho,p} = l_q^s(l_\Phi(X)) = b_{\Phi q}^s(X)\end{aligned}$$

Theorem 1.7. *Let $1 < p, q < \infty$, and $s \in R$. Then we have,for Banach spaces X_0, X_1*

$$\langle b^s_{pq}(X_0), b^s_{pq}(X_1)\rangle_{\rho,p} = b^s_{pq}(\langle X_0, X_1\rangle_{\rho,p}). \tag{7}$$

Proof: Using Theorem (1.2) and Remark (1.3) we have

$$\langle b^s_{pq}(X_0), b^s_{pq}(X_1)\rangle_{\rho,p} = \langle l^s_q(l_p(X_0)), l^s_q(l_p(X_1))\rangle_{\rho,p}$$

$$= l^s_q(l_p)\langle X_0, X_1\rangle_{\rho,p} = b^s_{pq}\langle X_0, X_1\rangle_{\rho,p}$$

Theorem 1.8. *Let $1 < p, q, u_0, u_1, v < \infty$ and $s \in R$. Then we have*

$$\langle b^s_{pq}(b^t_{u_0 v}), b^s_{pq}(b^t_{u_1 v})\rangle_{\rho,q} = b^s_{pq}(b^t_{\Phi v}), \tag{8}$$

with,

$$\rho(\tau) = \tau^{\frac{u_1}{u_1-u_0}} \quad \Phi^{-1}(\tau^{\frac{u_0u_1}{u_0-u_1}}).$$

Proof: Using Theorems (1.6) and (1.7) we have

$$\langle b^s_{pq}(b^t_{u_0 v}(X)), b^s_{pq}(b^t_{u_1 v}(X))\rangle_{\rho,p} = b^s_{pq}\langle b^t_{u_0 v}(X), b^t_{u_1 v}(X)\rangle_{\rho,p}$$

$$= b^s_{pq}(b^t_{\Phi v}(X))$$

Definition 1.9. *Let Φ be an Orlicz function, we define for $1 \leq q < \infty$, the Lorentz-Orlicz spaces $l_{\Phi q}$ as follows*

$$l_{\Phi q} = \left\{ x = (x_n)_n \quad : \|x\|_{l_{\Phi_q}} := \inf \left(c : (\sum_{n=1}^{\infty} \Phi(\frac{|x_n|}{c})^q)^{1/q} \leq 1 \right) < \infty \right\}, \tag{9}$$

and,

$$l_{\Phi\infty} = \left\{ x = (x_n)_n \quad : \|x\|_{l_{\Phi\infty}} := \inf \left(c : \sup_n \Phi(\frac{|x_n|}{c}) \leq 1 \right) < \infty \right\}. \tag{10}$$

Definition 1.10. *Let Φ be an Orlicz function and X be a Banach space. We define for $1 \leq q \leq \infty$, the Schatten-Orlicz classes $S^s_{\Phi q}$ as follows*

$$S^s_{\Phi q}(X) := \{T \in L(X) \quad : (s_n(T))_n \in l_{\Phi q}\}. \tag{11}$$

Theorem 1.11 (EF)**.** *Let $1 < p_0, p_1 < \infty$. Then we have*

$$\langle S^s_{p_0 q}(X), S^s_{p_1 q}(X)\rangle_{\rho,p} \subset S^s_{\Phi q}(X), \tag{12}$$

where ρ is given by

$$\rho(t) = \begin{cases} t^{\frac{p_1}{p_1-p_0}} \quad \Phi^{-1}(t^{\frac{p_0p_1}{p_0-p_1}}) & ,t > 0 \\ 0 & t = 0 \end{cases} \tag{13}$$

and $1 < p < \infty$.

2. Eigenvalue distribution of integral operators

Theorem 2.1 (C). *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} > 1$ and $K \in l_p(l_q)$ i.e., $K = (k_{ij})_{i,j\in\mathbb{N}}$ satisfy*

$$\|K\| = \left\{ \sum_i \left\{ \sum_j |k_{ij}|^q \right\}^{p/q} \right\}^{1/p} \quad < \quad \infty.$$

Define s by

$$\frac{1}{s} = \begin{cases} \frac{1}{p} + \frac{1}{q} - \frac{1}{2} & , p < 2 \quad and \quad q > 2 \\ \frac{1}{p} & p \geq 2 \quad or \quad p < 2, q \leq 2 \end{cases} \tag{14}$$

Then the integral operator $T_K : l_r \to l_r$ is well defined Riesz operator for $p < r < q'$ and its eigenvalues belong to the Lorentz sequence space $l_{s,p}$ with

$$\|(\lambda_n(T_K))_{n\in\mathbb{N}}\|_{l_{s,p}} \leq c(p,q)\|K\|_{pq}.$$

Theorem 2.2 (P1). *Let $1 \leq p, q, u, v \leq \infty$ and $\alpha, \beta > 0$ with $\alpha + \beta > 1 - \frac{1}{p} - \frac{1}{u}$. Define s by*

$$\frac{1}{s} = \alpha + \beta + \begin{cases} \frac{1}{p} + \frac{1}{u} - \frac{1}{2} & , 2 \leq u \leq \infty \\ \frac{1}{p} & 1 \leq u \leq 2 \end{cases} \tag{15}$$

Then $K \in b^{\alpha}_{pq}(b^{\beta}_{uv})$ implies $(\lambda_n(K))_{n\in\mathbb{N}} \in l_{s,q}$

Theorem 2.3 (P2, KW). *Let $\frac{1}{p} + \frac{1}{q} \leq 1$. Then the eigenvalues of any kernel $K \in L_p(L_q(X))$ belong to the Lorentz sequence space $l_{q^+} = l_{\max(2,q')}$.*

Theorem 2.4. *Let $1 < p, q_i < \infty$ with $\frac{1}{p} + \frac{1}{q_i} > 1$ $(i = 0, 1)$ and $K \in l_p(l_{q_i})$ i.e., $K = (k_{ij})_{i,j\in\mathbb{N}}$ satisfy*

$$\|K\| = \left\{ \sum_i \left\{ \sum_j |k_{ij}|^{q_i} \right\}^{p/q_i} \right\}^{1/p} \quad < \quad \infty.$$

Let

$$\frac{1}{s_i} = \begin{cases} \frac{1}{p} + \frac{1}{q_i} - \frac{1}{2} & , p < 2 \quad and \quad q_i > 2 \\ \frac{1}{p} & p \geq 2 \quad or \quad p < 2, q_i \leq 2 \end{cases} \tag{16}$$

Then the integral operator $T_K : l_{\Phi^*} \to l_p$ is well defined Riesz operator for $x^p \leq \Phi^*(x) \leq x^{q_i'}$ and its eigenvalues belong to the Lorentz - Orlicz sequence space $l_{\Psi,p}$ with

$$\|(\lambda_n(T_K))_{n\in\mathbb{N}}\|_{l_{\Psi,p}} \leq c(p,q)\|K\|_{p,\Phi} \qquad where$$

$$\Phi^{-1}(\tau) = \tau^{(\frac{1}{2} - \frac{1}{p}} \quad \Phi^{-1}(\tau).$$

Proof: Using the eigenvalue Theorem (2.1), we have

$$l_p(l_{q_i}) \subset S^s_{s_i,p}. \tag{17}$$

By Theorems (1.2), (1.4), (1.11) and Corollary (1.5) we have

$$\langle l_p(l_{q_0}), l_p(l_{q_1})\rangle_{\rho,q} = l_p(l_\Phi) \subset \langle S^s_{s_0p}, S^s_{s_1p}\rangle_{\rho,q} = S^s_{\Psi p}, \tag{18}$$

with,

$$\rho(\tau) = \tau^{\frac{q_1}{q_1-q_0}} \quad \Phi^{-1}(\tau^{\frac{q_0q_1}{q_0-q_1}}). \tag{19}$$

and

$$\rho(\tau) = \tau^{\frac{s_1}{s_1-s_0}} \quad \Psi^{-1}(\tau^{\frac{s_0s_1}{s_0-t_1}}). \tag{20}$$

(i) The case $p \geq 2$ or $p < 2$, $q_i \leq 2$:
In this case we have $s_0 = s_1 = p$, hence $l_p(l_\Phi) \subset S_p$.
(ii) The case $p < 2, q_i > 2$:
In this case we have

$$\frac{1}{s_0} - \frac{1}{s_1} = \frac{1}{q_0} - \frac{1}{q_1} \Rightarrow \qquad \frac{s_0s_1}{s_0 - s_1} = \frac{q_0q_1}{q_0 - q_1} = m$$

$$\text{and} \quad \frac{s_1}{s_1 - s_0} = -\frac{m}{t_0} \quad , \quad \frac{q_1}{q_1 - q_0} = -\frac{m}{q_0}.$$

Substituting in (19) and (20) we have

$$\Phi^{-1}(\tau^m) = \tau^{m(\frac{1}{q_0} - \frac{1}{s_0}} \quad \Phi^{-1}(\tau^m) = \Phi^{-1}(\tau^m) = \tau^{m(\frac{1}{2} - \frac{1}{p}} \quad \Phi^{-1}(\tau^m).$$

Then, we readily get the desired results.

Using the same method as above we can prove the following theorems.

Theorem 2.5. *Let $1 \leq p, q, u_0, u_1, v \leq \infty$ and $\alpha, \beta > 0$ with $\alpha + \beta > 1 - \frac{1}{p} - \frac{1}{u_i}$. Define s_i by*

$$\frac{1}{s_i} = \alpha + \beta + \begin{cases} \frac{1}{p} + \frac{1}{u_i} - \frac{1}{2}, & 2 \leq u_i \leq \infty \\ \frac{1}{p} & 1 \leq u_i \leq 2 \end{cases} \tag{21}$$

Then $K \in b^\alpha_{pq}(b^\beta_{\Phi v})$ implies $(\lambda_n(K))_{n\in\mathbb{N}} \in l_{\Psi,q}$ with

$$\Phi^{-1}(t) = t^m \Psi^{-1}(t) \quad , \quad m = \frac{1}{2} - (\alpha + \beta + \frac{1}{p})$$

Theorem 2.6. *Let $1 \leq p_0, p_1, q, u, v \leq \infty$ and $\alpha, \beta > 0$ with $\alpha + \beta > 1 - \frac{1}{p_i} - \frac{1}{u}$. Define s_i by*

$$\frac{1}{s_i} = \alpha + \beta + \begin{cases} \frac{1}{p_i} + \frac{1}{u} - \frac{1}{2}, & 2 \leq u \leq \infty \\ \frac{1}{p_i} & 1 \leq u \leq 2 \end{cases} \tag{22}$$

Then $K \in b^{\alpha}_{\Phi q}(b^{\beta}_{uv})$ *implies* $(\lambda_n(K))_{n\in\mathbb{N}} \in l_{\Psi,q}$ *with*

$$\Phi^{-1}(t) = t^m \Psi^{-1}(t) \quad , \quad m = \frac{1}{2} - (\alpha + \beta + \frac{1}{u})$$

Theorem 2.7. *Let* $\frac{1}{p} + \frac{1}{q_i} \leq 1 \quad (i = 0, 1)$. *Then the eigenvalues of any kernel* $K \in L_p(L_\Phi(X))$ *belong to the Lorentz sequence space* l_Ψ *with*

$$\Phi^{-1}(t) = t\Psi^{-1}(t^{-1}) \quad .$$

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Superconformal-Like Transformations and Nonlinear Realizations

Steven Duplij

Abstract

We consider various properties of $N = 1$ superconformal-like transformations which generalize conformal transformations to supersymmetric and noninvertible case. Alternative tangent space reduction in $N = 1$ superspace leads to some new transformations which are similar to the anti-holomorphic ones of the complex function theory, which gives new odd $N = 1$ superanalog of complex structure. They are dual to the ordinary superconformal transformations subject to the Berezinian addition formula presented, noninvertible, highly degenerated and twist parity of the tangent space in the standard basis, and they also lead to some "mixed cocycle condition". A new parametrization for the superconformal group is presented which allows us to extend it to a semigroup and to unify the description of old and new transformations. The nonlinear realization of invertible and noninvertible $N = 1$ superconformal-like transformations is studied by means of the odd curve motion technique and introduced clear diagrammatic method.

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1 Introduction

More than 25 years have passed after discovery of supersymmetry [1, 2], nevertheless its influence on abstract algebraic properties of the physical theory was mostly symbolical. The main constructions of the theory after several trivial and clear modifications were equipped by the suffix “super”. Then the building of quite supersymmetric model, excluding some negligible moments, was copied step by step from the corresponding nonsupersymmetric version, and the latter had to be its continuous limit in certain extent. In spite of the great success in the finite field models and superstring unified theories [3, 4], the supersymmetry by itself did not lead to significant changings or generalizations of the theory in abstract algebraic sense. In particular, the concept of superspace allowing to unify description of the bosonic and fermionic sectors of the theory was based on introduction of the additional nilpotent coordinates [5, 6]. They were used on a par with the ordinary “bosonic” coordinates, ignoring or avoiding in some way the fact that noninvertible objects and zero divisors inevitably arisen among the main variables of the theory. Therefore, a lot of mappings and functions turned out to be noninvertible, and indeed because of this fact, however it is strange and paradoxical from the mathematical viewpoint, they were artificially excluded by hand. So it was assumed that supergroups [7] are sufficient to be a super generalization of corresponding groups. This approach was named “factorizing by nilpotents” in physics [8, 9, 10] and applied in all previous supersymmetrizing approaches [11].

However, as a matter of fact, all transformations on a set into itself or all maps of a topological space preserving a definite structure form (without invertibility demands) indeed a semigroup[1] under composition [13, 14]. The above “factorizing” procedure in the semigroup theory is well-known and is named a Rees factorization [13].

In this paper we propose to reconsider the above ansatz and try to “factorize by non-nilpotents”, i.e. to study “non-group” properties of some supersymmetric models. We suggest that a consistent from abstract algebraic viewpoint way is simultaneous transition from space to superspace and from transformation or topological groups to corresponding supersemigroups: “su-

[1]Moreover, in physical applications “... impossibility to restrict ourselves by only invertible transformations is clear”([12], p.40)

per" generalization of the physical theory should be accompanied by "semi" generalization of its mathematics as a whole. Therefore, necessary super analogs of groups are really supersemigroups, and not supergroups. Thus we can paraphrase: super-nature implies semi-symmetry (cf. [4], p. 4).

In global sense supersymmetric theory should have semigroup structure, while the observable sector at present energies can be satisfactorily described by its invertible group part [3]. Nevertheless, it should not be restricted with the investigations of the latter[2], because its properties are connected with ones of the rest (ideal) part [13, 12].

¿From another side in superstring unified theories [3, 4] a special class of reduced mappings of two-dimensional (1|1) complex superspace, namely superconformal transformations [16, 17] play main and fundamental role [18, 19]. In the local approach to super Riemann surfaces represented as collections of open superdomains the superconformal transformations are used as gluing transition functions [16, 20]. They also can be defined as a result of the special reduction of the structure supergroup [21, 22]. Here using the invertibility weakening ansatz [23] we consider an alternative tangent space reduction, which leads to possible extensions of $N = 1$ superconformal transformations and to new transformations (see also [24]). We present a new parametrization of superconformal group is which allows us to extend it to a semigroup and to unify the description of old and new transformations.

We use the functional approach to superspace [25, 26] which admits existence of nontrivial topology in odd directions [27] and can be suitable for physical applications.

2 Preliminaries

A $(p|q)$-dimensional superspace $\mathbb{K}^{p|q}$ over Λ (in the sense of [25]) is the even sector of the direct product $\Lambda_0^p \times \Lambda_1^q$, where Λ be a commutative Banach $\mathbb{Z}_2$-graded superalgebra [7, 9] over a field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}_p$) with a decomposition into the direct sum: $\Lambda = \Lambda_0 \oplus \Lambda_1$. The elements a from Λ_0 and Λ_1 are homogeneous and have the fixed even and odd parity defined as $|a| \stackrel{def}{=} \{i \in \{0, 1\} = \mathbb{Z}_2 |\, a \in \Lambda_i\}$. The elements $X := (x_1 \ldots x_p) \in \Lambda_0^p$ and $\Theta := (\theta_1 \ldots \theta_q) \in \Lambda_0^q$ serve as coordinates in the superspace $\mathbb{K}^{p|q}$.

[2]For mathematical argumentation of replies to the question "why study semigroups" see [15].

The even homomorphism $\mathfrak{m}_b : \Lambda \to \mathbb{B}$, where $\mathbb{B}$ is a purely even algebra over $\mathbb{K}$, is called a body map [25]. If there exists an embedding $\mathfrak{n} : \mathbb{B} \hookrightarrow \Lambda$ such that $\mathfrak{m}_b \circ \mathfrak{n} = id$, then Λ admits the body and soul decomposition $\Lambda = \mathbb{B} \oplus \mathbb{S}$, and a soul map can be defined as $\mathfrak{m}_s : \Lambda \to \mathbb{S}$. Usually the isomorphism $\mathbb{B} \cong \mathbb{K}$ is implied (which is not necessary in general and can lead to very nontrivial behavior of the body).

In case that Λ is a Banach algebra (with a norm $||\cdot||$) soul elements are quasinilpotent [28], which means $\forall a \in \mathbb{S}, \lim\limits_{n\to\infty} ||a||^{1/n} = 0$. But quasinilpotency of the soul elements does not necessarily lead to their nilpotency ($\forall a \in \mathbb{S}\, \exists n,\, a^n = 0$) for the infinite-dimensional case [29]. In particular, if $\Lambda = \Lambda(N)$ is a Grassmann algebra with N anticommuting generators ξ_i such that $\xi_i \xi_k = -\xi_k \xi_i\ i, k \in N$, then $l_1$norm of an element $a \in \Lambda$, $a = \sum\limits_{i\in N} c_i \xi_i$ is defined by the expression $||a|| := \sum\limits_{i\in N} |c_i|$.

These facts allow us to consider noninvertible morphisms on a par with invertible ones (in some sense), which gives, in proper conditions, many interesting and nontrivial results (see [23, 30]).

3 Superanalytic $N = 1$ transformations

Locally (1|1)-dimensional superspace $\mathbb{C}^{1|1}$ in the coordinate language is described by the pair $Z = (z, \theta)$, where z is an even coordinate and θ is an odd one. In above definition of superspace there exist soul parts in the even coordinate $z = z_{body} + z_{soul},\ z_{body} = \mathfrak{m}_b(z),\ z_{soul} \stackrel{def}{=} z - z_{body}$, where $\mathfrak{m}_b$ is the body map [25] vanishing all nilpotent generators. The body map acts on the coordinates as follows $\mathfrak{m}_b(z) = z_{body},\ \mathfrak{m}_b(\theta) = 0$. This allows one to consider non-trivial soul topology in even directions on a par with odd ones [27].

Using holomorphy $\tilde{z} = \tilde{z}(z, \theta)$, conditions a general superanalytic (SA) transformation $\mathrm{T}_{SA} : \mathbb{C}^{1|1} \to \mathbb{C}^{1|1}$ can be presented as

$$\begin{cases} \tilde{z} = \tilde{z}(z, \theta), \\ \tilde{\theta} = \tilde{\theta}(z, \theta) \end{cases} \tag{1}$$

which does not contain dependence of complex conjugated coordinates. Taking into account nilpotence of the odd coordinate $\theta^2 = 0$ we obtain

$$\begin{cases} \tilde{z} = f(z) + \theta \cdot \chi(z), \\ \tilde{\theta} = \psi(z) + \theta \cdot g(z), \end{cases} \tag{2}$$

where four component functions $f(z), g(z) : \mathbb{C}^{1|0} \to \mathbb{C}^{1|0}$ and $\psi(z), \chi(z) : \mathbb{C}^{1|0} \to \mathbb{C}^{0|1}$ satisfy supersmooth conditions generalizing C^{∞} ([25]). Here and in the following we denote even functions and variables by Latin letters and odd ones by Greek letters, and remind that point $(\cdot)$ denotes a product in underlying Grassmann algebra.

By definition the odd functions $\psi(z), \chi(z)$ are noninvertible. So the invertibility of the whole superanalytic transformations transformation (1) is controlled by the even functions $f(z), g(z)$. Usually they are chosen invertible [16]. Here we will not restrict their invertibility and consider both cases on a par.

Definition 1 *Sets of invertible and noninvertible transformations $\mathbb{C}^{1|1} \to \mathbb{C}^{1|1}$ (2) form a semigroup of superanalytic transformations $\mathcal{T}_{SA}$.*

The invertible transformations are in its subgroup, while the noninvertible ones are in an ideal [23, 30].

Definition 2 *Invertible SA transformations are defined by the conditions $\mathfrak{m}_b[f(z)] \neq 0$, $\mathfrak{m}_b[g(z)] \neq 0$.*

Definition 3 *Halfinvertible SA transformations are defined by $\mathfrak{m}_b[f(z)] = 0$, $\mathfrak{m}_b[g(z)] \neq 0$.*

Definition 4 *Noninvertible SA transformations are defined by $\mathfrak{m}_b[f(z)] = 0$.*

Remark. The halfinvertible SA transformations can be resolved only under θ and not under z.

Obviously we can use the component functions from (2) for parametrization of the semigroup of SA transformations $\mathcal{T}_{SA}$.

