Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISBN 1083-0464
Issue 2 December 1998, pp. 1 – 6
Submitted: September 1, 1998. Published: February 1, 1999

EIGENVALUE DISTRIBUTION OF INTEGRAL OPERATORS DEFINED BY ORLICZ KERNELS

E. Elshobaky and M. Faragallah

Abstract

In this paper the new interpolation method (see [GP], [FG]) has been applied to drive some theorems, e.g., the interpolation of L_p spaces, l_p spaces, and b^{σ}_{pu} (see [EF]). Using the interpolation results in [EF] with the eigenvalue results we give a good estemate for the eigenvalues of integral operators defined by kernels belong to vector valued Orlicz spaces $L_p(L_\Phi)$, $l_p(l_\Phi)$ and vector valued Besov - Orlicz sequence spaces $b^s_{pu}(b^t_{\Phi v})$.

1991 A.M.S. Subject Classification Codes. 46B20, 47B10
Key Words and Phrases: Integral operator, Orlicz kernel, interpolation, Besov-Orlicz sequence

1. Interpolation spaces

In this section we summarize some well known definitions and theorems and we will also drive some interpolation results.

Theorem 1.1 (FG). *Let (X_0, X_1) and (Y_0, Y_1) be Banach couples, $\rho \in P^{+-}$ and $1 \le p, q < \infty$. Then $\langle X_0, X_1\rangle_{\rho,p} = \langle X_0, X_1\rangle_{\rho,q}$, with norm equivalence.*

Elshobaky: Department of Mathematics, Faculty of science,
Ain Shams University, Cairo, Egypt
Faragallah: Department of Mathematics, Faculty of Education,
Ain Shams University, Cairo, Egypt
E-mail: nagwamsa@frcu.eun.eg

Theorem 1.2 (FG). *Let (X_0, X_1) be a Banach couple. Then for all $p, 1 \leq p < \infty$, we have*

$$\langle L_p(X_0), L_p(X_1)\rangle_{\rho,p} = L_p(\langle X_0, X_1\rangle_{\rho,p}), \tag{1}$$

with equivalent norms.

Remark 1.3. *Using the fact that l_q^s-spaces are special L_q-spaces and the previous theorem, we have*

$$\langle l_q^s(X_0), l_q^s(X_1)\rangle_{\rho,p} = l_q^s(\langle X_0, X_1\rangle_{\rho,p}), \tag{2}$$

with the same assumption as in Theorem (1.2).

By an Orlicz function we mean a convex and continuous function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$.

Theorem 1.4 (FG). *Let Φ be an Orlicz function and let $L_\Phi(E)$ be the corresponding E-valued Orlicz space on a σ-finite measure space, where E is a Banach space. Suppose that Φ and the conjugate Φ^* satisfy the Δ_2-condition. Then there exist p_0 and p_1, with $1 < p_1 < p_0 < \infty$,*

$$\langle L_{p_0}(E), L_{p_1}(E)\rangle_{\rho,p} = L_\Phi(E), \qquad \textit{where} \tag{3}$$

$$\rho(t) = \begin{cases} t^{\frac{p_1}{p_1-p_0}} \quad \Phi^{-1}(t^{\frac{p_0p_1}{p_0-p_1}}), & t > 0 \\ 0 & t = 0 \end{cases} \tag{4}$$

and $1 < p < \infty$.

Theorem 1.5. *Let Φ be an Orlicz function and let $l_\Phi(E)$ be the corresponding E-valued Orlicz space on a σ-finite measure space, where E is a Banach space. Suppose that Φ and the conjugate Φ^* satisfy the Δ_2-condition. Then there exist p_0 and p_1, with $1 < p_1 < p_0 < \infty$,*

$$\langle l_{p_0}(E), l_{p_1}(E)\rangle_{\rho,p} = l_\Phi(E) \tag{5}$$

where ρ is given by (4).