Definition 5 *An element $\mathbf{s}$ of a superanalytic semigroup $\mathbf{S}_{SA}$ is parametrized by the four*

$$\left\{ \begin{array}{cc} f & \chi \\ \psi & g \end{array} \right\} \stackrel{def}{=} \mathbf{s} \in \mathbf{S}_{SA}, \tag{3}$$

and the action in $\mathbf{S}_{SA}$ is

$$\left\{ \begin{array}{cc} f_1 & \chi_1 \\ \psi_1 & g_1 \end{array} \right\} * \left\{ \begin{array}{cc} f_2 & \chi_2 \\ \psi_2 & g_2 \end{array} \right\} =$$

$$\left\{ \begin{array}{ll} f_1 \circ f_2 + \psi_2 \cdot \chi_1 \circ f_2 & f_1' \circ f_2 \cdot \chi_2 + g_2 \cdot \chi_1 \circ f_2 \\ & +\chi_1' \circ f_2 \cdot \chi_2 \cdot \psi_2 \\ \psi_1 \circ f_2 + \psi_2 \cdot g_1 \circ f_2 & \psi_1' \circ f_2 \cdot \chi_2 + g_2 \cdot g_1 \circ f_2 \\ & +g_1' \circ f_2 \cdot \chi_2 \cdot \psi_2 \end{array} \right\}. \tag{4}$$

where

$$f_1 \circ f_2 = f_1 \left(f_2 \left(z \right) \right) \tag{5}$$

and the prime ($'$) *denotes differentiation by argument.*

The associativity in $\mathbf{S}_{SA}$

$$\mathbf{s}_1 * (\mathbf{s}_2 * \mathbf{s}_3) = (\mathbf{s}_1 * \mathbf{s}_2) * \mathbf{s}_3 \tag{6}$$

is not trivial for (4) and needs to be proved.

Proposition 6 *The multiplication law (4) is associative.*

Proof. The relation (6) consists of four relations corresponding to four entries in (3). Using (4) for 1-1 element we find

$$\begin{aligned} \mathbf{s}_1 * (\mathbf{s}_2 * \mathbf{s}_3) \left|_{1-1} \right. &= f_1 \circ (f_2 \circ f_3 + \psi_3 \cdot \chi_2 \circ f_3) \\ &\quad + (\psi_2 \circ f_3 + \psi_3 \cdot g_2 \circ f_3) \cdot \chi_1 \circ (f_2 \circ f_3 + \psi_3 \cdot \chi_2 \circ f_3) \,. \end{aligned}$$

Opening brackets and expanding in Taylor series and taking into account nilpotence of entries we derive

$$\begin{aligned} \mathbf{s}_1 * (\mathbf{s}_2 * \mathbf{s}_3) \left|_{1-1} \right. &= f_1 \circ f_2 \circ f_3 + \psi_3 \cdot \chi_2 \circ f_3 \cdot f_1' \circ f_2 \circ f_3 \\ &\quad +\psi_2 \circ f_3 \cdot \chi_1 \circ f_2 \circ f_3 \\ &\quad +\psi_3 \cdot g_2 \circ f_3 \cdot \chi_1 \circ f_2 \circ f_3 \\ &\quad +\psi_2 \circ f_3 \cdot \chi_1' \circ f_2 \circ f_3 \cdot \psi_3 \cdot \chi_2 \circ f_3. \end{aligned}$$

Then we group the elements in different manner and obtain

$$\begin{aligned} \mathbf{s}_1 * (\mathbf{s}_2 * \mathbf{s}_3) \left|_{1-1} \right. &= (f_1 \circ f_2 + \psi_2 \cdot \chi_1 \circ f_2) \circ f_3 \\ &\quad +\psi_3 \cdot (f_1' \circ f_2 \cdot \chi_2 + \chi_1' \circ f_2 \cdot \chi_2 \cdot \psi_2 + g_2 \cdot \chi_1 \circ f_2) \circ f_3 \\ &= (\mathbf{s}_1 * \mathbf{s}_2) * \mathbf{s}_3 |_{1-1}. \end{aligned}$$

Analogous computations can be performed for other elements, which proves the associativity of (4) and the fact that the parametrization (3) gives actually a semigroup. $\square$

Remark. The multiplication (4) contains two products: superposition (5) and product in underlying Grassmann algebra $(\cdot)$. Therefore SA semigroup belongs nor to the class of continuous functions [31, 32, 33], neither to the class of multiplicative semigroups [34, 35].

The presence of two multiplications, zero divisors and nilpotents makes the analysis of abstract properties of SA semigroup (and superconformal semigroup considered below) much more complicated as against well studied function semigroups [31, 32, 33].

Proposition 7 *A two side identity in* $\mathbf{S}_{SA}$ *is*

$$\mathbf{e} = \left\{ \begin{array}{cc} z & 0 \\ 0 & 1 \end{array} \right\} \tag{7}$$

and a two side zero is the matrix (3) having zero entries.

Proof. It can be easily checked using (4). □

Consider the homomorphism φ of SA semigroup $\mathbf{S}_{SA}$ on the semigroup of superanalytic transformations $\mathcal{T}_{SA}$, i.e. $\varphi : \mathbf{S}_{SA} \rightarrow \mathcal{T}_{SA}$.

Proposition 8 *As it should be* $\ker \varphi = \mathbf{e}$.

In studies of supernumber systems containing zero divisors and nilpotents one usually says the magic words "factorizing by nilpotents" or "modulo nilpotents" and forget about additional exotic properties arising from thorough consideration of the latter. In function systems under study the situation is more delicate and needs additional abstract investigations.

For instance, in SA semigroup $\mathbf{S}_{SA}$ along the standard $\mathbf{e}$ and $\mathbf{z}$ we are able to introduce element dependent "local" identities and zeroes.

Definition 9 *For a given element* $\mathbf{s}$ *of SA semigroup local left, right and two sided identities* $\mathbf{e}_{\mathbf{s}}^{left}, \mathbf{e}_{\mathbf{s}}^{right}, \mathbf{e}_{\mathbf{s}} \in \mathbf{S}_{SA}$ *are defined by*

$$\mathbf{e}_{\mathbf{s}}^{left} * \mathbf{s} = \mathbf{s}, \tag{8}$$

$$\mathbf{s} * \mathbf{e}_{\mathbf{s}}^{right} = \mathbf{s}, \tag{9}$$

$$\mathbf{e}_{\mathbf{s}} * \mathbf{s} * \mathbf{e}_{\mathbf{s}} = \mathbf{s}. \tag{10}$$

Definition 10 *For a given element* $\mathbf{s}$ *of SA semigroup local left, right and two sided zeroes* $\mathbf{z}_{\mathbf{s}}^{left}, \mathbf{z}_{\mathbf{s}}^{right}, \mathbf{z}_{\mathbf{s}} \in \mathbf{S}_{SA}$ *are defined by*

$$\mathbf{z}_{\mathbf{s}}^{left} * \mathbf{s} = \mathbf{z}_{\mathbf{s}}^{left}, \tag{11}$$

$$\mathbf{s} * \mathbf{z}_{\mathbf{s}}^{right} = \mathbf{z}_{\mathbf{s}}^{right}, \tag{12}$$

$$\mathbf{z}_{\mathbf{s}} * \mathbf{s} * \mathbf{z}_{\mathbf{s}} = \mathbf{z}_{\mathbf{s}}. \tag{13}$$

The local identities and zeroes are sets of elements from $\mathbf{S}_{SA}$ and can be found from corresponding systems of functional differential equations, e.g. for $\mathbf{e}_{\mathbf{s}}^{left}$ from (8) in component form we have

$$\begin{array}{c} f_1 \circ f_2 + \psi_2 \cdot \chi_1 \circ f_2 = f_2, \\ \psi_1 \circ f_2 + \psi_2 \cdot g_1 \circ f_2 = \psi_2, \\ f_1' \circ f_2 \cdot \chi_2 + \chi_1' \circ f_2 \cdot \chi_2 \cdot \psi_2 + g_2 \cdot \chi_1 \circ f_2 = \chi_2, \\ \psi_1' \circ f_2 \cdot \chi_2 + g_1' \circ f_2 \cdot \chi_2 \cdot \psi_2 + g_2 \cdot g_1 \circ f_2 = g_2. \end{array} \tag{14}$$

Example. Let $\mathbf{s} = \left\{ \begin{array}{cc} z^2 & \beta \\ \alpha & z^{-1} \end{array} \right\}$, then $\mathbf{e}_{\mathbf{s}}^{left} = \left\{ \begin{array}{cc} z^2 & \beta \\ \alpha & z^{-1} \end{array} \right\}$.

To stress the difference from the function semigroup case we consider the left zeroes. From the multiplication law (5) it follows that for function semigroups the role of left zeroes play constant mappings

$$f_0(z) : z \to c_f = const. \tag{15}$$

because $\forall g(z)\,,\ f_0 \circ g = f_0(g(z)) = c_f = f_0$. Let us take an element $\mathbf{s}_0$ of SA semigroup having analogous to (15), i.e.

$$\mathbf{s}_0 = \left\{ \begin{array}{cc} f_0 & \chi_0 \\ \psi_0 & g_0 \end{array} \right\}. \tag{16}$$

Then from (4) we have

$$\mathbf{s}_0 * \mathbf{s} = \left\{ \begin{array}{cc} f_0 & \chi_0 \\ \psi_0 & g_0 \end{array} \right\} * \left\{ \begin{array}{cc} f & \chi \\ \psi & g \end{array} \right\} = \left\{ \begin{array}{cc} c_f + c_\chi \cdot g & c_\chi \cdot g \\ c_\psi + c_g \cdot \psi & c_g \cdot g \end{array} \right\}, \tag{17}$$

and so $\mathbf{s}_0 * \mathbf{s} \neq const$ as opposite to the function semigroup case [36, 33].

While comparing the SA multiplication (4) with matrix semigroup multiplication [37, 38], we notice that the set of lower triangle supermatrices (3), i.e. elements having $\chi = 0$, form a subsemigroup as usual, however the set of upper triangle ones having $\psi = 0$ does not form any subsemigroup due to presence of the middle term in 2-2 element of (4).

By means of SA transformations (2) one can construct a superanalytic supermanifold M_{SA} in the standard manner ([25, 8]). The component functions play the role of gluing transition functions. Thus, let $M_{SA} = \cup U_\alpha$, where U_α are superdomains covering M_{SA}. Its structure is fully determined by four transition functions $f_{\alpha\beta}\left(z_\beta\right), \chi_{\alpha\beta}\left(z_\beta\right), g_{\alpha\beta}\left(z_\beta\right), \psi_{\alpha\beta}\left(z_\beta\right)$ describing SA transformation $Z_\beta \to Z_\alpha$ on the intersection $U_\alpha \cap U_\beta$.

Proposition 11 *On triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$ the transition functions of SA supermanifold satisfy the consistency conditions*

$$\begin{array}{rcl} f_{\alpha\gamma} & = & f_{\alpha\beta} \circ f_{\beta\gamma} + \psi_{\beta\gamma} \cdot \chi_{\alpha\beta} \circ f_{\beta\gamma}, \\ \chi_{\alpha\gamma} & = & f'_{\alpha\beta} \circ f_{\beta\gamma} \cdot \chi_{\beta\gamma} + g_{\beta\gamma} \cdot \chi_{\alpha\beta} \circ f_{\beta\gamma} + \chi'_{\alpha\beta} \circ f_{\beta\gamma} \cdot \chi_{\beta\gamma} \cdot \psi_{\beta\gamma}, \\ g_{\alpha\gamma} & = & f'_{\alpha\beta} \circ f_{\beta\gamma} \cdot \chi_{\beta\gamma} + g_{\beta\gamma} \cdot g_{\alpha\beta} \circ f_{\beta\gamma} + g'_{\alpha\beta} \circ f_{\beta\gamma} \cdot \chi_{\beta\gamma} \cdot \psi_{\beta\gamma}, \\ \psi_{\alpha\gamma} & = & \psi_{\alpha\beta} \circ f_{\beta\gamma} + \psi_{\beta\gamma} \cdot g_{\alpha\beta} \circ f_{\beta\gamma}. \end{array} \tag{18}$$

Proof. Immediately follows from (4). □

3.1 Super Jacobian

Here we introduce an analog of Berezinian (super Jacobian) for noninvertible transformations.

Let us express the SA transformation (1) in the form of composition

$$1) \left\{ \begin{array}{rcl} \tilde{z} & = & F\left(z, \tilde{\theta}\right), \\ \tilde{\theta} & = & \tilde{\theta}, \end{array} \right. \qquad 2) \left\{ \begin{array}{rcl} z & = & z, \\ \tilde{\theta} & = & \tilde{\theta}\left(z, \theta\right), \end{array} \right. \tag{19}$$

where $F\left(z, \tilde{\theta}\right) = \tilde{z}\left(z, \theta\right)$. The super Jacobian of the first transformation is simply $J_1 = \partial F / \partial z$. If

$$\epsilon \left[\frac{\partial \tilde{\theta}}{\partial \theta} \right] \neq 0, \tag{20}$$

then, taking into account that θ is odd, we find $J_2 = \left(\partial \tilde{\theta} / \partial \theta\right)^{-1}$ [7]. So the total super Jacobian is

$$J_{SA} = J_1 J_2 = \frac{\partial F}{\partial z} \cdot \left(\frac{\partial \tilde{\theta}}{\partial \theta} \right)^{-1}. \tag{21}$$

To derive J_1 we can write $F\left(z,\tilde{\theta}\right)=\tilde{z}\left(z,\theta\left(z,\tilde{\theta}\right)\right)$, then we differentiate $\tilde{z}\left(z,\theta\left(z,\tilde{\theta}\right)\right)$ as a composite function

$$\frac{\partial F}{\partial z}=\frac{\partial \tilde{z}}{\partial z}+\frac{\partial \tilde{z}}{\partial \theta}\cdot\frac{\partial \theta}{\partial \tilde{\theta}}\cdot\frac{\partial \tilde{\theta}}{\partial z}. \tag{22}$$

Thus we obtain the full super Jacobian

$$J_{SA}=\frac{\frac{\partial \tilde{z}}{\partial z}-\frac{\partial \tilde{\theta}}{\partial z}\cdot\frac{\partial \theta}{\partial \tilde{\theta}}\cdot\frac{\partial \tilde{z}}{\partial \theta}}{\frac{\partial \tilde{\theta}}{\partial \theta}} \tag{23}$$

without the condition of invertibility of the transformation, i.e. without the standard requirement $\mathfrak{m}_b\left[\partial\tilde{z}/\partial z\right]\neq 0$ [9]. Nevertheless, in [7] it was shown that the expression of kind 23 (in matrix algebra) can be extended on the case $\mathfrak{m}_b\left[\partial\tilde{z}/\partial z\right]=0$ (halfinvertible in our classification).

Proposition 12 *Formula (23) gives a super Jacobian for invertible and halfinvertible SA transformations.*

Proof. From (2) we obtain

$$\frac{\partial \tilde{z}}{\partial z}=f'\left(z\right)+\theta\cdot\chi'\left(z\right), \tag{24}$$

$$\frac{\partial \tilde{\theta}}{\partial \theta}=g\left(z\right), \tag{25}$$

and therefore

$$\mathfrak{m}_b\left[\frac{\partial \tilde{z}}{\partial z}\right]=\mathfrak{m}_b\left[f'\left(z\right)\right]=\mathfrak{m}_b\left[f\left(z\right)\right],$$

$$\mathfrak{m}_b\left[\frac{\partial \tilde{\theta}}{\partial \theta}\right]=\mathfrak{m}_b\left[g\left(z\right)\right],$$

so according to the Definitions 2 and 3 the condition (20) covers invertible and halfinvertible transformations. □

Corollary 13 *For invertible and halfinvertible SA transformations we have*

$$J_{SA}^{inv,halfinv}=\mathrm{Ber}\left(\tilde{Z}/Z\right) \tag{26}$$

with

$$\mathrm{Ber}\,\left(\tilde{Z}/Z\right)=\mathrm{Ber}\,P_{SA}^{0}, \tag{27}$$

where

$$P_{SA}^{0}=\left(\begin{array}{cc}\frac{\partial\tilde{z}}{\partial z} & \frac{\partial\tilde{\theta}}{\partial z}\\ \frac{\partial\theta}{\partial z} & \frac{\partial\theta}{\partial\theta}\end{array}\right). \tag{28}$$

In the noninvertible case when (20) does not satisfied we cannot use (21) and (22), and the relation (26) is not more valid. So we are forced to extend the definitions (see also [39] for Jacobians of nonsupersymmetric nilpotent mappings). The Jacobian J_1 should be found from

$$J_1^{noninv}\cdot\frac{\partial\tilde{\theta}}{\partial\theta}=\frac{\partial\tilde{z}}{\partial z}\cdot\frac{\partial\tilde{\theta}}{\partial\theta}+\frac{\partial\tilde{z}}{\partial\theta}\cdot\frac{\partial\tilde{\theta}}{\partial z}, \tag{29}$$

and therefore instead of (23) and (26) we have

Definition 14 *The super Jacobian of noninvertible SA transformations is defined by the equation*

$$J_{SA}^{noninv}\cdot\left(\frac{\partial\tilde{\theta}}{\partial\theta}\right)^{2}=\frac{\partial\tilde{z}}{\partial z}\cdot\frac{\partial\tilde{\theta}}{\partial\theta}+\frac{\partial\tilde{z}}{\partial\theta}\cdot\frac{\partial\tilde{\theta}}{\partial z}. \tag{30}$$

Here the condition (20) is not necessary more . To find J_1^{noninv} and J_{SA}^{noninv} one should solve thoroughly the equations (29) and (30) (i.e. by expanding both sides in series of Grassmann algebra generators).

In terms of the component functions the super Jacobian of halfinvertible SA transformations (i.e. if $\mathfrak{m}_b\left[g\left(z\right)\right]\neq 0$) has the form

$$J_{SA}=\frac{f'\left(z\right)}{g\left(z\right)}+\frac{\chi\left(z\right)\cdot\psi'\left(z\right)}{g^{2}\left(z\right)}+\theta\left(\frac{\chi\left(z\right)}{g\left(z\right)}\right)' \tag{31}$$

which coincides with the Berezinian for the invertible and halfinvertible transformations. In case of noninvertible transformations we should use the following equation

$$\begin{aligned} J_{SA}^{noninv}\cdot g^{2}\left(z\right) &= f'\left(z\right)\cdot g\left(z\right)+\chi\left(z\right)\cdot\psi'\left(z\right)\\ &\quad+\theta\left(\chi'\left(z\right)\cdot g\left(z\right)-\chi\left(z\right)\cdot g'\left(z\right)\right) \end{aligned} \tag{32}$$

which can be solved by special methods dealing with nilpotents ([28, 39]).

Corollary 15 *For invertible SA transformations the Berezinian exists and invertible (*$\mathfrak{m}_b[f(z)] \neq 0,\ \mathfrak{m}_b[g(z)] \neq 0$*), for halfinvertible the Berezinian exists and noninvertible, for noninvertible SA transformations (*$\mathfrak{m}_b[f(z)] = 0$*) we are able to exploit the super Jacobian* J_{SA} *only.*

To classify all SA transformations we should introduce some numerical characteristic of noninvertibility.

Definition 16 *Noninvertibility index of SA transformation is defined by*

$$\operatorname{ind} J_{SA} \stackrel{def}{=} \left\{ n \in \mathbb{N} \mid J_{SA}^{n} = 0,\ J_{SA}^{n-1} \neq 0 \right\}. \tag{33}$$

Obviously that indeed the inverse variable gives numerical measure of noninvertibility.

Definition 17 *Noninvertibility degree of SA transformation is*

$$m \stackrel{def}{=} \frac{1}{\operatorname{ind} J_{SA}}. \tag{34}$$

Corollary 18 *Invertible SA transformations have* $\operatorname{ind} J_{SA} = \infty$ *and* $m = 0$.

We exclude from consideration the trivial case $J_{SA} = 0$.

Corollary 19 *The "most noninvertible" SA transformations have* $\operatorname{ind} J_{SA} = 2$ *and* $m = 1/2$.

3.2 Tangent superspace and its reduction

Let we consider action of noninvertible transformation in some analog of tangent superspace and consequences following from that. Among the latter there are new noninvertible transformations which are dual in some sense to the superconformal ones [23, 40]. The invertible case is studied in detail [16, 41, 4], therefore we concentrate our attention on new features connected with noninvertibility, trying simplify the consideration for clarity.

The tangent superspace in $\mathbb{C}^{1|1}$ is defined by the standard supersymmetric basis $\{\partial,\ D\}$, where $D = \partial_\theta + \theta\partial,\ \partial_\theta = \partial/\partial\theta,\ \partial = \partial/\partial z$. The dual cotangent space is spanned by 1-forms $\{dZ,\ d\theta\}$, where $dZ = dz + \theta d\theta$ (the signs as in [16]). In these notations the supersymmetry relations are $D^2 = \partial,\ dZ^2 = dz$.

The semigroup of SA transformations $\mathcal{T}_{SA}$ acts in tangent and cotangent superspaces by means of the tangent space matrix P_{SA} as

$$\begin{pmatrix} \partial \\ D \end{pmatrix} = P_{SA} \begin{pmatrix} \tilde{\partial} \\ \tilde{D} \end{pmatrix}$$

and

$$\begin{pmatrix} d\tilde{Z}, & d\tilde{\theta} \end{pmatrix} = \begin{pmatrix} dZ, & d\theta \end{pmatrix} P_{SA},$$

where

$$P_{SA} = \begin{pmatrix} \partial\tilde{z} - \partial\tilde{\theta}\cdot\tilde{\theta} & \partial\tilde{\theta} \\ D\tilde{z} - D\tilde{\theta}\cdot\tilde{\theta} & D\tilde{\theta} \end{pmatrix}. \tag{35}$$

Proposition 20 *The exterior differential $d = dZ\partial + d\theta D$ is invariant under SA transformations.*

Proof. We have

$$\begin{aligned} d &= \begin{pmatrix} dZ, & d\theta \end{pmatrix} \begin{pmatrix} \partial \\ D \end{pmatrix} = \begin{pmatrix} dZ, & d\theta \end{pmatrix} P_{SA} \begin{pmatrix} \tilde{\partial} \\ \tilde{D} \end{pmatrix} \\ &= \begin{pmatrix} d\tilde{Z}, & d\tilde{\theta} \end{pmatrix} \begin{pmatrix} \tilde{\partial} \\ \tilde{D} \end{pmatrix} = \tilde{d} \end{aligned} \tag{36}$$

□

Remark. We note that in (36) the invertibility is not used.

Proposition 21 $\mathrm{Ber}\left(\tilde{Z}/Z\right) = \mathrm{Ber}\, P_{SA}$.

Proof. We observe that

$$\begin{pmatrix} \frac{\partial\tilde{z}}{\partial z} & \frac{\partial\tilde{\theta}}{\partial z} \\ \frac{\partial\theta}{\partial z} & \frac{\partial\tilde{\theta}}{\partial\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\theta & 1 \end{pmatrix} \cdot \begin{pmatrix} \partial\tilde{z} - \partial\tilde{\theta}\cdot\tilde{\theta} & \partial\tilde{\theta} \\ D\tilde{z} - D\tilde{\theta}\cdot\tilde{\theta} & D\tilde{\theta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tilde{\theta} & 1 \end{pmatrix}. \tag{37}$$

Then from (37), (27), (28) and (35) it follows

$$\begin{aligned} \mathrm{Ber}\left(\tilde{Z}/Z\right) &= \mathrm{Ber}\, P_{SA}^{0} = \mathrm{Ber}\left(\begin{pmatrix} 1 & 0 \\ -\theta & 1 \end{pmatrix} \cdot P_{SA} \cdot \begin{pmatrix} 1 & 0 \\ \tilde{\theta} & 1 \end{pmatrix}\right) \\ &= \mathrm{Ber} \begin{pmatrix} 1 & 0 \\ -\theta & 1 \end{pmatrix} \cdot \mathrm{Ber}\, P_{SA} \cdot \mathrm{Ber} \begin{pmatrix} 1 & 0 \\ \tilde{\theta} & 1 \end{pmatrix} \\ &= \mathrm{Ber}\, P_{SA}. \end{aligned}$$

□

In case of invertible SA transformations the matrix P_{SA} defines the structure of a supermanifold for which these transformations play the part of transition functions. Therefore different reductions of the matrix P_{SA} give us various additional structures, but only one of them is usually considered [21], because only one of them can be invertible.

Taking into account noninvertibility we analyze *all* of them via vanishing *every* of the P_{SA} entries in turn, which gives in general 4 possibilities:

$$1)\ D\tilde{\theta} = 0, \tag{38}$$

$$2)\ \partial\tilde{\theta} = 0, \tag{39}$$

$$3)\ \Delta \equiv D\tilde{z} - D\tilde{\theta}\cdot\tilde{\theta} = 0, \tag{40}$$

$$4)\ Q \equiv \partial\tilde{z} - \partial\tilde{\theta}\cdot\tilde{\theta} = 0. \tag{41}$$

which are arranged according to the increasing of their nontriviality. The first two cases (38) and (39) are most simple, but they have some interesting peculiarities and will be considered separately.

4 Superconformal-like transformations

We here consider two other possible reductions (40) and (41) together.

In [24] it was shown that there exist two nontrivial reductions of any supermatrix (not one, triangle, as in the invertible case). We apply this result to P_{SA} (35).

Assertion 22 *The condition*

$$\mathfrak{m}_b\left[D\tilde{\theta}\right] \neq 0 \tag{42}$$

coincides with halfinvertibility of the SA transformation (3).

Proof. Indeed we observe from (37) that the Berezinian can be presented in two additive parts

$$\operatorname{Ber} P_A = \frac{\partial\tilde{z} - \partial\tilde{\theta}\cdot\tilde{\theta}}{D\tilde{\theta}} + \frac{\left(D\tilde{z} - D\tilde{\theta}\cdot\tilde{\theta}\right)\partial\tilde{\theta}}{\left(D\tilde{\theta}\right)^2} = \frac{Q}{D\tilde{\theta}} + \frac{\Delta\cdot\partial\tilde{\theta}}{\left(D\tilde{\theta}\right)^2}. \tag{43}$$

only if $\mathfrak{m}_b\left[D\tilde{\theta}\right]\neq 0$. Then from the component form (2) we derive $D\tilde{\theta}=g\left(z\right)+\theta\cdot\psi\left(z\right)$ and so $\mathfrak{m}_b\left[D\tilde{\theta}\right]=\mathfrak{m}_b\left[g\left(z\right)\right]$, therefore $\epsilon\left[D\tilde{\theta}\right]\neq 0\Rightarrow\epsilon\left[g\left(z\right)\right]\neq 0$ which is really the halfinvertibility condition (3). □

Proposition 23 *The Berezinian of superanalytic transformations in case* $D\tilde{\theta}\neq 0$ *is described by the formula*

$$\mathrm{Ber}\,P_{SA}=D\left(\frac{D\tilde{z}}{D\tilde{\theta}}\right). \tag{44}$$

Proof. After differentiating of the right hand side and using $D^2=\partial$ we derive

$$D\left(\frac{D\tilde{z}}{D\tilde{\theta}}\right)=\frac{\partial\tilde{z}\cdot D\tilde{\theta}+D\tilde{z}\cdot\partial\tilde{\theta}}{\left(D\tilde{\theta}\right)^2}=\frac{\partial\tilde{z}\cdot D\tilde{\theta}+\tilde{\theta}\cdot\partial\tilde{\theta}\cdot D\tilde{\theta}-\tilde{\theta}\cdot\partial\tilde{\theta}\cdot D\tilde{\theta}+D\tilde{z}\cdot\partial\tilde{z}}{\left(D\tilde{\theta}\right)^2}=$$

$$\frac{\left(\partial\tilde{z}+\tilde{\theta}\cdot\partial\tilde{\theta}\right)\cdot D\tilde{\theta}-\left(D\tilde{z}-\tilde{\theta}\cdot D\tilde{\theta}\right)\cdot\partial\tilde{\theta}}{\left(D\tilde{\theta}\right)^2}=\frac{Q\cdot D\tilde{\theta}-\Delta\cdot\partial\tilde{\theta}}{\left(D\tilde{\theta}\right)^2},$$

which coincides with (43). □

Using the Berezinian addition theorem [24] we obtain the formula

$$\mathrm{Ber}\,P_A=\mathrm{Ber}\,P_S+\mathrm{Ber}\,P_T, \tag{45}$$

where

$$P_S\overset{def}{=}\begin{pmatrix}\partial\tilde{z}-\partial\tilde{\theta}\cdot\tilde{\theta} & \partial\tilde{\theta}\\ 0 & D\tilde{\theta}\end{pmatrix}=\begin{pmatrix}Q & \partial\tilde{\theta}\\ 0 & D\tilde{\theta}\end{pmatrix}, \tag{46}$$

$$P_T\overset{def}{=}\begin{pmatrix}0 & \partial\tilde{\theta}\\ D\tilde{z}-D\tilde{\theta}\cdot\tilde{\theta} & D\tilde{\theta}\end{pmatrix}=\begin{pmatrix}0 & \partial\tilde{\theta}\\ \Delta & D\tilde{\theta}\end{pmatrix}. \tag{47}$$

Denote sets of the matrices (46) and (47) by $\mathbf{P}_S$ and $\mathbf{P}_T$ respectively. We stress that till now there are no conditions imposed on the transformations and they are superanalytic (1).

Superconformal-like transformations arise when we project the Berezinian on one of the terms in (43) or the set of superanalytic matrices $\mathbf{P}_{SA}$ on $\mathbf{P}_S$ or $\mathbf{P}_T$. It is seen that there exit two kinds of superconformal-like transformations.

Definition 24 *Invertible, halfinvertible and noninvertible superconformal transformations (SCf) are defined by the condition*

$$\Delta = D\tilde{z} - D\tilde{\theta} \cdot \tilde{\theta} = 0. \tag{48}$$

Definition 25 *Halfinvertible and noninvertible transformations twisting parity of tangent space*[3] *(TPt) are defined by*

$$Q = \partial\tilde{z} - \partial\tilde{\theta} \cdot \tilde{\theta} = 0. \tag{49}$$

If we apply the conditions (49) and (48) to the matrices P_S and P_T we derive

$$P_{SCf} \stackrel{def}{=} P_S|_{\Delta=0} = \begin{pmatrix} Q_{SCf} & \partial\tilde{\theta} \\ 0 & D\tilde{\theta} \end{pmatrix}, \tag{50}$$

$$P_{TPt} \stackrel{def}{=} P_T|_{Q=0} = \begin{pmatrix} 0 & \partial\tilde{\theta} \\ \Delta_{TPt} & D\tilde{\theta} \end{pmatrix}, \tag{51}$$

where $Q_{SCf} = Q|_{\Delta=0}$ and $\Delta_{TPt} = \Delta|_{Q=0}$.

The condition $\Delta = 0$ (48) gives us in the invertible case the ordinary superconformal (SCf) transformations T_{SCf} [16] and the reduced matrix P_{SCf} (50) is a result of the standard reduction of structure supergroup (see e.g. [21]). Another condition $Q = 0$ (49) leads to the degenerated noninvertible transformations T_{TPt} twisting parity of the standard tangent space (TPt) (see [23] and below). The alternative reduction [24] of the tangent space supermatrix P_A gives us the antitriangle supermatrix P_{TPt} (51). The dual role of SCf and TPt transformations is clearly seen from the Berezinian addition theorem (45) (see [24]) and the projections (50) and (51). Since SCf transformations can be viewed as a superanalog of complex structure [42, 43] , we can treat TPt transformations as another odd $N = 1$ superanalog of complex structure in a certain extent.

Remark. It is more natural to call TPt transformations anti-SCf transformations due to the following analogy with the nonsupersymmetric case. For an ordinary 2×2 matrix $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we obviously have the following identity $\det P = \det \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + \det \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = \det P_{Diag} + \det P_{Antidiag}$, which can

[3] *The reason of such name will be seen below.*

be called a "determinant addition formula". In the complex function theory the first matrix describes the tangent space matrix of holomorphic mappings and the second one—of antiholomorphic mappings. In supersymmetric case the supermatrices P_S and P_T play the role similar to one of the nonsupersymmetric diagonal and antidiagonal matrices in ordinary theory as it is seen from (45). Therefore, if P_{SCf} generalizes the tangent space matrix of holomorphic mappings, supermatrices P_{TPt} could be considered as respective generalization for antiholomorphic mappings.

Corollary 26 *Evidently,*

$$\operatorname{Ber} P_S|_{Q=0} = \operatorname{Ber} P_T|_{\Delta=0} = \operatorname{Ber} \begin{pmatrix} 0 & \partial\tilde{\theta} \\ 0 & D\tilde{\theta} \end{pmatrix} = 0. \tag{52}$$

Using this relation together with (50) and (51) we can project the Berezinian addition equality (45) on the superconformal-like transformations T_{SCf} and T_{TPt} as follows

$$\begin{aligned} \operatorname{Ber} P_A &= \begin{cases} \operatorname{Ber} P_S + \operatorname{Ber} P_T, & \Delta = 0, \\ \operatorname{Ber} P_S + \operatorname{Ber} P_T, & Q = 0. \end{cases} = \\ \begin{cases} \operatorname{Ber} P_{SCf} + 0, \\ 0 + \operatorname{Ber} P_{TPt}, \end{cases} &= \begin{cases} \operatorname{Ber} P_{SCf}, & (\mathrm{SCf}\ ) \\ \operatorname{Ber} P_{TPt}, & (\mathrm{TPt}\ ) \end{cases} \end{aligned} \tag{53}$$

After corresponding projections for Q and Δ we have

$$Q_{SCf} = \left(\partial\tilde{z} - \partial\tilde{\theta}\cdot\tilde{\theta}\right)|_{\Delta=0} = \left(D\tilde{\theta}\right)^2, \tag{54}$$

$$\Delta_{TPt} = \left(D\tilde{z} - D\tilde{\theta}\cdot\tilde{\theta}\right)|_{Q=0} = \partial_\theta\tilde{z} - \partial_\theta\tilde{\theta}\cdot\tilde{\theta}. \tag{55}$$

It is remarkable to notice the similarity of the formulas (54) and (55), which proves us once more the duality between SCf and TPt transformations.

Using (54) one obtains [21]

$$P_{SCf} = \begin{pmatrix} \left(D\tilde{\theta}\right)^2 & \partial\tilde{\theta} \\ 0 & D\tilde{\theta} \end{pmatrix}. \tag{56}$$

If $\mathfrak{m}_b \left[D\tilde{\theta}\right] \neq 0$ then Ber P_{SCf} can be simply determined from (56), and it is [16]

$$\operatorname{Ber} P_{SCf} = D\tilde{\theta}. \tag{57}$$

In noninvertible case $\mathfrak{m}_b \left[D\tilde{\theta}\right] = 0$ the Berezinian cannot be defined, but we can accept (57) as a definition of the Jacobian of noninvertible SCf transformations (see [23, 44]).

Definition 27 *The Berezinian of noninvertible SCf transformations is*

$$\operatorname{Ber} P_{SCf}^{noninv} = D\tilde{\theta}. \tag{58}$$

¿From (55) we derive

$$P_{TPt} = \begin{pmatrix} 0 & \partial\tilde{\theta} \\ \Delta_{TPt} & D\tilde{\theta} \end{pmatrix} \tag{59}$$

(cf. (46)). If $\epsilon \left[D\tilde{\theta}\right] \neq 0$ the Berezinian of P_{TPt} can be determined as

$$\operatorname{Ber} P_{TPt} = \frac{\Delta_{TPt} \cdot \partial\tilde{\theta}}{\left(D\tilde{\theta}\right)^2}. \tag{60}$$

¿From (55) it follows that $D\Delta_{TPt} = -\left(D\tilde{\theta}\right)^2$, and therefore $\partial\Delta_{TPt} = -2 \cdot D\tilde{\theta} \cdot \partial\tilde{\theta}$, which gives

$$\operatorname{Ber} P_{TPt} = \frac{\partial\Delta_{TPt} \cdot \Delta_{TPt}}{2\left(D\tilde{\theta}\right)^3}. \tag{61}$$

Since Δ_{TPt} is odd and so nilpotent, Ber P_{TPt} is also nilpotent and pure soul.

We observe that the even and odd superfunctions $Q = Q(z,\theta)$ and $\Delta = \Delta(z,\theta)$ play an important role in possible reductions of superanalytic structure, and therefore it is worth to study them in more detail. A general relation between Q and Δ is

$$Q - D\Delta = \left(D\tilde{\theta}\right)^2. \tag{62}$$

¿From it and (43) we derive another useful formula for the Berezinian of general SA transformations (if $\epsilon\left[D\tilde{\theta}\right]\neq 0$)

$$\operatorname{Ber} P_{SA}=D\tilde{\theta}+D\left(\frac{\Delta}{D\tilde{\theta}}\right)=D\left(\tilde{\theta}+\frac{\Delta}{D\tilde{\theta}}\right) \tag{63}$$

in which the SCf condition $\Delta=0$ is seen manifestly.

5 Degenerated transformations

Let us consider the intersection $\mathbf{P}_D=\mathbf{P}_S\cap\mathbf{P}_T$ which is a set of the degenerated matrices P_D of the form

$$P_D\stackrel{def}{=}\left(\begin{array}{cc}0 & \partial\tilde{\theta}\\ 0 & D\tilde{\theta}\end{array}\right), \tag{64}$$

which depend on the odd coordinate θ transformation only. The degenerated matrix of the shape (64) can be obtained by projection from P_S and P_T matrices. It means that, if the transformation of the odd sector (second line in (2)) is given, i.e. the functions $\psi(z)$ and $g(z)$ are fixed, the conditions (49) and (48) determine behavior of the even sector (functions $f(z)$ and $\chi(z)$). In this case, since the degenerated matrix P_D depends on the odd sector transformation only, we obtain

$$P_D=P_S|_{Q=0}=P_T|_{\Delta=0}. \tag{65}$$

If $D\tilde{\theta}=0$, then $\tilde{\theta}=\alpha=const$ and also $\partial\tilde{\theta}=D\left(D\tilde{\theta}\right)=0$, therefore the odd sector becomes degenerated being a left zero and constant mapping similarly to 15 (such mappings form restrictive semigroups, see e.g. [45]).

Nevertheless, the whole SA transformation (2) is not a left zero due to (17) and has the following form

$$\left\{\begin{array}{rcl}\tilde{z} & = & f(z)+\theta\cdot\chi(z),\\ \tilde{\theta} & = & \alpha.\end{array}\right. \tag{66}$$

These transformations are noninvertible (due to degenerated odd sector) and form a semigroup Deg

It is remarkable that under the degenerated (Deg) transformations defined by (65) the both cocycle relations hold valid simultaneously. Also, Deg transformations form a subsemigroup $\mathcal{T}_{Deg}$ in $\mathcal{T}_{SA}$, because of $\mathbf{P}_D \cdot \mathbf{P}_D \subseteq \mathbf{P}_D$. Moreover, $\mathcal{T}_{Deg}$ is an ideal in $\mathcal{T}_{SA}$, $\mathcal{T}_{SCf}$ and $\mathcal{T}_{TPt}$ since $\mathbf{P}_D \cdot \mathbf{P}_A \subseteq \mathbf{P}_D$, $\mathbf{P}_D \cdot \mathbf{P}_S \subseteq \mathbf{P}_D$ and $\mathbf{P}_D \cdot \mathbf{P}_T \subseteq \mathbf{P}_D$. The degenerated transformations are characterized by one odd function $\psi(z)$ only and by the absence of the θ-dependence of the transformation $Z \to \tilde{Z}$ (see (55)), so that

$$\begin{cases} \tilde{z}_{Deg} = f(z)\,, \\ \tilde{\theta}_{Deg} = \psi(z)\,, \end{cases} \tag{67}$$

where $f'(z) = \psi'(z)\,\psi(z)$. The multiplication in $\mathcal{T}_{Deg}$ coincides with the second row of (69).

6 Alternative parametrization

The reduction conditions (49) and (48) fix 2 of 4 component functions form (2) in each case. Usually [16] SCf transformations T_{SCf} are parametrized by $\begin{pmatrix} f \\ \psi \end{pmatrix}$, while other functions are found from (49) and (48). However, the latter can be done for invertible transformations only. To avoid this difficulty we introduce an alternative parametrization by the pair $\begin{pmatrix} g \\ \psi \end{pmatrix}$, which allows us to consider SCf and TPt transformations in a unified way and include noninvertibility. Indeed, fixing $g(z)$ and $\psi(z)$ we find for other component functions of (2) the equations

$$\begin{cases} f_n'(z) &= \psi'(z)\,\psi(z) + \frac{1+n}{2} g^2(z)\,, \\ \chi_n'(z) &= g'(z)\,\psi(z) + n g(z)\,\psi'(z)\,, \end{cases} \tag{68}$$

where $n = \begin{cases} +1, & \text{SCf}, \\ -1, & \text{TPt}, \end{cases}$ can be treated as a projection of some "reduction spin" switching the type of transformation. So the reduced transformation of the even coordinate (see (2)) should contain this additional index, i.e. $z \to \tilde{z}_n$ (at this point some additional to (45) analogy with complex structure is transparent). Since $f'_{-1}(z) = \psi'(z)\,\psi(z)$ is nilpotent, TPt transformations are always noninvertible and high degenerated after the body mapping. The unified multiplication law is

$$\left(\begin{array}{c} h \\ \varphi \end{array}\right)_n * \left(\begin{array}{c} g \\ \psi \end{array}\right)_m = \left(\begin{array}{c} g \cdot h \circ f_m + \chi_m \cdot \psi \cdot h' \circ f_m + \chi_m \cdot \varphi' \circ f_m \\ \varphi \circ f_m + \psi \cdot h \circ f_m \end{array}\right), \quad (69)$$

where $(*)$ is transformation composition and $(\circ)$ is function composition. For "reduction spin" projections we have only two definite products $(+1) * (+1) = (+1)$ and $(+1) * (-1) = (-1)$. The first formula is a consequence of $\mathbf{P}_S \cdot \mathbf{P}_S \subseteq \mathbf{P}_S$ (see (46)), which is simple manifestation of the fact that SCf transformations T_{SCf} form a substructure [21], i.e. a subsemigroup $\mathcal{T}_{SCf}$ of SA semigroup $\mathcal{T}_{SA}$ (in the invertible case—a subgroup [16]).

7 Nonlinear realizations of superconformal-like transformations

The study of "nonlinear" SCf and TPt transformations is interesting and worthwhile due to several reasons. ¿From one side, some of the first papers on supersymmetry [1, 46] were written in terms of its nonlinear realization (for nonsupersymmetric background of the method see [47, 48, 49]). Further, there were hopes that in the framework of nonlinear realizations one could solve the problems with superpartners and spontaneously supersymmetry breaking in realistic models [50, 51]. From another side, the nonlinearly realized two dimensional superconformal symmetry [52, 53] were used in the theory of superstrings [54]. In addition to these investigations we will study finite transformations and include noninvertibility. We also consider the connection between "linear" and "nonlinear" realizations [55, 56, 57], but from the pure kinematical viewpoint, and give a transparent diagram presentation for it in our special case.

7.1 The motion of odd curve in $\mathbb{C}^{1|1}$

According to the interpretation of [58] we can study the motion of the curve $\theta = \lambda(z)$ in $\mathbb{C}^{1|1}$. So that using the superanalytic transformations (2) we obtain

$$\tilde{z} = f(z) + \lambda(z) \cdot \chi(z), \quad (70)$$

$$\tilde{\lambda}(\tilde{z}) = \psi(z) + \lambda(z) \cdot g(z), \tag{71}$$

where the second equation reflects the Einstein style of transformations. In the next paragraph we will derive the equation (71) from the general diagrammatic approach.

In four dimensional case the function $\lambda(z)$ is usually called Akulov-Volkov field [50, 58] and in physical applications it plays a role of Nambu-Goldstone fermion [46, 59] (and therefore it is also called goldstino).

As it is seen from (71) the transformation of $\lambda(z)$ is highly nonlinear. The relations of such kind always appear in nonlinear group realizations, and the goldstino $\lambda(z)$ describes supersymmetry breaking [60, 61, 62].