Theorem 1.6. *Let Φ be an Orlicz function which satisfies, together with its conjugate Φ^*, the Δ_2-condition. Then there exist p_0 and p_1, with $1 < p_0, p_1 < \infty$, such that*

$$\langle b_{p_0q}^s(X), b_{p_1q}^s(X)\rangle_{\rho,p} = b_{\Phi q}^s(X), \tag{6}$$

where ρ is given by (4).

Proof: Using Remark (1.3) and Theorem (1.5) we have

$$\langle b_{p_0q}^s(X), b_{p_1q}^s(X)\rangle_{\rho,p} = \langle l_q^s(l_{p_0}(X)), l_q^s(l_{p_1}(X))\rangle_{\rho,p}$$

$$= l_q^s\langle l_{p_0}(X), l_{p_1}(X)\rangle_{\rho,p} = l_q^s(l_\Phi(X)) = b_{\Phi q}^s(X)$$

Theorem 1.7. *Let $1 < p, q < \infty$, and $s \in R$. Then we have,for Banach spaces X_0, X_1*

$$\langle b^s_{pq}(X_0), b^s_{pq}(X_1)\rangle_{\rho,p} = b^s_{pq}(\langle X_0, X_1\rangle_{\rho,p}). \tag{7}$$

Proof: Using Theorem (1.2) and Remark (1.3) we have

$$\langle b^s_{pq}(X_0), b^s_{pq}(X_1)\rangle_{\rho,p} = \langle l^s_q(l_p(X_0)), l^s_q(l_p(X_1))\rangle_{\rho,p}$$

$$= l^s_q(l_p)\langle X_0, X_1\rangle_{\rho,p} = b^s_{pq}\langle X_0, X_1\rangle_{\rho,p}$$

Theorem 1.8. *Let $1 < p, q, u_0, u_1, v < \infty$ and $s \in R$. Then we have*

$$\langle b^s_{pq}(b^t_{u_0 v}), b^s_{pq}(b^t_{u_1 v})\rangle_{\rho,q} = b^s_{pq}(b^t_{\Phi v}), \tag{8}$$

with,

$$\rho(\tau) = \tau^{\frac{u_1}{u_1-u_0}} \quad \Phi^{-1}(\tau^{\frac{u_0 u_1}{u_0-u_1}}).$$

Proof: Using Theorems (1.6) and (1.7) we have

$$\langle b^s_{pq}(b^t_{u_0 v}(X)), b^s_{pq}(b^t_{u_1 v}(X))\rangle_{\rho,p} = b^s_{pq}\langle b^t_{u_0 v}(X), b^t_{u_1 v}(X)\rangle_{\rho,p}$$

$$= b^s_{pq}(b^t_{\Phi v}(X))$$

Definition 1.9. *Let Φ be an Orlicz function, we define for $1 \le q < \infty$, the Lorentz-Orlicz spaces $l_{\Phi q}$ as follows*

$$l_{\Phi q} = \left\{ x = (x_n)_n \quad : \|x\|_{l_{\Phi q}} := \inf\left(c : (\sum_{n=1}^{\infty} \Phi(\frac{|x_n|}{c})^q)^{1/q} \le 1 \right) < \infty \right\}, \tag{9}$$

and,

$$l_{\Phi\infty} = \left\{ x = (x_n)_n \quad : \|x\|_{l_{\Phi\infty}} := \inf\left(c : \sup_n \Phi(\frac{|x_n|}{c}) \le 1 \right) < \infty \right\}. \tag{10}$$

Definition 1.10. *Let Φ be an Orlicz function and X be a Banach space. We define for $1 \le q \le \infty$, the Schatten-Orlicz classes $S^s_{\Phi q}$ as follows*

$$S^s_{\Phi q}(X) := \{T \in L(X) \quad : (s_n(T))_n \in l_{\Phi q}\}. \tag{11}$$

Theorem 1.11 (EF). *Let $1 < p_0, p_1 < \infty$. Then we have*

$$\langle S^s_{p_0 q}(X), S^s_{p_1 q}(X)\rangle_{\rho,p} \subset S^s_{\Phi q}(X), \tag{12}$$

where ρ is given by

$$\rho(t) = \begin{cases} t^{\frac{p_1}{p_1-p_0}} \quad \Phi^{-1}(t^{\frac{p_0 p_1}{p_0-p_1}}) & ,t > 0 \\ 0 & t = 0 \end{cases} \tag{13}$$

and $1 < p < \infty$.