To find the goldstino transformation we expand $\tilde{\lambda}(\tilde{z})$ in series and iterate exploiting nilpotency

$$\tilde{\lambda}(f(z)) = \psi(z) + \lambda(z) \cdot g(z) - \tilde{\lambda}'(f(z)) \cdot \lambda(z) \cdot \chi(z) \ . \tag{72}$$

In case f^{-1} exists, we derive the finite superanalytic transformation of $\lambda(z)$

$$\tilde{\lambda} = \psi \circ f^{-1} + \lambda \circ f^{-1} \cdot g \circ f^{-1} - \tilde{\lambda}' \cdot \lambda \circ f^{-1} \cdot \chi \circ f^{-1}, \tag{73}$$

where $f \circ g = f(g(z))$.

It is not possible to find a general solution of the equation (72), and therefore we consider some particular cases.

Example. (Global SUSY) The global supersymmetry in $\mathbb{C}^{1|1}$ corresponds to the following choice

$$f(z) = z,\ g(z) = 1,\ \chi(z) = \varepsilon,\ \psi(z) = \varepsilon, \tag{74}$$

where ε is a constant odd parameter. Then from (70) and (71) we have

$$\tilde{\lambda}_{Glob}(z) = \varepsilon + \lambda(z) - \tilde{\lambda}'_{Glob}(z) \cdot \lambda(z) \cdot \varepsilon \,. \tag{75}$$

This equation is also difficult to solve manifestly without any additional requirements. But for infinitesimal transformations we obtain

$$\delta_\varepsilon \lambda_{Glob}(z) = \tilde{\lambda}_{Glob}(z) - \lambda(z) = \varepsilon \cdot [1 + \lambda(z) \cdot \lambda'(z)] \tag{76}$$

which satisfy the conventional supersymmetry algebra

$$[\delta_\varepsilon, \delta_\eta]\, \lambda_{Glob}(z) = 2\varepsilon\eta \cdot \lambda(z) \cdot \lambda'(z) \tag{77}$$

in accordance with [46, 59].

Remark. In finite global case we put

$$\tilde{\lambda}^{fin}_{Glob}(z) = \tilde{\lambda}_{Glob}(z) + \Delta(z), \tag{78}$$

where $\tilde{\lambda}_{Glob}(z)$ is given by (76). Inserting (78) into (75) one derives the equation for $\Delta(z)$ as follows

$$\Delta'(z) \cdot \varepsilon \cdot \lambda(z) = \Delta(z) \tag{79}$$

which can be solved by expanding on nilpotents in a given underlying superalgebra.

Let us consider superconformal-like transformations parametrized by two functions $g(z)$, $\psi(z)$ (see Section 4). Then starting from the same function $\lambda(z)$ we can in general find $\tilde{\lambda}_n(z)$ from (71) as two separate solutions (corresponding to the "reduction spin" (68) projection n) of the following system of equations

$$\begin{cases} \tilde{\lambda}_n\left(f_n^{(g\psi)}(z)\right) & = & \psi(z) + \lambda(z) \cdot g(z) - \tilde{\lambda}'_n\left(f_n^{(g\psi)}(z)\right) \cdot \lambda(z) \cdot \chi_n^{(g\psi)}(z)\ , \\ f_n^{(g\psi)\prime}(z) & = & \psi'(z)\,\psi(z) + \frac{1+n}{2} g^2(z)\,, \\ \chi_n^{(g\psi)\prime}(z) & = & g'(z)\,\psi(z) + ng(z)\,\psi'(z)\,, \end{cases} \tag{80}$$

where prime denotes derivative by argument, $n = +1$ corresponds to SCf transformations and $n = -1$ - to TPt transformations, and TPt are so called *noninvertible transformations twisting parity of tangent space* (see (68) and [23, 40]).

Definition 28 *We call* $\tilde{\lambda}_{SCf}(z) = \tilde{\lambda}_{n=+1}(z)$ *a SCf goldstino, and* $\tilde{\lambda}_{TPt}(z) = \tilde{\lambda}_{n=-1}(z)$ *a TPt goldstino.*

As previously, it is not possible to solve the system (80) manifestly in general case.

Remark. It is necessary to stress that equations (80) do not depend on invertibility properties of superconformal-like transformations [44, 23] and only they can be used to find TPt goldstino evolution ($n = -1$ case).

Example. (Infinitesimal SCf) Let we parametrize infinitesimal SCf transformations by

$$f(z) = z + r(z)\,,\; g(z) = 1 + \frac{1}{2} r'(z)\,,\; \chi(z) = \varepsilon(z)\,,\; \psi(z) = \varepsilon(z)\,, \tag{81}$$

where $r(z)\,,\varepsilon(z)$ are infinitesimal. Then, from (80) we obtain

$$\delta_{r,\varepsilon}\lambda_{SCf}(z) = \varepsilon(z)\cdot\left[1 + \lambda(z)\cdot\lambda'(z)\right] + \frac{1}{2} r'(z)\cdot\lambda(z) - r(z)\cdot\lambda'(z) \tag{82}$$

in agreement with [53].

8 Connection between linear and nonlinear realizations from diagrammatic viewpoint

The relationship between linear and nonlinear realizations [55, 57] plays an important role in understanding of the spontaneously supersymmetry breaking mechanisms [56]. The interest to the study of $N = 1$ superconformal-like transformations is stipulated for the fact that nonlinearly realized infinitesimal SCf transformations [53, 63] are widely used in superstring embeddings [54, 64] and hierarhies [65, 66]. Here we investigate them in the noninvertible finite case and from some another kinematical viewpoint using a clear diagrammatic approach (which is applicable to any general multidimensional case as well).

Let us consider the following diagram

$$\begin{tikzcd} Z_A \arrow[r, "\mathcal{G}", "\text{W-Z}"'] & \tilde{Z} \\ Z \arrow[u, "\mathcal{A}"] \arrow[r, "\mathcal{H}", "\text{A-V}"'] & Z_H \arrow[u, "\mathcal{B}"'] \end{tikzcd} \tag{83}$$

where $\mathcal{A} : Z \to Z_A$, $\mathcal{G} : Z_A \to \tilde{Z}$, $\mathcal{B} : Z_H \to \tilde{Z}$, $\mathcal{H} : Z \to Z_H$ (and $Z = (z, \theta)$) are superanalytic transformations (1). The transformation $\mathcal{G}$ plays the role of the linear transformation of Wess-Zumino type and the nonlinear transformation $\mathcal{H}$ (from a subgroup) is of Akulov-Volkov type, while $\mathcal{A}$ and $\mathcal{B}$ correspond to the transformations with Goldstone fields as parameters (describing cosets) [49, 47].

8.1 Global 2D supersymmetry

According to the general prescriptions [55, 48] we can take $\mathcal{G}$ as a global linear supersymmetry transformation in two-dimensional case

$$\mathcal{G}:\left\{\begin{array}{lcl}\tilde{z} & = & z_A+\theta_A\cdot\varepsilon,\\ \tilde{\theta} & = & \varepsilon+\theta_A,\end{array}\right. \tag{84}$$

then we take $\mathcal{H}$ as an ordinary conformal transformation with composite parameters to be find and interpret $\mathcal{A}$ and $\mathcal{B}$ as coset transformations with the local odd parameters $\lambda(z)$ and $\tilde{\lambda}_{Glob}(z_H)$

Example.

$$\mathcal{A}:\left\{\begin{array}{lcl}z_A & = & z+\theta\cdot\lambda(z),\\ \theta_A & = & \lambda(z)+\theta,\end{array}\right.\qquad \mathcal{B}:\left\{\begin{array}{lcl}\tilde{z} & = & z_H+\theta_H\cdot\tilde{\lambda}_{Glob}(z_H),\\ \tilde{\theta} & = & \tilde{\lambda}_{Glob}(z_H)+\theta_H,\end{array}\right. \tag{85}$$

Indeed, the commutativity of the diagram (83) gives us the equation of $\lambda(z)$ evolution similar to (71) and (75) and equations for parameters of $\mathcal{H}$ in the following way.

Definition 29 *A "linear" transformation $\mathcal{G}$ is representable by a "nonlinear" transformation $\mathcal{H}$, iff the diagram (83) is commutative*

$$\mathcal{G}\circ\mathcal{A}=\mathcal{B}\circ\mathcal{H}. \tag{86}$$

Remark. In the group theory this construction is related to the induced representation [67]. But here we, in general, do not demand invertibility of the entries in (86) and consider finite transformations.

Using (86) we obtain the relations

$$\begin{array}{lcl}\tilde{z}_{\mathcal{G}\circ\mathcal{A}} & = & \tilde{z}_{\mathcal{B}\circ\mathcal{H}},\\ \tilde{\theta}_{\mathcal{G}\circ\mathcal{A}} & = & \tilde{\theta}_{\mathcal{B}\circ\mathcal{H}}\end{array} \tag{87}$$

which are the representability condition (86) in coordinate language (as 4 component equations after expanding in θ).

In the particular case of global supersymmetry (84) the equations 87 are

$$\begin{array}{l}z_A+\theta_A\cdot\varepsilon=z_H+\theta_H\cdot\tilde{\lambda}_{Glob}(z_H),\\ \theta_A+\varepsilon=\tilde{\lambda}_{Glob}(z_H)+\theta_H.\end{array} \tag{88}$$

After exploiting (85) we derive parameters of the conformal transformation

$$\mathcal{H}:\left\{\begin{array}{lcl} z_H & = & z+\lambda\left(z\right)\cdot\varepsilon, \\ \theta_H & = & \theta, \end{array}\right. \tag{89}$$

and the evolution equation for

$$\tilde{\lambda}_{Glob}\left(z_H\right)=\varepsilon+\lambda\left(z\right). \tag{90}$$

then expanding on nilpotents

$$\varepsilon+\lambda\left(z\right)=\tilde{\lambda}_{Glob}\left(z\right)+\tilde{\lambda}'_{Glob}\left(z\right)\cdot\lambda\left(z\right)\cdot\varepsilon \tag{91}$$

which coincides with (75). Thus, the relations (86) and (87) are initial in determining Goldstone field evolution.

8.2 The $-\lambda$-rule in two dimensions

If $\mathcal{A}$ is invertible, the representability condition (86) becomes

$$\mathcal{G}=\mathcal{B}\circ\mathcal{H}\circ\mathcal{A}^{-1}. \tag{92}$$

In the global case invertibility of $\mathcal{A}$ is evident, then from (85) we derive

$$\mathcal{A}^{-1}:\left\{\begin{array}{lcl} z & = & z_A-\theta_A\cdot\lambda\left(z_A\right), \\ \theta & = & -\lambda\left(z_A\right)+\theta_A\left[1+\lambda\left(z_A\right)\cdot\lambda'\left(z_A\right)\right]. \end{array}\right. \tag{93}$$

This explains nature of the well-known "$-\lambda$ rule" [55, 68] while comparing superfields of linear and nonlinear realizations [69]. The relation (92) is a general form of the "splitting trick" [55, 56] according to which any linear superfield can be presented as a set of nonlinear transforming components. The analog of this trick for a noninvertible finite case is the representability condition (86), and it is not solved under $\mathcal{A}$. Thus, for a superfield $\Phi\left(z,\theta\right)$ we can write

$$\delta_{\mathcal{H}}\Phi\left(z,\theta\right)=\Phi\left(z+\lambda\left(z\right)\cdot\varepsilon,\theta\right)-\Phi\left(z,\theta\right)=\varepsilon\cdot\lambda\left(z\right)\cdot\frac{\partial\Phi\left(z,\theta\right)}{\partial z}, \tag{94}$$

where $\delta_{\mathcal{H}}$ is infinitesimal "nonlinear" transformation $\mathcal{H}$ corresponding to $\mathcal{G}$. If we use (93) and put

$$\begin{array}{lcl} \Phi\left(z,\theta\right) & = & \Phi\left(z_A-\theta_A\cdot\lambda\left(z_A\right),-\lambda\left(z_A\right)+\theta_A\left[1+\lambda\left(z_A\right)\cdot\lambda'\left(z_A\right)\right]\right) \\ & \stackrel{def}{=} & \Phi_A\left(z_A,\theta_A\right), \end{array} \tag{95}$$

then for infinitesimal "linear" transformation $\mathcal{G}$ we obtain the standard supersymmetry relation

$$\delta_{\mathcal{G}}\Phi_A(z_A,\theta_A)=\Phi(z_A+\varepsilon\cdot\theta_A,\theta_A+\varepsilon)-\Phi_A(z_A,\theta_A)=\varepsilon\cdot Q_A\Phi_A(z_A,\theta_A), \tag{96}$$

where Q_A is an ordinary supertranslation (cf. [55]).

Now we are ready to prove the "reversed" splitting trick which manifestly follows from the representability condition (86) applied to global two dimensional supersymmetry.

Proposition 30 *Any superfield $\Phi(z,\theta)$ transforming nonlinearly as in (94) together with $\lambda(z)$ transforming as in (76) give a linearly (globally) transformed superfield (96).*

Proof. We should prove that $\Delta\Phi(z,\theta)=\delta_{\mathcal{G}}\Phi_A(z_A,\theta_A)$, where

$$\Delta\Phi(z,\theta)\overset{def}{=}\delta_{\mathcal{H}}\Phi(z,\theta)+\delta_{\mathcal{B}}\Phi(z,\theta)-\delta_{\mathcal{A}}\Phi(z,\theta),$$

and $\delta_{\mathcal{H}}$ is given by (94). It follows from (85) that $\delta_{\mathcal{B}}-\delta_{\mathcal{A}}$ describes changing of $\lambda(z)$, therefore

$$\delta_{\mathcal{B}}\Phi(z,\theta)-\delta_{\mathcal{A}}\Phi(z,\theta)=\delta_{\varepsilon}\lambda_{Glob}(z)\cdot\frac{\partial\Phi(z,\theta)}{\partial\lambda}.$$

So that from (76) we have

$$\Delta\Phi(z,\theta)=\varepsilon\cdot\left(\lambda(z)\cdot\frac{\partial\Phi(z,\theta)}{\partial z}+(1+\lambda(z)\cdot\lambda'(z))\cdot\frac{\partial\Phi(z,\theta)}{\partial\lambda}\right).$$

Making change of variables $(z,\theta)\to(z_A,\theta_A)$ and using the relations

$$\frac{\partial\Phi(z,\theta)}{\partial z}=(1+\theta\cdot\lambda'(z))\cdot\frac{\partial\Phi_A(z_A,\theta_A)}{\partial z_A}+\lambda'(z)\cdot\frac{\partial\Phi_A(z_A,\theta_A)}{\partial\theta_A}$$

and

$$\frac{\partial\Phi(z,\theta)}{\partial\lambda}=-\theta\cdot\frac{\partial\Phi_A(z_A,\theta_A)}{\partial z_A}+\frac{\partial\Phi_A(z_A,\theta_A)}{\partial\theta_A}$$

following from (85), we obtain

$$\begin{aligned}\Delta\Phi(z,\theta) &= (\theta+\lambda(z))\cdot\varepsilon\cdot\frac{\partial\Phi_A(z_A,\theta_A)}{\partial z_A}+\varepsilon\cdot\frac{\partial\Phi_A(z_A,\theta_A)}{\partial\theta_A}\\ &= \delta_{\mathcal{G}}z_A\cdot\frac{\partial\Phi_A(z_A,\theta_A)}{\partial z_A}+\delta_{\mathcal{G}}\theta_A\cdot\frac{\partial\Phi_A(z_A,\theta_A)}{\partial\theta_A}\\ &= \delta_{\mathcal{G}}\Phi_A(z_A,\theta_A).\end{aligned}$$

□

9 Nonlinear realization of general finite $N=1$ superconformal transformations

Let us consider the representability condition (86) for a general $N=1$ superconformal-like transformations $Z_A \to \tilde{Z}$ which now play the role of "linear" ones. According to Section 4 they can be parametrized by two functions $g(z_A)$ and $\psi(z_A)$ and have the form

$$\mathcal{G}:\begin{cases}\tilde{z} &= f_n^{(g\psi)}(z_A)+\theta_A\cdot\chi_n^{(g\psi)}(z_A),\\ \tilde{\theta} &= \psi(z_A)+\theta_A\cdot g(z_A),\end{cases}\tag{97}$$

where

$$\begin{aligned}f_n^{(g\psi)\prime}(z_A) &= \psi'(z_A)\,\psi(z_A)+\tfrac{1+n}{2}\cdot g^2(z_A),\\ \chi_n^{(g\psi)\prime}(z_A) &= g'(z_A)\,\psi(z_A)+n\cdot g(z_A)\,\psi'(z_A),\end{aligned}\tag{98}$$

where $n=\begin{cases}+1, & \text{SCf transformation,}\\ -1, & \text{TPt transformation,}\end{cases}$ is a projection of "reduction spin" switching the type of transformation (see also [40] for more details).

Then while trying to represent $\mathcal{G}$ in terms of nonlinear composition similarly to the diagram (83) we face with the following restriction which is consequence of the $N=1$ superconformal-like multiplication law [40]. If $\mathcal{T}$ is a superconformal-like transformation, then there are only two possibilities in the composition $z\xrightarrow{\mathcal{T}}\tilde{z}\xrightarrow{\tilde{\mathcal{T}}}\tilde{\tilde{z}}$

$$\begin{aligned}\tilde{\mathcal{T}}_{SCf}*\mathcal{T}_{SCf} &= \tilde{\tilde{\mathcal{T}}}_{SCf},\\ \tilde{\mathcal{T}}_{TPt}*\mathcal{T}_{SCf} &= \tilde{\tilde{\mathcal{T}}}_{TPt}.\end{aligned}\tag{99}$$

Therefore, we have only two possibilities to include TPt transformations into the diagrammatic representation (83) as

$$\mathcal{G}_{SCf}\circ\mathcal{A}_{SCf}=\mathcal{B}_{SCf}\circ\mathcal{H}_{SCf}, \tag{100}$$

$$\mathcal{G}_{TPt}\circ\mathcal{A}_{SCf}=\mathcal{B}_{TPt}\circ\mathcal{H}_{SCf}. \tag{101}$$

The first one is the nonlinear representation of $N=1$ superconformal group in analogy with the ordinary infinitesimal invertible four-dimensional case [55, 70] (and 86) in which $\mathcal{A}_{SCf}$ and $\mathcal{B}_{SCf}$ play the role of cosets.

Let us consider (100) in more detail. The exact shape of cosets $\mathcal{A}_{SCf}$ and $\mathcal{B}_{SCf}$ can be taken as

$$\mathcal{A}_{SCf}:\begin{cases} z_A &= z+\theta\cdot\lambda\left(z\right), \\ \theta_A &= \lambda\left(z\right)+\theta\sqrt{1+\lambda\left(z\right)\cdot\lambda'\left(z\right)}, \end{cases} \tag{102}$$

$$\mathcal{B}_{SCf}:\begin{cases} \tilde{z} &= z_H+\theta_H\cdot\tilde{\lambda}\left(z_H\right), \\ \tilde{\theta} &= \tilde{\lambda}\left(z_H\right)+\theta_H\sqrt{1+\tilde{\lambda}\left(z_H\right)\cdot\tilde{\lambda}'\left(z_H\right)}, \end{cases} \tag{103}$$

and for $\mathcal{H}$ we choose the following general parametrization

$$\mathcal{H}_{SCf}:\begin{cases} z_H &= p\left(z\right), \\ \theta_H &= \rho\left(z\right)+\theta\cdot q\left(z\right) \end{cases} \tag{104}$$

Then, expanding the coordinate form (87) into components we obtain 4 corresponding equations for 4 unknown functions $p\left(z\right), q\left(z\right), \rho\left(z\right), \tilde{\lambda}\left(z\right)$

$$p\left(z\right)+\rho\left(z\right)\cdot\tilde{\lambda}\left(p\left(z\right)\right) = f_{+1}^{(g\psi)}\left(z\right)+g\left(z\right)\cdot\lambda\left(z\right)\cdot\psi\left(z\right), \tag{105}$$

$$\tilde{\lambda}\left(p\left(z\right)\right)+\rho\left(z\right)\cdot\sqrt{1+\tilde{\lambda}\left(p\left(z\right)\right)\cdot\tilde{\lambda}'\left(p\left(z\right)\right)} = \psi\left(z\right)+g\left(z\right)\cdot\lambda\left(z\right), \tag{106}$$

$$\begin{aligned} q\left(z\right)\cdot\tilde{\lambda}\left(p\left(z\right)\right) &= \lambda\left(z\right)\cdot f_{+1}^{(g\psi)\prime}\left(z\right)+ \\ &\quad g\left(z\right)\cdot\psi\left(z\right)\cdot\sqrt{1+\lambda\left(z\right)\cdot\lambda'\left(z\right)}, \end{aligned} \tag{107}$$

$$\begin{aligned} q\left(z\right)\cdot\sqrt{1+\tilde{\lambda}\left(p\left(z\right)\right)\cdot\tilde{\lambda}'\left(p\left(z\right)\right)} &= \lambda\left(z\right)\cdot\psi'\left(z\right)+ \\ &\quad g\left(z\right)\cdot\sqrt{1+\lambda\left(z\right)\cdot\lambda'\left(z\right)}, \end{aligned} \tag{108}$$

where $f_{+1}^{(g\psi)}\left(z\right)$ is determined from (98).

In case $q(z)$ and $g(z)$ are invertible, these equations have the following solution for parameters of nonlinear $\mathcal{H}$ transformation in terms of parameters of "linear" $\mathcal{G}$ transformation as

$$\begin{aligned} p(z) &= f_{+1}^{(g\psi)}(z) + g(z) \cdot \lambda(z) \cdot \psi(z), && (109) \\ q(z) &= \sqrt{p'(z)}, && (110) \\ \rho(z) &= 0, && (111) \end{aligned}$$

and for goldstino transformation rule

$$\tilde{\lambda}(p(z)) = \psi(z) + g(z) \cdot \lambda(z), \tag{112}$$

that naturally coincides with the previous approach (72) with $f(z) = f_{+1}^{(g\psi)}(z)$ and $\chi(z) = g(z) \cdot \psi(z)$.

Therefore, $\mathcal{H}$ is the split $N = 1$ SCf transformation [71, 20]

$$\mathcal{H}_{SCf} : \begin{cases} z_H = p(z), \\ \theta_H = \theta \cdot \sqrt{p'(z)} \end{cases} \tag{113}$$

with the composite parameter $p(z)$ from (109), which can be presented as the following commutative diagram

$$\begin{array}{ccc} Z_A & \xrightarrow[\text{full}]{\mathcal{G}_{SCf}} & \tilde{Z} \\ \uparrow{\scriptstyle \mathcal{A}_{SCf}} & & \uparrow{\scriptstyle \mathcal{B}_{SCf}} \\ Z & \xrightarrow[\text{split}]{\mathcal{H}_{SCf}} & Z_H \end{array} \tag{114}$$

Thus, using the SCf goldstino field $\lambda(z)$ we have manifestly obtained a nonlinear realization of general finite SCf transformations.

Second relation (101) and the corresponding commutative diagram

$$\begin{array}{ccc} Z_A & \xrightarrow{\mathcal{G}_{TPt}} & \tilde{Z} \\ \uparrow{\scriptstyle \mathcal{A}_{SCf}} & & \uparrow{\scriptstyle \mathcal{B}_{TPt}} \\ Z & \xrightarrow{\mathcal{H}_{SCf}} & Z_H \end{array} \tag{115}$$

have no such transparent meaning, because $\mathcal{B}_{TPt}$ is noninvertible, and so it cannot be a standard coset. Nevertheless, since the final answer for the nonlinear transformation $\mathcal{H}_{SCf}$ is known from the previous approach (80), the noninvertible analog of coset $\mathcal{B}_{TPt}$ can be found in principle from the system of equations analogous to (105)–(108).Let we write $\mathcal{B}_{TPt}$ in the form

$$\mathcal{B}_{SCf}:\left\{\begin{array}{rcl}\tilde{z} & = & f_{-1}^{(b\widetilde{\lambda})}(z_H)+\theta_H\cdot\chi_{-1}^{(b\widetilde{\lambda})}(z_H),\\ \tilde{\theta} & = & \widetilde{\lambda}(z_H)+\theta_H\cdot b(z_H),\end{array}\right. \tag{116}$$

where

$$\begin{array}{rcl} f_n^{(b\widetilde{\lambda})\prime}(z_H) & = & \widetilde{\lambda}'(z_H)\cdot\widetilde{\lambda}(z_H)+\frac{1+n}{2}\cdot b^2(z_H),\\ \chi_n^{(b\widetilde{\lambda})\prime}(z_H) & = & b'(z_H)\cdot\widetilde{\lambda}(z_H)+n\cdot b(z_H)\cdot\widetilde{\lambda}'(z_H),\end{array} \tag{117}$$

and prime denotes derivative by argument. So the corresponding system of equations now is

$$f_{-1}^{(b\widetilde{\lambda})}(p(z))+\rho(z)\cdot\chi_{-1}^{(b\widetilde{\lambda})}(p(z)) = f_{+1}^{(g\psi)}(z)+\lambda(z)\cdot\chi_{+1}^{(g\psi)}(z), \tag{118}$$

$$\widetilde{\lambda}(p(z))+\rho(z)\cdot b(p(z)) = \psi(z)+g(z)\cdot\lambda(z), \tag{119}$$

$$\begin{array}{rcl}\rho(z)\cdot f_{-1}^{(b\widetilde{\lambda})\prime}(p(z))+q(z)\cdot\chi_{-1}^{(b\widetilde{\lambda})}(p(z)) & = & \lambda(z)\cdot f_{+1}^{(g\psi)\prime}(z)+\\ & & \chi_{+1}^{(g\psi)}(z)\cdot\sqrt{1+\lambda(z)\cdot\lambda'(z)},\end{array} \tag{120}$$

$$\begin{array}{rcl}\rho(z)\cdot q(z)\cdot\widetilde{\lambda}'(p(z))+q(z)\cdot b(p(z)) & = & \lambda(z)\cdot\psi'(z)+\\ & & g(z)\cdot\sqrt{1+\lambda(z)\cdot\lambda'(z)}.\end{array} \tag{121}$$

In case $\mathcal{A}_{SCf}$ is invertible we can obtain

$$\mathcal{G}_{TPt}=\mathcal{B}_{TPt}\circ\mathcal{H}_{SCf}\circ\mathcal{A}_{SCf}^{-1} \tag{122}$$

which gives an analog of nonlinear realization for noninvertible TPt transformations.

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A UNIFYING SEMILOCAL CONVERGENCE THEOREM FOR NEWTON-LIKE METHODS IN BANACH SPACE

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Abstract. We provide a unifying semilocal convergence theorem for Newton-like methods, in order to approximate a solution of a nonlinear equation in Banach space. We use a certain positive functional as an upper bound on the distances involved. This way we can obtain sharper error bounds than before. Earlier results are special cases of our theorem. Several choices of the majorizing functional are also provided.

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Key Words and Phrases. Newton-like methods, Banach space, majorizing functional.

1. INTRODUCTION.

In this study we are concerned with approximating a solution x^* of an equation

$$F(x) = 0, \tag{1}$$

where F is a continuous nonlinear operator defined on an open convex subset D of a Banach space E_1 with values in a Banach space E_2.

Newton-like methods of the form

$$x_{n+1} = x_n - A(x_n)^{-1}F(x_n) \quad (n \geq 0) \tag{2}$$

have been used by many authors to approximate x^*. For a survey of such results we refer the reader to [2], [3], [4]. Here, $A(x) \in L(E_1, E_2)$ $(x \in D)$ which denotes the space of all bounded linear operators from E_1 into E_2. For $A(x) = F'(x)$ $(x \in D)$ we obtain the famous Newton–Kantorovich method [3], [4], [5]. Several other choices for operator A are also possible [1], [2], [4].

Here, we introduce a majorizing functional and use it to provide a unifying semilocal convergence theorem for Newton-like method (2). Results already in

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the literature are considered special cases of our theorem if we simply choose the functional appropriately. Moreover, we show that in case this functional exists, then our error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ are sharper than earlier ones.

2. CONVERGENCE ANALYSIS.

We state and prove the following semilocal convergence theorem for Newton-like method (2).

Theorem. *Let $F: D \subseteq E_1 \to E_2$ be a continuous operator, defined on a convex subset D of a Banach space E_1 with values in a Banach space E_2.*

Assume

(a) *for some $x_0 \in D$ there exists $R \geq 0$ such that*

$$\bar{U}(x_0, R) = \{x \in E_1 \mid \|x - x_0\| \leq R\} \subseteq D;$$

(b) *$A(x)^{-1} \in L(E_2, E_1)$ for all $x \in U(x_0, R)$;*

(c) *there exists a functional $\alpha: \bar{U}(x_0, R) \to [0, +\infty)$ such that*

$$\|A(x)^{-1}F(x)\| \leq \alpha(x) \quad \text{for all } x \in \bar{U}(x_0, R); \tag{3}$$

(d) *iteration $\{t_n\}$ $(n \geq 0)$ given by*

$$\begin{aligned} t_{n+2} &= t_{n+1} + \alpha_{n+1}, \quad (n \geq 0),\ t_0 = 0, \\ t_1 &\geq \|A(x_0)^{-1}F(x_0)\|,\ \alpha_n = \alpha(t_n) \quad (n \geq 1) \end{aligned} \tag{4}$$

is monotonically increasing and bounded above by R.

Then

(i) *Iteration $\{t_n\}$ $(n \geq 0)$ generated by (4) converges to some limit $r_0 \in [0, R]$.*

(ii) *Newton-like iteration $\{x_n\}$ $(n \geq 0)$ generated by (2) is well defined, remains in $\bar{U}(x_0, r_0)$ for all $n \geq 0$ and converges to a solution x^* of equation $F(x) = 0$. Moreover, the following error bounds hold for all $n \geq 0$:*

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \tag{5}$$

and

$$\|x^* - x_n\| \leq r_0 - t_n. \tag{6}$$

Proof. (i) Scalar iteration $\{t_n\}$ $(n \geq 0)$ is monotonically increasing and bounded above by R (hypothesis (d)), and as such it converges to some limit $r_0 \in [0, R]$.

(ii) By hypothesis (b) Newton-like method (2) is well defined for all $n \geq 0$. We will show estimate (5) for all $n \geq 0$. For $n = 0$, (2), (4) give $\|x_1 - x_0\| \leq t_1 = t_1 - t_0$, which shows (5) for $n = 0$, and $x_1 \in \bar{U}(x_0, r_0)$. Let us assume that (5) holds and $x_{n+1} \in \bar{U}(x_0, r_0)$ for $n = 0, 1, 2, \ldots, k$. Using (2), (3) and (4) we get

$$\begin{aligned} \|x_{k+2} - x_0\| &\leq \|x_{k+2} - x_{k+1}\| + \|x_{k+1} - x_0\| \\ &\leq \cdots \leq (t_{k+2} - t_{k+1}) + (t_{k+1} - t_k) + \cdots + (t_1 - t_0) \\ &\leq t_{k+2} - t_0 = t_{k+2} \leq r_0, \end{aligned}$$

and

$$\|x_{k+2} - x_{k+1}\| = \| - A(x_{k+1})^{-1} F(x_{k+1})\| \leq \alpha(x_{k+1}) = \alpha_{k+1} = t_{k+2} - t_{k+1},$$

which completes the induction.

It follows from estimate (5) that Newton-like method $\{x_n\}$ ($n \geq 0$) is Cauchy in a Banach space E_1, and as such it converges to some limit $x^* \in \bar{U}(x_0, r_0)$ (since $\bar{U}(x_0, r_0)$ is a closed set). By letting $n \to \infty$ in (2) and using the continuity of $A(x)^{-1}$ and F we get $F(x^*) = 0$. That is x^* is a solution of equation (1).

Finally estimate (6) follows from (5) by using standard majorization techniques [3], [4], [5].

That completes the proof of the Theorem. ♠

Remark. The selection of the functional α is not easy in general. Here are some choices:

(1) An "optimum" choice is given by

$$\alpha(x) = \sup_{\substack{x \in \bar{U}(x_0, r) \\ r \in [0, R]}} \|A(x)^{-1} F(x)\|. \tag{7}$$

If such a functional can be found then since all upper bounds (see (3)) on the expression $\|A(x)^{-1}F(x)\|$ ($x \in D$) already in the literature are "larger" than α, it follows that our error bounds on the distances are smaller than all earlier ones (see [1]–[5] and the references there).

(2) Let $\gamma_0 : \bar{U}(x_0, R) \to [0, +\infty)$ be a functional; γ be a nonnegative nondecreasing, continuous function defined on $[0, R]$ such that:

$$\|A(x_0)^{-1}(A(x) - A(x_0))\| \leq \gamma(x) \leq \gamma_0(\|x - x_0\|) \quad \text{for all } x \in \bar{U}(x_0, r) \subseteq \bar{U}(x_0, R) \tag{8}$$

and

$$\gamma_0(R) < 1; \tag{9}$$

β be a functional such that $\beta : \bar{U}(x_0, R) \to [0, +\infty)$; β_0 be a nonnegative, nondecreasing continuous function of three variables defined on $[0, R]^3$ such that:

$$\left\| \int_0^1 A(x_0)^{-1}[F'(x + t(y - x) - A(x)](y - x)dt \right\| \leq \beta(x) \leq \beta_0(\|x - x_0\|, \|y - x_0\|, \|x - y\|) \tag{10}$$

for all $x, y \in \bar{U}(x_0, r) \subseteq \bar{U}(x_0, R)$ and

$$y = x - A(x)^{-1}F(x). \tag{11}$$

It then follows from the estimate

$$\|A(x)^{-1}F(x)\| \leq \|A(x)^{-1}A(x_0)\| \cdot \|A(x_0)^{-1}F(x)\|, \tag{12}$$

the Banach Lemma on invertible operators [4], [5], (3), (8)–(11) that functional α can be chosen by

$$\alpha(x) = \delta(\|x-x_0\|)\beta_0(\|x-x_0\|, \|x-x_0-A(x)^{-1}F(x)\|, \|A(x)^{-1}F(x)\|), \tag{13}$$

where

$$\delta(t) = \frac{1}{1-\gamma_0(t)}, \quad t \in [0,R], \tag{14}$$

and that $A(x)^{-1}$ is invertible for all $x \in \bar{U}(x_0, R)$. Note that functionals β, γ can be given by

$$\beta(x) = \sup_{\substack{x\in\bar{U}(x_0,r),\ r\in[0,R]\\ y=x-A(x)^{-1}F(x)}} \left\| \int_0^1 A(x_0)^{-1}[F'(x)+t(y-x)) - A(x)](y-x)dt \right\| \tag{15}$$

and

$$\gamma(x) = \sup_{\substack{x\in\bar{U}(x_0,r)\\ r\in[0,R]}} \|A(x_0)^{-1}(A(x)-A(x_0))\|. \tag{16}$$

It follows from (12), (13), and (14) that if α is chosen as in (7) then

$$\begin{aligned} \alpha(x) &\le \beta(x)\frac{1}{1-\gamma(x)} \\ &\le \beta_0(\|x-x_0\|, \|x-x_0-A(x)^{-1}F(x)\|, \|A(x)^{-1}F(x)\|)\delta(\|x-x_0\|), \end{aligned} \tag{17}$$

for all $x \in \bar{U}(x_0, r)$, $r \in [0,R]$. That is, error bounds on the distances $\|x_{n+1}-x_n\|$, $\|x_n - x^*\|$ $(n \ge 0)$ with α chosen as in (7) are smaller or equal than the corresponding ones when α is chosen as in (13) or by

$$\alpha(x) = \beta(x)\frac{1}{1-\gamma(x)},\ x \in \bar{U}(x_0, r),\ r \in [0,R] \tag{18}$$

and β, γ given by (15) and (16) respectively.

Our theorem can be reduced to any theorem on Newton methods already in the literature for special choices of the functional α. We will not show that for any such theorem in existence. However we will only show it under Newton–Kantorovich hypotheses [4], [5] for Newton's method. That is we set $A(x) = F'(x)$ $(x \in D)$ in (2). Indeed, let ℓ, b be such that

$$\|F'(x) - F'(y)\| \le \ell\|x-y\| \quad \text{for all } x, y \in D \tag{19}$$

$$\|F'(x_0)^{-1}\| \le b. \tag{20}$$

Then it can easily be seen that we can choose α as in (18) for

$$\gamma(x) = \ell b\|x-x_0\| \ \text{ and } \ \beta(x) = \frac{1}{2}\ell b\|F'(x)^{-1}F(x)\|^2 \ \ (x \in D). \tag{21}$$

Finally, for a list of choices for the functional α other than (7) and relevant ideas we refer the reader to [1], [2], [4].

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ON FAMILIES OF CAUCHY TRANSFORMS

Yusuf Abu Muhanna and El-Bachir Yallaoui

Abstract

In this paper we prove a number of results on Cauchy transforms of generalized type given by Borel measures supported on the class of analytic functions mapping the unit disc into the unit disk.

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1. Introduction

Let $\Delta = \{z \in C : |z| < 1\}$, $\Gamma = \partial\Delta$ and let B (equipped with the topology of uniform convergence on compact subsets) denote the set of functions ϕ that are analytic in Δ such that $|\phi(z)| < 1$ and $\phi(0) = 0$. Let $\mathcal{M}, \mathcal{N}$ denote the sets of complex–valued Borel measures on Γ and B respectively. Here, $\mathcal{M}$ is equivalent to the subset of $\mathcal{N}$ consisting of all those measures supported on the set $\{xz : |x| = 1\}$.

For $z \in \Delta$ and $\alpha \geq 0$, let A_α denote the family of functions f for which there exists a measure $\mu \in \mathcal{N}$ such that

$$f(z) = \begin{cases} \displaystyle\int_B \frac{1}{(1-\phi(z))^\alpha}\, d\mu(\phi) & \text{for } \alpha > 0 \\ \displaystyle\int_B \log\frac{1}{1-\phi(z)}\, d\mu(\phi) + f(0) & \text{for } \alpha = 0 \end{cases} \tag{1.1}$$

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The classes F_α consisting of those functions g for which there exists a measure $\mu \in \mathcal{M}$ such that,

$$g(z) = \begin{cases} \displaystyle\int_B \frac{1}{(1-\overline{x}z)^\alpha}\, d\mu(x) & \text{for } \alpha > 0 \\ \displaystyle\int_B \log\frac{1}{1-\overline{x}z}\, d\mu(x) + f(0) & \text{for } \alpha = 0 \end{cases} \tag{1.2}$$

have been well studied [3], [4], [9] and [10]. The classes F_α are subsets of A_α when the measures μ in (1.1) are in $\mathcal{M}$.

The class A_α is a Banach space with respect to the norm

$$\|f\|_{A_\alpha} = \begin{cases} \inf \|\mu\|, & \text{for } \alpha > 0 \\ \inf \|\mu\| + |f(0)|, & \text{for } \alpha = 0 \end{cases} \tag{1.3}$$

where μ varies over all measures in $\mathcal{N}$ for which (1.1) holds and where $\|\mu\|$ denotes the total variation norm of μ .

Clearly, for $f \in F_\alpha$, $\|f\|_{F_\alpha} \geq \|f\|_{A_\alpha}$. It is also known from [1] that for $\alpha \geq 1$, $F_\alpha = A_\alpha$.

We will show in this paper that for $0 \leq \alpha < \beta$,

$$A_\alpha \subset A_\beta \quad \text{and} \quad \|f\|_{A_\beta} \leq \|f\|_{A_\alpha} \tag{1.4}$$

This generalizes similar results for F_α in [6] and [7].

We will also show that $A_0 = BMOA$ and that the norm $\|.\|_{A_0}$ is equivalent to well known BMO norms. Furthermore we will show that, for all $n \geq 1$ and $\alpha \geq 0$

$$\|z^n\|_{A_\alpha} \leq k$$

where the constant k is independent of n or α.

2. The Classes A_α

In this section we will establish for $0 \leq \alpha < \beta$ the relationship between A_α and A_β as well as their respective norms.

Theorem 1. *If $0 \leq \alpha < \beta$, then $A_\alpha \subset A_\beta$ and $\|f\|_{A_\beta} \leq \|f\|_{A_\alpha}$*

Proof. Note that since $A_\alpha = F_\alpha$ for $\alpha \geq 1$ [1], and for $0 < \alpha < \beta, F_\alpha \subset F_\beta$ and $\|f\|_{F_\beta} \leq \|f\|_{F_\alpha}$ [7], then all we have to prove is the case $0 \leq \alpha < \beta < 1$.

(i) Let $f \in A_\alpha$ where $0 < \alpha < \beta$, then we can write

$$f(z) = \int_B \frac{1}{(1-\psi(z))^\alpha} d\mu(\psi), \tag{2.1}$$

Since $\dfrac{1}{(1-z)^\alpha} \in F_\alpha \subset F_\beta$, we can write

$$\frac{1}{(1-z)^\alpha} = \int_\Gamma \frac{1}{(1-\overline{x}z)^\beta}\, d\nu(x) \tag{2.2}$$

and

$$\left\| \frac{1}{(1-z)^\alpha} \right\|_{F_\beta} \leq \left\| \frac{1}{(1-z)^\alpha} \right\|_{F_\alpha} = 1 \tag{2.3}$$

Now by replacing z in (2.2) by $\psi(z)$ and putting the result in (2.1) we get

$$f(z) = \int_B \int_\Gamma \frac{1}{(1-\overline{x}\psi(z))^\beta} d\nu(x) d\mu(\psi). \tag{2.4}$$

Suppose, without loss of generality that ν is a positive measure and let

$$g_n(z) = \sum_{k=1}^{n} \frac{\nu_k}{(1-\overline{x}z)^\beta}.$$

Then by (2.2),

$$\int_B g_n(\psi)\ d\mu(\psi)$$

converges locally uniformly to

$$f(z) = \int_B \int_\Gamma \frac{1}{(1-\overline{x}\psi(z))^\beta} d\nu(x) d\mu(\psi).$$

Let $\eta_n(\psi) = \sum_{k=1}^{n} \nu_k \mu(\psi)$ then,

$$\int_B g_n(\psi)\ d\mu(\psi) = \int_B \frac{1}{(1-\psi)^\beta} d\eta_n(\psi),$$

where $\|\eta_n\| \leq \|\nu\|\|\mu\|$ for all n. Hence by compactness, there exists a measure σ, such that,

$$f(z) = \int_B \frac{1}{(1-\psi(z))^\beta} d\sigma(\psi) \tag{2.5}$$

which shows that $f \in A_\beta$. Furthermore, $\|\sigma\| \leq \|\nu\|\|\mu\|$. Consequently,

$$\|f\|_{A_\beta} \leq \|\sigma\| \leq \|\mu\|\|\nu\| \quad \text{for all } \mu$$

however since μ and ν are arbitrary measures that give (2.1) and (2.2) then,

$$\|f\|_{A_\beta} \leq \inf\{\|\mu\|\} \inf\{\|\nu\|\}$$

Hence by (2.3)

$$\|f\|_{A_\beta} \leq \|f\|_{A_\alpha}$$

(ii) Now let $f \in A_0$. We want to show that $f \in A_\alpha$ for any $\alpha > 0$. By definition,

$$f(z) = \int_B \log \frac{1}{1-\phi(z)}\ d\mu(\phi) + f(0) \tag{2.6}$$

Since $\log \dfrac{1}{1-z} \in F_0 \subset F_\alpha$ see [5], then

$$\log \frac{1}{1-z} = \int_\Gamma \frac{1}{(1-\overline{x}z)^\alpha}\ d\nu(x)$$

where ν depends only on α. Hence (2.6) becomes

$$f(z) = \int_B \int_\Gamma \frac{1}{(1-\overline{x}\phi(z))^\alpha} d\nu(x)\ d\mu(\phi) + f(0) \tag{2.7}$$

where the integral in (2.7) looks exactly like the one in (2.3) with α replacing β and hence using an argument similar to the one in (i) we will get that

$$f(z) = \int_B \int_\Gamma \frac{1}{(1-\psi(z))^\alpha}\, d\sigma(\psi) + f(0) \tag{2.8}$$

which shows that $f \in A_\alpha$. Furthermore $\|\sigma\| \le \|\nu\|\|\mu\|$, and hence

$$\|f\|_{A_\alpha} \le \inf\{\|\mu\|\} + |f(0)| = \|f\|_{A_0} \tag{2.9}$$

$\square$

3. CHARACTERIZATION OF A_0

It is known from ([2], p248) that a function $\phi \in$ BMO if and only if there exists functions ϕ_1 and ϕ_2 in L^∞ such that

$$\phi = \phi_1 + \widetilde{\phi_2} + \alpha$$

where both $\|\phi_1\|_\infty$ and $\|\phi_2\|_\infty$ are less than $C\|\phi\|_\star$, C is a constant and $\|\ \|_\star$ is the classical BMO norm ([2], p248).

Consequently $f \in$ BMOA if and only if there are analytic functions f_1 and f_2 such that

$$f = f_1 + f_2 + \alpha \tag{3.1}$$

where $\|\mathrm{Re} f_1\|_\infty \le C$ and $\|\mathrm{Im}\ f_2\|_\infty \le C$.

If we define on BMOA the norm

$$\|f\|^\star = \inf\{\ \|\mathrm{Re} f_1\|_\infty + \|\mathrm{Im} f_2\|_\infty : f = f_1 + f_2 + \alpha\ \} \tag{3.2}$$

then by ([2], p248), the norms $\|f\|^\star$ and $\|f\|_\star$ are equivalent.

Now we have the following proposition which establishes a set equality between A_0 and BMOA.

Theorem 2. $A_0 = BMOA$

Proof. Suppose that $f \in A_0$, then according to (1.1) and (1.2) there exists a measure $\mu \in \mathcal{N}$ such that,

$$f(z) = \int_B \log \frac{1}{1-\phi(z)}\, d\mu(\phi) + f(0) \tag{3.3}$$

Assume without loss of generality, that μ is a probability measure. Then f is subordinate to $\log \dfrac{1}{1-z} + f(0)$ and consequently by (3.1), $f \in$ BMOA. The proof of the other inclusion follows from (3.1) and subordination.

Theorem 3. *The norms $\|.\|_\star$ and $\|.\|_{A_0}$ are equivalent, namely there exists positive constants c_1 and c_2 such that*

$$c_1\|f\|_\star \le \|f\|_{A_0} \le c_2\|f\|_\star \tag{3.4}$$

Proof. Suppose $f \in$ BMOA, then f can be decomposed as in (3.1). Let d_1 denote $\|\mathrm{Re} f_1\|_\infty$ and d_2 denote $\|\mathrm{Im}\ f_2\|_\infty$. Then

$$\left|\frac{\pi}{2d_1}\mathrm{Im} i f_1(z)\right| \leq \frac{\pi}{2} \tag{3.5}$$

and

$$\left|\frac{\pi}{2d_2}\mathrm{Im} f_2(z)\right| \leq \frac{\pi}{2} \tag{3.6}$$

for all $z \in \Delta$. Consequently, by subordination

$$i f_1(z) = \frac{2d_1}{\pi}\left(\log\frac{1}{1-\phi(z)} - \log\frac{1}{1+\phi(z)}\right) + i f_1(0) \tag{3.7}$$

$$f_2(z) = \frac{2d_2}{\pi}\left(\log\frac{1}{1-\psi(z)} - \log\frac{1}{1+\psi(z)}\right) + f_2(0) \tag{3.8}$$

for all $z \in \Delta$ and where $\phi, \psi \in B$. Therefore

$$\|f\|_{A_0} \leq \frac{4}{\pi}(d_1 + d_2), \tag{3.9}$$

and hence

$$\|f\|_{A_0} \leq \frac{4}{\pi}\|f\|^\star \leq c_2\|f\|_\star \tag{3.10}$$

which gives the right inequality in (3.3).

Next, we show the left inequality. Let us write f as in (1.2) and assume without loss of generality that μ is a positive measure. Then,

$$|\mathrm{Im} f(z)| \leq c\|\mu\| \tag{3.11}$$

where $c > 1$. Thus

$$|\mathrm{Im} f(z)| \leq c\|f\|_{A_0} \tag{3.12}$$

and since

$$\|\mathrm{Im} f\|_\infty \geq \|\mathrm{Im} f\|_\star \tag{3.13}$$

we have

$$\|f\|_\star \leq k_1\|\mathrm{Im} f\|_\star \leq k_2\|f\|_{A_0} = \frac{1}{c_1}\|f\|_{A_0}. \tag{3.14}$$

where $c_1 = \dfrac{1}{k_2}$ and the left inequality in (3.13) follows by ([2], p235) and this concludes the proof. $\square$

Theorem 4. $\|z^n\|_{A_\alpha} \leq k$ *for* $n \geq 1$

Proof. It is enough to show that $\|z^n\|_{A_0} \leq k$ for $n \geq 1$. Since we showed that $\| \, . \, \|_\star$ and $\| \, . \, \|_{A_0}$ are equivalent, let us approximate $\|z^n\|_\star$. It is known from ([2], p240)that,

$$\|g\|_\star^2 \approx \sup_\psi \iint_\Delta | \bigtriangledown g|^2 (1-|z|^2)|\psi'(z)| dA, \tag{3.15}$$

where $\psi(z) = \dfrac{z+z_0}{1+\overline{z_0}z}$ is a mobius transformation.
Replace g in (3.15) by z^n and $|z|$ by r to get,

$$\begin{aligned}
I &= \iint_\Delta \left|\bigtriangledown(z^n)\right|^2 (1-r^2)|\psi'(z)| \, dA \\
&\leq \iint_\Delta n^2 r^{2n-2}(1-r^2)|\psi'(z)| dA \\
&\leq \int_0^{2\pi} \int_0^1 n^2 r^{2n-2}(1-r^2)|\psi'(z)| dr d\theta \\
&\leq 2\pi n^2 \int_0^1 r^{2n-2}(1-r) \, dr, \text{ because } \int_0^{2\pi} |\psi'(z)| d\theta \leq 2\pi \\
&\leq 2\pi n^2 \int_0^1 (r^{2n-2} - r^{2n}) \, dr = \frac{4\pi n^2}{(4n^2-1)} \leq \frac{4\pi}{3} \qquad (3.16)
\end{aligned}$$

which gives us the desired result and completes the proof. □

The following theorem is a direct consequence of Proposition 4.

Theorem 5. *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is analytic and if* $\sum_{n=1}^{\infty} |a_n| < \infty$ *then* $f \in A_\alpha$ *for all* $\alpha \geq 0$.

Proof. It is sufficient to prove that $f \in A_0$ since $A_0 \subset A_\alpha$. To show that $f \in A_0$ all we have to show is that the norm $\|f(z)\|_{A_0}$ is bounded.

$$\|f(z)\|_{A_0} = \left\| \sum_{n=0}^{\infty} a_n z^n \right\|_{A_0} \leq \sum_{n=0}^{\infty} |a_n| \, \|z^n\|_{A_0} \leq \frac{4\pi}{3} \sum_{n=0}^{\infty} |a_n| < \infty. \quad \square$$

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On locally nilpotent endomorphisms of Gorenstein injective modules

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Abstract

In this paper, we generalize the idea of injective envelopes to the Ω-envelopes. We find more general conditions for local nilpotent homorphisms with the Ω-envelopes. Also we define the Gorenstein injective envelopes and prove similar theorems about the envelopes and find relations among the injective envelopes, Gorenstein injective envelopes and Ω-envelopes. Finally we can prove that if the base ring R is an n-Gorenstein and $\{(G_i), \psi_j^i\}$ is a directed system of Gorenstein injective R-modules, then $\varinjlim G_i$ is Gorenstein injective.

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1. Ω-Envelopes

Let Ω be a collection of R-modules of a left Noetherian ring R.

Definition 1. *By an Ω-envelope of a left R-module N we mean a linear map $\phi : N \longrightarrow E(N)$ where $E(N)$ is in Ω such that*

1. *if $f : N \longrightarrow F$ is a linear map where $F \in \Omega$, then there is a linear map $g : E(N) \longrightarrow F$ such that $f = g \circ \phi$, i.e., the diagram is commutative.*

$$\begin{tikzcd} N \arrow[r, "\phi"] \arrow[dr, "f"'] & E(N) \arrow[d, dotted, "g"] \\ & F \end{tikzcd}$$

2. *if $F = E(N)$ and $f = \phi$ then g is necessarily an automorphism.*

We note that an Ω-envelope of M, when it exists, is unique up to isomorphism.

Theorem 1. *If every left R-module has an Ω-envelope and if Ω is closed under direct sums of a countable number of modules in Ω, then any direct sum $\oplus_{i=1}^{\infty} M_i \longrightarrow \oplus_{i=1}^{\infty} E(M_i)$ of Ω-envelopes $M_i \longrightarrow E_i$ is an Ω-envelope.*

Proof. The proof of corollary 1 [3] gives that if

$$\begin{tikzcd} \oplus_{i\in I} M_i \arrow[r] \arrow[dr] & \oplus_{i\in I} E_i \arrow[d, dotted] \\ & \oplus_{i\in I} E_i \end{tikzcd}$$

is commutative, then $\oplus_{i\in I} E_i \longrightarrow \oplus_{i\in I} E_i$ is an injection. But by hypothesis $\oplus_{i\in I} M_i$ has an Ω-envelope $\oplus_{i\in I} M_i \longrightarrow E$. But then

$$\begin{tikzcd} \oplus_{i\in I} M_i \arrow[r] \arrow[dr] & \oplus_{i\in I} E_i \arrow[d, dotted] \\ & E \end{tikzcd}$$

can be completed to a commutative diagram since each $M_i \longrightarrow E_i$ is an Ω-envelope and $\oplus_{i\in I} E_i \in \Omega$. Since $\oplus_{i\in I} M_i \longrightarrow E$ is an envelope, the diagram

$$\begin{tikzcd}
\oplus_{i\in I}M_i \arrow[r] \arrow[dr] & E \arrow[d, dotted] \\
& \oplus_{i\in I}E_i
\end{tikzcd}$$

can be completed to a commutative diagram. Then consider the following diagram

$$\begin{tikzcd}
& E \arrow[d] \\
\oplus_{i\in I}M_i \arrow[ur] \arrow[r] \arrow[dr] & \oplus_{i\in I}E_i \arrow[d] \\
& E
\end{tikzcd}$$

shows $E \longrightarrow \oplus_{i\in I}E_i \longrightarrow E$ is an automorphism of E. So it makes E isomorphically to a summand of $\oplus_{i\in I}E_i$. If $E \longrightarrow \oplus_{i\in I}E_i$ were not an isomorphism, then $ker(\oplus_{i\in I}E_i \longrightarrow E) \neq 0$. But then $ker(\oplus_{i\in I}E_i \longrightarrow \oplus_{i\in I}E_i) \neq 0$ which is impossible by the corollary 1 [3]. $\square$

Let Ω be some class of left R-modules such that every left R-module has an Ω-envelope and Ω is closed under a countable direct sum of left R-modules. Then we can extend the idea about locally nilpotent endomorphisms of the class of injective modules to an arbitrary class Ω of modules defined in definition 1.

Lemma 1. *If $N \xrightarrow{\phi} E(N)$ is an Ω-envelope of N and $N \xrightarrow{g} N$ is an automorphism of N, then $g(N) \xrightarrow{\phi\cdot g^{-1}} E(N)$ is an Ω-envelope of $g(N)$.*

Proof. Consider the following diagram :

$$\begin{tikzcd}
N \arrow[r, "\phi"] \arrow[d, "g"'] & E(N) \\
g(N) \arrow[r, "\phi\circ g^{-1}"] \arrow[ur, dotted, "\phi\circ g^{-1}"] & E(N)
\end{tikzcd}$$

The map $\phi \circ g^{-1}$ satisfies all the conditions of Ω-envelope. $\square$

Lemma 2. *Let $N \in \Omega$ and $h : N \longrightarrow E(N)$ be a Ω-envelope of N. If $\phi : E(N) \longrightarrow E(N)$ is a linear map such that $\phi|_N$ is an automorphism of N, then ϕ is an automorphism of $E(N)$.*

Proof. Consider the map

$$\begin{tikzcd} N \arrow[r, "h"] \arrow[d, "\phi|_N"'] \arrow[dr, dotted, "h"] & E(N) \arrow[d, "\phi"] \\ N \arrow[r, "h \circ (\phi|_N)^{-1}"] & E(N) \end{tikzcd}$$

Since $\phi|_N$ is an automorphism, $(\phi|_N)^{-1}$ is defined. So the left triangle is commutative. By the definition of Ω-envelope, ϕ is an automorphism of $E(N)$. $\square$

Theorem 2. *If $f \in End(E)$ where $E \in \Omega$ such that $ker(f) \subset E$ is an Ω-envelope, then f is locally nilpotent on E.*

Proof. Let K be the kernel of f, then $K \subset E$ is an Ω-envelope. Then by the hypothesis of Ω and by Thoerem 1, $K \oplus K \oplus \cdots \subset E \oplus E \oplus \cdots$ is an Ω-envelope. So define a map

$$\begin{array}{rcl} \phi : E \oplus E \oplus \cdots & \longrightarrow & E \oplus E \oplus \cdots \\ (x_1, x_2, \cdots) & \mapsto & (x_1, x_2 - f(x_1), x_3 - f(x_2), \cdots) \end{array}$$

Then ϕ is a homomorphism and $\phi|_{K \oplus K \oplus \cdots} = id_{K \oplus K \oplus \cdots}$. So by lemma 2, ϕ is an automorphism of $E \oplus E \oplus \cdots$ and in particular ϕ is onto. Let $x \in E$ and consider $(x, 0, 0, \cdots)$. Then $\phi(x_1, x_2, x_3, \cdots) = (x, 0, 0, \cdots)$ for some $(x_1, x_2, x_3, \cdots) \in E \oplus E \oplus \cdots$. Then $x_1 = x, x_2 - f(x_1) = 0, x_3 - f(x_2) = 0$ and so on. So $x_n = f^{n-1}(x)$ for all $n \geq 2$. But since $x_{n+1} = 0$ for some n, $f^n(x) = 0$ i.e., f is locally nilpotent on E. $\square$

¿From the fact that $ker(f) \subset ker(f^2) \subset \cdots \subset ker(f^n) \subset \cdots$ we can generalize the hypothesis of theorem 2.

Theorem 3. *If R is left Noetherian ring, E is in Ω, and $f \in End(E)$ is such that $\bigcup_{n=1}^{\infty} ker(f^n) \subset E$ is an Ω-envelope, then f is locally nilpotent on E .*

Proof. Let $K = \cup_{n=1}^{\infty} ker(f^n) \subset E$. Define

$$\begin{array}{rcl} \psi : E \oplus E \oplus \cdots & \longrightarrow & E \oplus E \oplus \cdots \\ (x_1, x_2, \cdots) & \longmapsto & (x_1, x_2 - f(x_1), x_3 - f(x_2), \cdots) \end{array}$$

Then easily ψ is an endomorphism. If $(x_1, x_2, \cdots)$ in $K \oplus K \oplus \cdots$, then there exist $a_i, b_i \geq 1$ such that $f^{a_i}(x_i) = 0$ and $f^{b_i}(x_{i-1}) = 0$. So $f^{m_i}(x_i - f(x_{i-1})) = 0$ for some $m_i \geq max\{a_i, b_i\}$ for each i. So ψ maps $K \oplus K \oplus \cdots$ into $K \oplus K \oplus \cdots$. And

$\psi|_{K\oplus K\oplus\cdots}$ is injective since its kernel is 0. Now let $(y_1, y_2, \cdots) \in K\oplus K\oplus\cdots$. Since each $y_i \in K = \bigcup_{n=1}^{\infty} Ker(f^n)$ there exists $t_i \in \{1, 2, \cdots\}$ such that $f^{t_i}(y_i) = 0$. Given $n \in \{1, 2, \cdots\}$ choose m such that

$$m \geq max\{t_n, t_{n-1} - 1, t_{n-2} - 2, \cdots, t_1 - (n-1)\}.$$

Then

$$\begin{aligned} & f^m(y_n + f(y_{n-1}) + f^2(y_{n-2}) + \cdots + f^{n-1}(y_1)) \\ = \; & f^m(y_n) + f^{m+1}(y_{n-1}) + f^{m+2}(y_{n-2}) + \cdots + f^{m+n-1}(y_1) \\ = \; & 0 \end{aligned}$$

so for every $n \in \{1, 2, \cdots\}$

$$y_n + f(y_{n-1}) + f^2(y_{n-2}) + \cdots + f^{n-1}(y_1) \in K$$

Then

$$\begin{aligned} & \psi(y_1, y_2 + f(y_1), \cdots, y_n + f(y_{n-1}) + f^2(y_{n-2}) + \cdots + f^{n-1}(y_1), \cdots) \\ = \; & (y_1, y_2 + f(y_1) - f(y_1), y_3 + f(y_2) + f^2(y_1) - f(y_2) - f^2(y_1), \cdots) \\ = \; & (y_1, y_2, \cdots, y_n, \cdots). \end{aligned}$$

So $\psi|_{K\oplus K\oplus\cdots}$ is onto. Hence $\psi|_{K\oplus K\oplus\cdots}$ is an isomorphism on $K \oplus K \oplus \cdots$ and ψ is an automorphism of $E \oplus E \oplus \cdots$. Let $x \in E$ and consider $(x, 0, 0, \cdots)$. Then $\psi(x_1, x_2, \cdots) = (x, 0, 0, \cdots)$ for some $(x_1, x_2, \cdots) \in E \oplus E \oplus \cdots$. Then

$$\begin{aligned} x_1 &= x, \\ x_2 - f(x_1) &= 0 \\ x_3 - f(x_2) &= 0 \\ &\cdots \\ x_n - f(x_{n-1}) &= 0 \\ &\cdots \end{aligned}$$

So $x_n = f(x_{n-1}) = f^2(x_{n-2}) = \cdots = f^{n-1}(x_1) = f^{n-1}(x)$. But for some n, $x_{n+1} = f^n(x) = 0$. Therefore f is a locally nilpotent on E. □

Lemma 3. *If R is left Noetherian ring and $f : E \longrightarrow E$ is an endomorphism of a left R-module in Ω such that $\cup_{n=1}^{\infty} ker(f^n) \subset E$ is an Ω-envelope, then $\cup_{n=1}^{\infty} ker(f^n) = E$.*

Proof. Obvious. □

Lemma 4. *Let R be a left Noetherian. Let E be a left R-module in Ω and let $f_0, f_1, f_2, \cdots$ be a sequence of elements in $End(E)$. Consider those $x \in E$ such that $f_n \circ f_{n-1} \circ \cdots f_1 \circ f_0(x) = 0$ for some n(depending on x). If K is the set of such x, then K is a submodule of E.*

Proof. Obvious. □

Theorem 4. *If K is closed for all $f_0, f_1, \cdots$ i.e., $f_i(x) \in K$ for all $x \in K$ and $K \subset E$ is an Ω-envelope, then $K = E$.*

Proof. Consider the direct sum $K \oplus K \oplus \cdots$ of countable number of Ks. Since $K \subset E$ is an Ω-envelope, $K \oplus K \oplus \cdots \subset E \oplus E \oplus \cdots$ is an Ω-envelope. So define a map

$$\begin{aligned} \phi : E \oplus E \oplus \cdots &\longrightarrow E \oplus E \oplus \cdots \\ (x_1, x_2, x_3, \cdots) &\mapsto (x_1, x_2 - f_0(x_1), x_3 - f_1(x_2), \cdots) \end{aligned}$$

Then ϕ is a homomorphism, and $\phi|_{K\oplus K\oplus\cdots}$ is an injection since $\ker\phi|_{K\oplus K\oplus\cdots} = 0$. And if $(y_1, y_2, y_3, \cdots) \in K \oplus K \oplus \cdots$, then let

$$\begin{aligned} x_1 &= y_1 \\ x_2 &= y_2 + f_0(x_1) \\ x_3 &= y_3 + f_1(x_2) \\ &\cdots \end{aligned}$$

Then

$$\begin{aligned} \phi(x_1, x_2, \cdots) &= (x_1, x_2 - f_0(x_1), x_3 - f_1(x_2), \cdots) \\ &= (y_1, y_2 + f_0(x_1) - f_0(x_1), y_3 + f_1(x_2) - f_1(x_2), \cdots) \\ &= (y_1, y_2, y_3, \cdots) \end{aligned}$$

So ϕ is onto on $K \oplus K \oplus \cdots$. Hence $\phi|_{K\oplus K\oplus\cdots}$ is an isomorphism of $K \oplus K \oplus \cdots$. So by lemma 2, ϕ is an isomorphism of $E \oplus E \oplus \cdots$ and in particular ϕ is onto. Let $x \in E$ and consider $(x, 0, 0, \cdots)$. Then $\phi(x_1, x_2, x_3, \cdots) = (x, 0, 0, \cdots)$ for some $(x_1, x_2, x_3, \cdots) \in E \oplus E \oplus \cdots$. Then $x_1 = x, x_2 - f_0(x_1) = 0, x_3 - f_1(x_2) = 0$ and so on. So $x_n = f_{n-2}(x_{n-1})$ for all $n \geq 2$. But for some n, $x_{n+2} = 0$. Choose any $x \in E$ then

$$\begin{aligned} 0 &= x_{n+2} \\ &= f_n(x_{n+1}) \\ &= f_n(f_{n-1}(x_n)) \\ &= f_n(f_{n-1}(f_{n-2}(x_{n-1}))) \\ &= \cdots \\ &= (f_n \circ f_{n-1} \circ \cdots \circ f_0)(x) \end{aligned}$$

So $x \in K$ for all $x \in E$. This ends the proof. $\square$

For examples, let Ω be the collection of injective modules then Ω is closed under direct sums of a countable number of modules in Ω. The definition of Ω-envelopes is equivalent to the definition of the injective envelopes of modules.

Corollary 1. *If R is left Noetherian, E is an injective left R-module, and $f \in End(E)$ is such that E is an essential extension of $ker(f)$ denoted $ker(f) \subset' E$, then f is locally nilpotent on E.*

Corollary 2. *If R is left Noetherian, E is an injective left R-module, and $f \in End(E)$ is such that $\bigcup_{n=1}^{\infty} ker(f^n) \subset' E$, then f is locally nilpotent on E .*

Corollary 3. *Let R be a left Noetherian ring, E be an injective left R-module and $f_0, f_1, f_2, \cdots$ be a sequence of elements in $End(E)$. Let $K = \{x \in E | f_n \circ f_{n-1} \circ \cdots \circ f_0(x) = 0$ for some $n\}$. If $f_i(K) \subset K$ for all $i \geq 0$ and $K \subset' E$ then $K = E$.*

2. Gorenstein injective modules over Gorenstein rings

Definition 2. *A ring R is said to be Iwanaga-Gorenstein (or simply Gorenstein) if R is left and right Noetherian and if R has finite self-injective dimension on both sides.*

Zaks [8] has shown that when R is Gorenstein the two self-injective dimensions are the same.

Definition 3. *A left R-module G is said to be Gorenstein injective if there is an exact sequence*
$$\cdots \to E^{-2} \to E^{-1} \to E^{0} \to E^{1} \to E^{2} \to \cdots$$
of injective left R-modules such that $G = ker(E^0 \to E^1)$ and such that the functor $Hom(E, -)$ leaves the sequence exact whenever E is an injective left R-module.

All injective modules are trivially Gorenstein injective. Over a Gorenstein ring R the injective modules are the only Gorenstein injective modules precisely when R has finite global dimension. The following result gives a manageable criterion in order that a module G be Gorenstein injective over a Gorenstein ring.

Proposition 1. *If R is a Gorenstein ring and $n = inj.dim._R R$ (also $= inj.dim.R_R$) then a left R-module G is Gorenstein injective if and only if there is an exact sequence*
$$0 \to M \to E^0 \to \cdots \to E^{n-1} \to G \to 0$$
with $E^0, \cdots, E^{n-1}$ injective left R-modules, i.e. if and only if G is an n-th cosyzygy.

Proof. [9], Proposition 4.4 □

Let Ω_G be the collection of Gorenstein injective left R-modules of a Gorenstein ring R. Then the Ω_G-envelope is also called the Gorenstein injective envelope. In [9] all modules over a Gorenstein ring were proved to have such envelopes.

Lemma 5. *The collection Ω_G of Gorenstein injective modules over a Gorenstein ring is closed under direct sums. And if $M_i \longrightarrow G_i$ are Ω_G-envelopes over a Gorenstein ring for all $i \in I$, then $\oplus M_i \longrightarrow \oplus G_i$ is an Ω_G-envelope.*

Proof. Suppose $n = inj.dim_R R$. Let $(G_i)_{i\in I}$ be a family of Gorenstein injective modules. By Proposition 1 above there exist exact sequences
$$0 \longrightarrow M_i \longrightarrow E_i^0 \longrightarrow E_i^1 \longrightarrow \cdots \longrightarrow E_i^{n-1} \longrightarrow G_i \longrightarrow 0$$
such that $E_i^0, E_i^1, \cdots, E_i^{n-1}$ are all injective. Then
$$0 \longrightarrow \oplus M_i \longrightarrow \oplus E_i^0 \longrightarrow \oplus E_i^1 \longrightarrow \cdots \longrightarrow \oplus E_i^{n-1} \longrightarrow \oplus G_i \longrightarrow 0$$
is an exact sequence. Since R is left Noetherian, each $\oplus E_i^j$ is an injective module. So again by Proposition 1, $\oplus G_i$ is Gorenstein injective. Hence Ω_G is closed under the direct sums. □

Corollary 4. *If R is a Gorenstein ring, and $f \in End(G)$ is such that $ker(f) \subset G$ is an Ω_G-envelope, then f is locally nilpotent on E .*

Proof. Obvious from Theorem 2. $\square$

Corollary 5. *If R is a Gorenstein ring, and $f \in End(G)$ is such that $\bigcup_{n=1}^{\infty} ker(f^n) \subset G$ is an Ω_G-envelope, then f is locally nilpotent on E .*

Proof. Obvious from Theorem 3. $\square$

Corollary 6. *Let R be a Gorenstein ring and let $K \subset G$ be an Ω_G-envelope. Let $f_0, f_1, f_2, \cdots$ be elements of $End(G)$ such that $f_n(K) \subset K$ for all $n \geq 0$. If for each $x \in K$, there is an $n \geq 0$ such that $(f_n \circ f_{n-1} \circ \cdots \circ f_0)(x) = 0$, then the same is true for all $x \in G$.*

Proof. Obvious from Theorem 4. $\square$

Remark: If in the corollary 4 it were true that $ker(f)$ is essential in E (as is the case of an injective envelope) then the result would follow from the proof of the corresponding result for injective envelopes. However there are Gorenstein injective envelopes which are not essential extensions (see [10]).

Theorem 5. *Let R be an n-Gorenstein ring and $\{(G_i), \psi_j^i\}$ be a directed system of Gorenstein injective R-modules. Then $\lim\limits_{\rightarrow} G_i$ is Gorenstein injective.*

Proof. Let $\mathcal{X}$ be a representative set of indecomposable injective left R-modules. For left R-modules M, we form the injective modules $E^* = \oplus_{Hom(E,M)} E$, and $E' = \oplus_{E \in \mathcal{X}} E^*$ and let $E' \to M$ be defined by the evaluation maps. Then it is easy to see that $E' \to M$ is an injective precover and that this procedure is functorial. We now appeal to Proposition 4.3 [9]. That result says that if G is Gorenstein injective and if $E \to G$ is an injective cover then $E \to G$ is surjective and $ker(E \to G)$ is Gorenstein injective. But then if $E' \to G$ is an injective precover, it is easy to see that $E' = E_1 \oplus E_2$ where $E_1 \to G$ is an injective cover and where $E_2 \subset ker(E' \to G)$. So the result holds if we only assume $E \to G$ an injective precover. From this it follows that if G is Gorenstein injective, we can functorially construct an exact sequence

$$0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^{n-1} \longrightarrow G \longrightarrow 0$$

with $E^0, E^1, \cdots, E^{n-1}$ injective. Now let $\{(G_i), \psi_j^i\}$ be a directed system of Gorenstein injective R-modules. Then we functorially construct such exact sequences

$$0 \longrightarrow N_i \longrightarrow E_i^0 \longrightarrow E_i^1 \longrightarrow \cdots \longrightarrow E_i^{n-1} \longrightarrow G_i \longrightarrow 0$$

with $E_i^0, E_i^1, \cdots, E_i^{n-1}$ injective for each i. Then

$$0 \longrightarrow \lim_{\rightarrow} N_i \longrightarrow \lim_{\rightarrow} E_i^0 \longrightarrow \cdots \longrightarrow \lim_{\rightarrow} E_i^{n-1} \longrightarrow \lim_{\rightarrow} G_i \longrightarrow 0$$

is also exact. Since the ring R is Gorenstein, each $\lim\limits_{\rightarrow} E_i^j$ is injective module. Hence $\lim\limits_{\rightarrow} G_i$ is an n-th cosyzygy of $\lim\limits_{\rightarrow} N_i$ and so $\lim\limits_{\rightarrow} G_i$ is Gorenstein injective. $\square$

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MATHEMATICAL BASIS OF A SYSTEM SUPPORTING INTERCONNECTION DESIGN

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Abstract. A software system is constructed for the computation of characteristic model parameters of multi-conductor strip and microstrip lines which are needed in design of high-speed packaging and Very Large Scale Integration (VLSI) interconnections. New mathematical algorithms are developed which assure high accuracy and an excellent computational efficiency. Three major mathematical improvements are presented in this paper: (1) a new matrix square root algorithm which is used in computing an admittance matrix; (2) a new matrix algorithm based on the theory of M-matrices which provides monotonic global convergence to the exact solution in calculating a diagonally matched load impedance matrix of transmission lines; and (3) a multi-dimensional cubic spline algorithm which is used in computing intermediate values from the database of transmission line parameters. Model parameters of transmission lines are stored in a database, which can be used to determine a proper transmission line structure for a set of desired line electrical characteristics. Databases for three and five conductor transmission line systems were constructed. It was found that five conductor systems can effectively represent the seven, nine or more conductor systems for most practical applications. Use of a special spline interpolation in our algorithms involving interconnection database resulted in a substantial reduction of database size, in comparison to algorithms with linear interpolation without compromising accuracy. Consequently, computer time needed for calculations involving database is also significantly reduced. The expert system has special importance in repeated applications and in cases when several design alternatives have to be considered.

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1. INTRODUCTION.

Unrelenting increase in operating speed and size of integrated circuits and systems require careful analysis of signal transmission through interconnections. Important components in modeling these interconnections are multiple, coupled transmission lines. Transmission line models depict such significant phenomena in high speed circuits as signal delay, reflections at terminations or discontinuities, and noise which are due to electro-magnetic coupling between the adjacent lines (cross-talk) and current surges occurring on power

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and ground lines during the switching of gates [1,2]. Another trend observed in technology is the decreasing level of supply voltage, which is enforced by a dramatic increase in portable equipment applications. Noise analysis in signal transmission becomes more important in the circuits with lower operating voltage because the noise margins and noise immunities do not scale down linearly with the supply voltage. In this situation the inclusion of transmission line phenomena in the analysis and design of circuits and systems is critical. However, such phenomena are quite complicated and their analysis involves sophisticated algorithms, which require the usage of substantial computer resources [2,3].

The most critical features of software support for design are the ease of use and its response time. In most applications several design alternatives should be considered in order to find the most appropriate interconnection structure. Previous experience with design data bases for interconnections [9] indicated that efficiency of techniques is very critical in synthesis which involves repeated calculations. Given some requirements many design alternatives can be generated immediately by applying proper approximation schemes. The systematic use of previously obtained data is the basis of an "expert system,"the fundamentals of which will be also described in this paper.

Currently available interconnection support software is both complicated in use and slow in turn-around time. This paper presents new, fast algorithms for calculating interconnection model quantities such as characteristic impedance and diagonally matched load impedances. These new algorithms are very efficient in terms of computer time and thus are suitable for application in constructing an interconnection design support software.

In selecting a final model for the interconnections there is an additional trade-off between the accuracy and efficiency of computation. Lossy transmission line depicts the transmission related phenomena such as signal delay, cross-talk, and reflections at the terminations and discontinuities with adequate accuracy, but this is obtained at the price of complicated calculations requiring specialized algorithms and computer programs, significant processing time, and involved post-processing analysis. Much more efficient in this respect is a simplified modeling which neglects material losses and uses lossless transmission line models. Analysis of signal transmission with such a simplification yields upper bounds for cross-talk and reflections, which is acceptable in design. The only problem requiring caution is a signal delay which is underestimated in the analysis based on lossless transmission lines. Nevertheless, the noise estimates, which as stated above are becoming most critical, are provided by the lossless line models with sufficient accuracy. In many applications signal delay in interconnections presents only a small fraction of clock cycle. Thus in view of these considerations it was decided to base the system development on modeling interconnections by lossless transmission lines.

2. PROBLEM FORMULATION.

Characterization and analysis of transmission line system begin with a comprehensive modeling of interconnect systems.

2.1 Model of Interconnections.

A simplified model of multiple, parallel interconnections is shown in Figure 2.1. It is assumed for simplicity that the signal sending circuits (sources) are at the near-end ($x = 0$). These circuits are represented by the ideal voltage sources (e_α) with their internal impedances (r_{Gi}); $i = 1, 2, \ldots, N$. The receiving circuits located at $x = D$ are represented by resistors (r_{Li}). An interpretation of those models and receiving circuits built in specific technologies are discussed below.

The transmission line equations in terms of voltage waves are

$$\begin{aligned} u(x,t) &= u_+(x,t) + u_-(x,t) \\ i(x,t) &= Y_0 u_+(x,t) - Y_0 u_-(x,t) \end{aligned} \tag{2.1}$$

where u, i are vectors of line voltages and currents respectively, u_+ and u_- are vectors representing respective voltage waves traveling in positive and negative directions of the x-axis, and Y_0 is the characteristic admittance matrix of the interconnection system. The characteristic admittance matrix has positive diagonal and non-positive off-diagonal elements. This matrix is computed using the inductance, L, and capacitance, C, matrices of interconnecting structures. The L, C matrices are determined from the interconnect geometry and material properties using novel approach with the use of expert database. The details are provided below. Some new mathematical algorithms for computing the relevant electrical quantities of interconnects are described further in this paper. At near-end ($x = 0$) we have the following circuit relations in the vector form:

$$Y_G E_G - i_G = Y_G u(0, t) \tag{2.2}$$

where E_G, i_G are respective vectors of voltage sources, and their currents, and Y_G is a diagonal matrix

$$Y_G = \mathrm{diag}(\frac{1}{r_{G1}}, \ldots, \frac{1}{r_{GN}}). \tag{2.3}$$

At the far-end ($x = D$) the circuit relations are

$$Y_L u(D, t) = i(D, t) \tag{2.4}$$

where Y_L is a diagonal matrix

$$Y_L = \mathrm{diag}(\frac{1}{r_{L1}}, \ldots, \frac{1}{r_{LN}}). \tag{2.5}$$

2.2 Coupling Noise Estimators.

Noting that $i_G = i(0, t)$ and utilizing the transmission line equations (2.1) and (2.2) yield

$$u_+(0, t) = \tau_G E_G + \rho_G u_-(0, t) \tag{2.6}$$

where $\tau_G = (Y_0 + Y_G)^{-1} Y_G$ is the matrix of source injection coefficients and $\rho_G = (Y_0 + Y_G)^{-1}(Y_0 - Y_G)$ is the matrix of reflection coefficients at the source end.

Equation (2.6) gives basis for building an estimator of near-end cross-talk. For the limited time ($t \leq 2D\tau_{min}$ where τ_{min} is the smallest modal delay per unit of length), $u_-(0, t) = 0$ and the formula for near-end cross-talk is

$$V_{NE} = \tau_G E_G \tag{2.7}$$

and it permits to obtain the worst case estimate (one line passive with all neighbors active) of cross-talk.

Analogously the application of transmission line equations (2.1) to the far-end ($x = D$) relation yields

$$u_-(D, t) = \rho_L u_+(D, t) \tag{2.8}$$

where $\rho_L = (Y_L + Y_0)^{-1}(Y_0 - Y_L)$ is the matrix of load (far-end) reflection coefficients. Formula (2.8) gives simple estimation of far-end cross talk (V_{FE}) for $D\tau_{min} \leq t \leq 3D\tau_{min}$, because for that time the incoming voltage is $u_+(D, t) = \tau_G E_G$ and thus

$$V_{FE} = (I + \rho_L)\tau_G E_G, \tag{2.9}$$

where I is the identity matrix.

For $t < D\tau_{min}$ there is no voltage at the far-end and thus $V_{FE} = 0$. For other time intervals the formulas are slightly more complicated, because it is necessary to account for reflections.

2.3 Noise Estimation for CMOS and BJT Circuits.

In Complementary Metal-Oxide-Semiconductor (CMOS) circuits commonly used in practice, the far-end can be considered open, i.e. $Y_L = 0$, resulting in perfect reflections because the load consists of small terminating capacitances. This means that the matrix of reflection coefficients at the far-end is the identity matrix:

$$\rho_L = I. \tag{2.10}$$

The near-end in such circuits is terminated by the properly designed drivers forming so called diagonally matched load impedances which bring the entries on the main diagonal of reflection coefficients matrix to zero.

From (2.9) it is deduced that for CMOS circuits the estimate of far-end noise is

$$V_{FE} = 2\tau_G E_G. \tag{2.11}$$

It should be noted that this estimate is valid for limited time $D\tau_{min} \leq t \leq 3D\tau_{min}$, because it does not account for reflections from the near-end which may arrive after the time $3D\tau_{min}$.

The Bipolar Junction Transistors (BJT) circuits are typically terminated by the diagonally matched resistances at the far-end while the near-end circuits are driven by powerful drivers with negligible output impedances such that the near-ends are unterminated. In this case the near-end voltages (including cross-talk) may be determined by the formula (2.7). Far-end may be considered terminated, $\rho_L \cong 0$, and the far-end voltages are approximately equal to the near-end ones but shifted in time. The operation of properly designed BJT circuits is called "first incident switching "as the line voltages reach threshold levels at the first incident.

3. DETAILS OF MATHEMATICAL ALGORITHMS.

In this section the algorithms for numerical solutions to the basic computational problems are outlined and their applicability is discussed.

3.1 Matrix Square Root Algorithm.

The characteristic admittance matrix of lossless transmission line systems, composed of N-signal conductors and a reference conductor is computed using the formula

$$Y_0 = L^{-1}(LC)^{\frac{1}{2}}.$$

In order to apply this equation to find Y_0, we need a fast method to compute the square root of a matrix.

The definition of square root of and $N \times N$ matrix A can be easily rewritten as a system of N^2 nonlinear equations for the N^2 unknown elements of a square root matrix X. If a_{ij} and x_{ij} denote the (i, j) elements of A and X, respectively, then these equations are as follows:

$$a_{ij} = \sum_{l=1}^{N} X_{il} X_{lj} \qquad (i, j = 1, 2, \ldots, N). \tag{3.1}$$

The two most commonly used techniques for solving such equations are the multi-variable Newton method and eigenvalue analysis. It is well known [4] that the Newton method has only local convergence, and therefore a sufficiently good initial approximation is needed to guarantee convergence. Such initial approximations are difficult to obtain. Eigenvalue analysis is a good known method, but it needs more computation than the method suggested below [4]. This computation is particularly difficult and the algorithms may be even ambiguous in case of multiple eigenvalues. The computation speed is essential when real time computation is performed and large size data base construction is needed. In this section an algorithm based on matrix triangularization is introduced, and a closed form is presented for the square roots of triangular matrices.

Assume first that U is an upper triangular matrix with positive diagonal elements.

Theorem 1. *The square root of U is upper triangular, and it can be obtained in a finite, closed form.*

Proof. We are looking for the square root V of matrix U in an upper triangular form. Let u_{ij} and $v_{ij} (i \leq j)$ denote the (i, j) elements of U and V, respectively.

Then equation $U = V^2$ can be rewritten as

$$u_{ij} = \sum_{l=1}^{j} v_{il} v_{lj} \qquad (i \le j). \tag{3.2}$$

If $i = j$, then

$$u_{ij} = v_{ii}^2,$$

therefore for all i,

$$v_{ii} = \sqrt{u_{ii}}. \tag{3.3}$$

If $j = i + 1$, then

$$u_{ij} = v_{ii} v_{ij} + v_{ij} v_{jj},$$

which implies that

$$v_{ij} = \frac{u_{ij}}{v_{ii} + v_{jj}} \tag{3.4}$$

In the general case, when $j \ge i + 2$,

$$u_{ij} = v_{ii} v_{ij} + \sum_{l=i+1}^{j-1} v_{il} v_{lj} + v_{ij} v_{jj},$$

from which we conclude that

$$v_{ij} = \frac{1}{v_{ii} + v_{jj}} (u_{ij} - \sum_{l=i+1}^{j-l} v_{il} v_{lj}). \tag{3.5}$$

Equations (3.3), (3.4) and (3.5) give a finite algorithm to compute matrix V. Use equation (3.3) with $i = N$ to obtain v_{NN}. Select $i = N - 1$, and use equation (3.3) again with $i = N - 1$ to get $v_{N-1,N-1}$ and then use equation (3.4) with $j = N$ to compute $v_{N-1,N}$. By selecting sequentially $i = N - 2, N - 3, \ldots, 2, 1$, all other rows of V are determined in the following way. The element V_{ii} is obtained from equation (3.3), the element $v_{i,i+1}$ is computed by equation (3.4) with $j = i + 1$, and all other elements of this row are obtained by using equation (3.5) in the order, $j = i + 2, \ldots, N$. Thus the theorem is proven.

It is well known from matrix theory [4] that all real square matrices can be transformed into upper triangular form, and for example, the QR algorithm is very efficient to find this transformation. Assume that $A = T^{-1} U T$, where U is upper triangular and T is invertible. Then

$$X = T^{-1} V T, \tag{3.6}$$

where V is the square root of U, since

$$X^2 = T^{-1} V T T^{-1} V T = T^{-1} V^2 T = T^{-1} U T = A.$$

In summary, if A is a square matrix, then in the first step we find U of the triangular transformation of A. The second step is the application of the algorithm presented in the proof of the theorem for matrix U. And finally, X is obtained by using equation (3.6).

3.2 A New Matrix Algorithm.

In finding diagonally matched load impedance (impedance of drivers in CMOS circuits and far-end terminating impedances in bipolar circuits) the following problem has to be solved. Assume that Y is a given $N \times N$ real matrix with positive diagonal and non-positive off-diagonal elements. It is also assumed that Y^{-1} exists and is non-negative. Find an $N \times N$ real diagonal matrix X with positive diagonal such that matrix

$$\rho = (Y + X)^{-1}(Y - X)$$

has zero diagonal.

Let y_{ij} and r_{ij} denote the (ij) elements of Y and ρ, respectively, and let x_i denote the ith diagonal element of X. Notice first that the defining equation of matrix ρ can be rewritten as

$$(Y + X)\rho = Y - X,$$

therefore this problem is equivalent to the solution of the following system of nonlinear equations:

$$y_{ii} - x_i = \sum_{l=1,l\neq i,j}^{N} y_{il} r_{li} (i = 1, 2, \ldots, N)$$

and

$$y_{ij} = \sum_{l=1,l\neq i,j}^{N} y_{il} r_{lj} + (y_{ii} + x_i) r_{ij} \qquad (i = 1, 2, \ldots, N; j \neq i). \tag{3.7}$$

There are $N + N(N-1) = N^2$ equations for the N^2 unknowns: x_i and $r_{ij}(j \neq i)$. It can be proved that this system has at least one solution, which can be obtained by using the multi-variable Newton method, however the disadvantages of this method being discussed in the previous section still apply. An alternative approach based on diagonalization was proposed in [5], and a method based on optimization was applied in [6].

The proposed new solution method does not require the knowledge of good initial approximation, it always converges monotonically to a solution if starting from the zero matrix, and is computationally more efficient than earlier methods. In practice however we recommend the combination of our method and Newton's method in the following way. Apply our method until the approximation becomes good enough to serve as an initial approximation for a convergent Newton's sequence. Then switch to the Newton's method, which will converge very fast to the solution.

Our algorithm is based on the special mapping $F : R^{N\times N} \to R^{N\times N}$ of real matrices such that $F(A)$ is defined as the diagonal part of A for all matrices. The defining equation for ρ can be rewritten as

$$\rho = (Y + X)^{-1}(Y + X - 2X) = I - 2(Y + X)^{-1}X.$$

Therefore

$$F((Y + X)^{-1}X) = \frac{1}{2}I,$$

and consequently,

$$F((Y + X)^{-1}) = \frac{1}{2}X^{-1}.$$

This equation can be rewritten as

$$X = \frac{1}{2}[F((Y + X)^{-1})]^{-1}, \tag{3.8}$$

which is an N-dimensional fixed point problem, since X is diagonal. Consider now the iteration procedure

$$X^{(k+1)} = \frac{1}{2}[F((Y + X^{(k)})^{-1})]^{-1}, \tag{3.9}$$

starting from the initial approximation $X^{(0)} = 0$.

Theorem 2. *The iteration sequence defined by (3.9) is monotonic and bounded, therefore it converges to the smallest non-negative fixed point.*

Proof. The assumptions imply that Y is an M-matrix [7]. Therefore $X^{(1)} \geq 0 = X^{(0)}$. Assume that $X^{(k)} \geq X^{(k-1)}$ with some $k > 0$. The monotonicity of the inverse of M-matrices implies that the mapping defined by the right hand side of equation (3.8) increases in X. Therefore

$$X^{(k+1)} = \frac{1}{2}[F((Y + X^{(k)})^{-1})]^{-1} \geq \frac{1}{2}[F(Y + X^{(k-1)})^{-1})]^{-1} = X^{(k)},$$

hence $X^{(k+1)} \geq X^{(k)}$ for all $k \geq 0$. That is, the iteration sequence is monotonic.

Assume next that X^* is a non-negative fixed point, which exists. We will next prove that for all $k, X^{(k)} \leq X^*$. Obviously $X^{(0)} = 0 \leq X^*$. Assume that for some $k, X^{(k)} \leq X^*$ Then

$$X^{(k+1)} = \frac{1}{2}[F((Y + X^{(k)})^{-1})]^{-1} \leq \frac{1}{2}[F((Y + X^*)^{-1})]^{-1} = X^*,$$

therefore $X^{(k)} \leq X^*$ for all $k \geq 0$.

Since the iteration sequence is bounded, it converges to a non-negative diagonal matrix. The continuity of the right hand side of equation (3.8) in X implies that the limit is a fixed point. Since $X^{(k)}$ is bounded from above by all non-negative fixed points, the limit is the smallest non-negative fixed point, which completes the proof.

3.3 A Special Multi-Dimensional Cubic Spline Algorithm.

The practical way to reduce the memory requirement (influenced by the database size) in constructing the interconnect database must consider a good approximation schemes between the nodes. In the literature the most commonly used methods of function approximations are the interpolating polynomials, the least squares fits, and splines. In our case the first two methods cannot be applied. Interpolating polynomials have very high degree if the number of nodes is large, as it is in our case. It is well known that interpolation polynomials do not necessarily converge to the target functions and they are subject to error cummulation. Least squares polynomials do not interpolate, therefore there are always errors at the nodes. Cubic splines offer several advantages in this application. In each segment the splines are cubic polynomials, they are differentiable and smooth functions which interpolate at all nodes. For more details, see, for example [8]. The differentiality and smoothness are required in numerical simulation with the use of database. The basis for multi-dimensional spline functions is the construction of a special one-dimensional spline, which is first discussed. Consider four points $(x_{i-1}, f_{i-1}), (x_i, f_i), (x_{i+1}, f_{i+1})$ and (x_{i+2}, f_{i+2}). In segment $[x_i, x_{i+1}]$

$$s_i(x) = A_i\omega^3 + B_i\omega^2 + C_i\omega + D_i \tag{3.10}$$

where A_i, B_i, C_i, D_i are unknown coefficients and

$$\omega = \frac{x - x_i}{h_i}$$

is a new variable with $h_i = x_{i+1} - x_i$. We do not require that the step sizes $h_{i-1} = x_i - x_{i-1}, h_i = x_{i+1} - x_i$, and $h_{i+1} = x_{i+2} - x_{i-1}$ are the same as in most publications.

The following conditions are assumed:

$$\begin{aligned} s_i(x_i) &= f_i \\ s_i(x_{i+1}) &= f_{i+1} \\ s_i'(x_i) &= p_i'(x_i) \\ s_i'(x_{i+1}) &= p_{i+1}'(x_{i+1}) \end{aligned} \tag{3.11}$$

where p_i and p_{i+1} are the quadratic interpolation polynomials fitting the points $(x_{i-1}, f_{i-1}), (x_i, f_i), (x_{i+1}, f_{i+1})$, and $(x_i, f_i), (x_{i+1}, f_{i+1}), (x_{i+2}, f_{i+2})$, respectively. The first and second conditions mean that s_i interpolates at the two end-points of the segment, the third and fourth equations imply that

$$s_i'(x_i) = s_{i-1}'(x_i)$$

and

$$s_i'(x_{i+1}) = s_{i+1}'(x_{i+1}),$$

that is, the spline is differentiable at the nodes. Between the nodes it is obviously differentiable, since it is a cubic polynomial. Conditions (3.11) can be rewritten as

$$\begin{aligned} &D_i = f_i \\ &A_i + B_i + C_i + D_i = f_{i+1} \\ &\frac{1}{h_i} C_i = -\frac{h_i}{h_{i-1}(h_{i-1} + h_i)} f_{i-1} + \frac{-h_{i-1} + h_i}{h_{i-1} h_i} f_i + \frac{h_{i-1}}{h_i(h_{i-1} + h_i)} f_{i+1} \\ &\frac{1}{h_i}(3A_i + 2B_i + C_i) = -\frac{h_{i+1}}{h_i(h_i + h_{i+1})} f_i + \frac{-h_i + h_{i+1}}{h_i h_{i+1}} f_{i+1} + \frac{h_i}{h_{i+1}(h_i + h_{i+1})} f_{i+2} \end{aligned}$$

where we use the well known Lagrange form of interpolation polynomials [4]. We have four linear equations for four unknown coefficients. After solving for A_i, B_i, C_i, D_i substitute the results into the cubic polynomial form of $s_i(x)$ to obtain the formula

$$s_i(x) = f_{i-1} P_{i-1}(\omega) + f_i P_i(\omega) + f_{i+1} P_{i+1}(\omega) + f_{i+2} P_{i+2}(\omega), \tag{3.12}$$

where

$$\begin{aligned} P_{i-1}(\omega) &= \alpha_i \omega^3 - 2\alpha_i \omega^2 + \alpha_i \omega \\ P_i(\omega) &= (2 + \beta_i + \gamma_i)\omega^3 + (-3 - 2\beta_i - \gamma_i)\omega^2 + \beta_i \omega + 1 \\ P_{i+1}(\omega) &= (-2 + \delta_i + \eta_i)\omega^3 + (3 - 2\delta_i - \eta_i)\omega^2 + \delta_i \omega \\ P_{i+2}(\omega) &= \phi_i \omega^3 - \phi_i \omega^2 \end{aligned} \tag{3.13}$$

with

$$\begin{aligned} \alpha_i &= \frac{h_i^2}{h_{i-1}(h_{i-1} + h_i)}, \beta_i = \frac{-h_{i-1} + h_i}{h_{i-1}}, \gamma_i = -\frac{h_{i+1}}{h_i + h_{i+1}}, \\ \delta_i &= \frac{h_{i-1}}{h_{i-1} + h_i}, \eta_i = \frac{-h_i + h_{i+1}}{h_{i+1}}, \quad \text{and} \quad \phi_i = \frac{h_i^2}{h_{i+1}(h_i + h_{i+1})}. \end{aligned}$$

For simple calculation we notice that

$$\delta_i = -\gamma_{i-1} \frac{h_{i-1}}{h_i}, \eta_i = \beta_{i+1} \frac{h_i}{h_{i+1}}, \alpha_i = \gamma_{i-1} \frac{h_i}{h_{i-1}},$$

and

$$\phi_i = -\gamma_i(\frac{h_i}{h_{i+1}})^2.$$

Based on equations (3.12) and (3.13) multi-dimensional splines are constructed in the same way as B-splines are defined [8]. Consider and n-dimensional net, where in coordinate $k(k = 1, 2, \ldots, n)$ the nodes are $x_{k1}, x_{k2}, \ldots, x_{kn_k}$. Define $h_{ki} = x_{k,i+1} - x_{ki}$ and assume that for a point $(x_1, \ldots, x_n), x_{ki_k} < x_k \leq x_{k,i_k+1}$ for all k. Let $\omega_k = \frac{x_k - x_{ki_k}}{h_{ki_k}}$, and define polynomials $P_{k,i_k-1}, P_{k,i_k}, P_{k,i_k+1}$, and P_{k,i_k+2} similarly to equation (3.13) where $\alpha_i, \beta_i, \gamma_i, \delta_i, \eta_i$, and ϕ_i are computed by using step sizes in direction x_k. Then the spline segment is given as

$$S_{i_1 i_2 \ldots i_n}(x_1, \ldots, x_n) = \sum_{l_1=i_1-1}^{i_1+2} \sum_{l_2=i_2-1}^{i_2+2} \cdots \sum_{l_n=i_n-1}^{i_n+2} P_{1l_1}(\omega_1) P_{2l_2}(\omega_2) \ldots P_{nl_n}(\omega_n) f_{l_1 l_2 \ldots l_n} \tag{3.14}$$

where $f_{l_1 l_2 \ldots l_n}$ is the function value at the node $(x_{1l_1}, x_{2l_{2\ldots,}} x_{nl_n})$.

4. IMPLEMENTATION AND CASE STUDIES.

We now turn to the problem of implementation of the new algorithms for computing capacitance and inductance matrices of multi-conductor transmission line systems and other quantities relevant to signal transmission on packaging interconnections. The algorithms described in Section 3 have been implemented in a prototype design support system, named DIME (DIagonally Matched load impedance Estimator), written in C-language. The system DIME is currently running on SPARC work stations.

4.1 System Structure.

The software package consists of two main parts: routine computation of characteristic quantities in signal transmission, and analysis/design synthesis routine. Figure 4.1 shows the block diagram of the entire system. The routine for computation of transmission lines characteristic quantities includes a graphic editor accepting the pictorial input of the interconnecting structure. The program can accept both ASCII data and graphical representations of an interconnecting system cross section as its input data for analysis. The graphic editor captures the cross section of the structure. The geometry of cross section is specified in terms of grid point numbers, where geometrical resolution is controlled by the number of grid points used to represent the minimum feature of a transmission line system. The input data from the graphic editor is used to compute transmission line characteristics such as capacitance, inductance and diagonally matched load impedance matrices. The results are stored in a database for analysis or synthesis of interconnections.

Using the outputs from the parameter computation routine, a database of transmission line characteristics is generated. The order of complexity of the database is $O(n^m)$ where n is the number of the database entries (nodes) generated for interconnection technology, and m is the number of conductor dimensions representing a given transmission line geometry. For a homogeneous strip line structure, $m = 3$, since the following three independent variables are used: normalized conductor width, normalized conductor thickness, and normalized inter-conductor separation. The stated quantities are normalized with respect to the thickness of dielectric layer supporting the conductors.

The databases similar to the one reported in [9] are used to obtain desired transmission line characteristics and their physical dimensions. The database entries are used to calculate the quantities not found directly in the database using an interpolation routine. A set of conductor dimensions (thickness, width, and inter-conductor separation) is a key to access generated stripline databases; whereas the thicknesses and dielectric constants of dielectrics are additional (secondary) quantities added to the access key data for computation of actual dimensions of designed structure.

Database generation sequence is shown in Figure 4.2. It consists of three parts which are performed repeatedly for all the combinations of dimension values during database generation: node selection for each dimension of transmission line structure, calculation of characteristics for each combination of dimension value selected, and insertion into database.

4.2 Case Studies.

The program package has been applied to many cases of multi-conductor strip and micro-strip line structures with a specific epoxy-glass (glass fabric soaked with an epoxy-resin, trade name FR-4) and alumina (Al_2O_3) as their dielectric layers. The data reported here are for structures with FR-4. We assume that the width of all conductors in a transmission line system is the same. Figure 4.3 shows the diagonally matched load impedance, Z_{dm}, curves of the center line with fixed conductor thickness of three parallel transmssion strip line systems, where the relative dielectric constant of FR-4 is 4.4. The curves show monotonous decrease as the width of the conductor increases and the interconductor separation decreases. The same types of curves are shown in Figure 4.4, where fixed width (curves a) and fixed separation (curves b) are used. The diagonally matched load impedance values decrease as the thickness increases (curves b) and increase as the separation increases monotonically (curves a). The impedance curves of other lines of the three conductor transmission line system are slightly higher than the ones presented here.

Cases with five parallel conductors are illustrated in Figure 4.5, where the center line impedances with fixed conductor thickness (Figure 4.5a), and fixed conductor width and fixed inter-conductor separation (Figure 4.5b) are used. The shape of the curves are very similar to the three conductor cases, but the values of the diagonally matched load impedance are lower than those of the three conductor cases. These differences in diagonally matched load impedances are due to couplings from the additional (fourth and fifth) conductors.

As the number of conductors in the interconnecting system increases, the diagonally matched load impedance values decrease. The differences in the impedance values between N conductor systems and $N+1$ conductor systems decreases as N increases. Figure 4.6 shows the changes in diagonally matched load impedance due to the increase in the number of conductors. The curves were calculated for the conductor width of 1 mil, inter-conductor separation of 1 mil, and 20 mil in dielectric layer thickness, respectively. As shown in Figure 4.6, the impedance value levels off beyond 5 conductors. The same computation was carried out for many different conductor and dielectric dimensions, and similar behavior was always observed.

As an extreme case, systems with different heights were examined. The biggest difference in impedance between 5 and 7 conductor system does not exceed 15%. The differences between 7 and 9 conductor system do not exceed 6% in magnitude for FR-4 dielectric layers as shown in Figure 4.7. As the number of conductors increases, the additional conductors are located far from the conductor under consideration, resulting in relatively smaller contribution in impedance than nearby ones. In this study it was found that modeling with a 5 conductor system can adequately represent most practical cases of multi-conductor transmission line systems in diagonally matched load impedance computation. Databases for 3600 dimension sets for 3 and 5 conductor systems were established using the program. Dimensions not found in the database entries are interpolated using our three-dimensional cubic spline algorithm.

Several cases of multi-conductor microstrip line systems have been examined with several layers of dielectric material. Figure 4.8 shows an example of a three conductor system with two layers of dielectric, where the first dielectric layer represents FR-4 and the second one represent anti-corrosion coating material such as polyamide with a dielectric constant $\varepsilon = 3.0$. The system can handle structures with up to four layers of different dielectrics.

The developed expert system was applied to a three conductor structure described in [10]. The system produces results which are about 5% different from those reported in [10]. Figure 4.9 presents the diagonally matched load impedance of the center line of the three conductor system: Figure 4.9a shows the impedance curves for a fixed conductor thickness; and Figure 4.9b includes curves for a fixed conductor width and curves for a fixed inter-conductor separation. These values are slightly higher than those for strip line cases shown earlier in Figure 4.3.

Microstrip line cases with five conductors are illustrated in Figure 4.10: Figure 4.10a presents the cases with a fixed conductor thickness; Figure 4.10b shows cases with a fixed conductor width and cases with a fixed inter-conductor separation; and cases with the second dielectric thickness are presented in Figure 4.10c. The diagonally matched load impedances of the center conductor of the five conductor system are slightly lower than those for the three conductor system, but are higher than those for the strip line cases shown in Figure 4.5. This is due to the impedance contribution from two additional parallel conductors of the five conductor system. Notice that in Figure 4.10c the impedance drops sharply as the thickness begins to increase and rises slightly as the thickness becomes large. This implies that the thickness of a thin coating layer should be controlled precisely, whereas the thick layer can be used for crude process environment. As the number of conductors increases, the diagonal load impedance value of the center conductor decreases.

Figure 4.11 describes the changes of the diagonally matched load impedance of the center conductor as the number of conductors increases. Significant changes are observed among two, three, four, and five conductor systems, but the differences become negligible for the cases beyond five conductor cases. It implies that five conductor systems can effectively represent seven, and nine conductor systems, as in strip line cases. The impedance difference between seven and nine conductor systems does not exceed 3% as shown in Figure 4.12. This impedance difference can be interpreted in terms of near-end and far-end noise defined in the equations (2.7) and (2.9).

Figure 4.13 shows the changes in noise levels as the number of conductors increases. This is computed by applying 5V step pulses to all parallel conductors except the center conductor where the noise levels are measured. Here again, very little changes in noise level are observed beyond seven conductors. It is our conclusion that seven, nine, and eleven conductor systems can be represented by five conductor systems.

4.3 Accuracy of Computation and Database Size.

Enhancing computational accuracy and efficiency where the major goals in developing the new mathematical algorithms. These goals were achieved using new efficient methods for solving field problems to generate database, and a cubic spline algorithm for very fast interpolation between data values allowing the interactive use of the system in the interconnection design process.

Figure 4.14 shows a strip line structure with two conductors. The conductor dimensions are the same as those used by Scheinfein and Palusinski [11]. As shown in Figure 4.14a, the normalized thickness τ of the conductor is 0.25, the normalized width ω is 0.5, and the inter-conductor separation σ is 0.5, and the height h is 1.0. The computed results presented in Figure 4.14b show that they are very close to the measured values.

A study on accuracy and database size reduction was carried out for two interpolation schemes: linear and spline interpolation. Interpolation with three-dimensional cubic spline algorithm can generate diagonally matched load impedances with much higher accuracy than linear interpolation. Figure 4.15 shows an accuracy comparison between spline interpolation (solid line) and linear interpolation (dotted line). The difference between the exact computation and the cubic spline interpolation is negligible when 8 interpolation nodes are used. About 10% difference between the exact computation and the linear interpolation was observed when the same number of interpolation nodes were used.

To reduce the size of the database by reducing the number of interpolation nodes, some nodes were dropped from the database and spline interpolation. In order to provide the rationale for database size reduction an accuracy degradation resulting from the size reduction was examined. Figure 4.16 includes three cases of different node selections: 12 nodes (solid line), 6 nodes (dotted line) and 4 nodes (broken line). The impedance values were computed for fixed conductor width and fixed inter-conductor separation (differences are not well visible in this scale). For better illustration of consequences of size reduction the differences in Z_{dm} are plotted in Figure 4.17. The differences between the case of 12 nodes and of 6 nodes are negligible: the differences do not exceed 1.5%, which is acceptable for most practical applications. More substantial differences between the case of 12 nodes and that of 4 nodes were observed in some thickness ranges. Larger differences were observed when the impedance values change significanly; the smaller the impedance values the smaller the differences between the interpolated results and the exact calculation results. From the above observations, it was concluded that the number of spline nodes of each dimension can be reduced by half, in comparison to a case with linear interpolation without significant accuracy degradation.

Substantial database size reduction can be achieved using spline interpolation without significant accuracy degradation. With 50% of spline nodes for each of three dimensions, the database size can be reduced to 12.5% of the original size. The database size is even a more crucial factor in building an expert system for designing interconnects with layer dielectrics, since the number of dimensions increases.

5. Conclusion.

With the new development in microelectronics and rapidly expanding applications in portable equipment, high speed digital circuits with low operating voltage are gaining in importance. Such circuits require accurate estimation of noise because their noise margin (noise immunity) are getting reduced. Such circuits are very susceptible to signal perturbations and signal integrity is a principal requirement in their design. This paper presents three major mathematical improvements in calculating interconnection characteristics essential for the estimation of noise in high speed digital circuits. A new matrix square root algorithm is more efficient than other algorithms known in the literature. A new efficient matrix algorithm based on the theory M-matrices was implemented to determine diagonally matched load impedance. A special multidimensional cubic spline algorithm gives a very accurate interpolation scheme for interconnect database and also allows usage of database with significantly reduced size. These algorithms are successfully exploited in a system supporting interconnection design.

The optimum size of the database can be determined by finding the best location of the nodes to be used for spline large-scale interpolation with a given error tolerance. This otimization problem is nonlinear, non-convex, and therefore only heuristic algorithms can be developed for its solution. In our next paper, this problem will be addressed in detail and a particular solution method will be introduced. Database of 3 and 5 conductor systems are constructed using the new program, each comprising 1800 sets of transmission line characteristics. The computation results obtained from the new algorithms were verified using available measured data. The comparisons of calculated and measured values were satisfactory.

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