2. Eigenvalue distribution of integral operators

Theorem 2.1 (C). *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} > 1$ and $K \in l_p(l_q)$ i.e., $K = (k_{ij})_{i,j\in\mathbb{N}}$ satisfy*

$$\|K\| = \left\{ \sum_i \left\{ \sum_j |k_{ij}|^q \right\}^{p/q} \right\}^{1/p} \quad < \quad \infty.$$

Define s by

$$(14) \qquad \frac{1}{s} = \begin{cases} \frac{1}{p} + \frac{1}{q} - \frac{1}{2} & , p < 2 \quad \text{and} \quad q > 2 \\ \frac{1}{p} & p \geq 2 \quad \text{or} \quad p < 2, q \leq 2 \end{cases}$$

Then the integral operator $T_K : l_r \to l_r$ is well defined Riesz operator for $p < r < q'$ and its eigenvalues belong to the Lorentz sequence space $l_{s,p}$ with

$$\|(\lambda_n(T_K))_{n\in\mathbb{N}}\|_{l_{s,p}} \leq c(p,q)\|K\|_{pq}.$$

Theorem 2.2 (P1). *Let $1 \leq p, q, u, v \leq \infty$ and $\alpha, \beta > 0$ with $\alpha + \beta > 1 - \frac{1}{p} - \frac{1}{u}$. Define s by*

$$(15) \qquad \frac{1}{s} = \alpha + \beta + \begin{cases} \frac{1}{p} + \frac{1}{u} - \frac{1}{2} & , 2 \leq u \leq \infty \\ \frac{1}{p} & 1 \leq u \leq 2 \end{cases}$$

Then $K \in b^{\alpha}_{pq}(b^{\beta}_{uv})$ implies $(\lambda_n(K))_{n\in\mathbb{N}} \in l_{s,q}$

Theorem 2.3 (P2, KW). *Let $\frac{1}{p} + \frac{1}{q} \leq 1$. Then the eigenvalues of any kernel $K \in L_p(L_q(X))$ belong to the Lorentz sequence space $l_{q^+} = l_{\max(2,q')}$.*

Theorem 2.4. *Let $1 < p, q_i < \infty$ with $\frac{1}{p} + \frac{1}{q_i} > 1$ $(i = 0, 1)$ and $K \in l_p(l_{q_i})$ i.e., $K = (k_{ij})_{i,j\in\mathbb{N}}$ satisfy*

$$\|K\| = \left\{ \sum_i \left\{ \sum_j |k_{ij}|^{q_i} \right\}^{p/q_i} \right\}^{1/p} \quad < \quad \infty.$$

Let

$$(16) \qquad \frac{1}{s_i} = \begin{cases} \frac{1}{p} + \frac{1}{q_i} - \frac{1}{2} & , p < 2 \quad \text{and} \quad q_i > 2 \\ \frac{1}{p} & p \geq 2 \quad \text{or} \quad p < 2, q_i \leq 2 \end{cases}$$

Then the integral operator $T_K : l_{\Phi^*} \to l_p$ is well defined Riesz operator for $x^p \leq \Phi^*(x) \leq x^{q_i'}$ and its eigenvalues belong to the Lorentz - Orlicz sequence space $l_{\Psi,p}$ with

$$\|(\lambda_n(T_K))_{n\in\mathbb{N}}\|_{l_{\Psi,p}} \leq c(p,q)\|K\|_{p,\Phi} \qquad \text{where}$$

$$\Phi^{-1}(\tau) = \tau^{(\frac{1}{2} - \frac{1}{p}} \quad \Phi^{-1}(\tau).$$

Proof: Using the eigenvalue Theorem (2.1), we have

$$l_p(l_{q_i}) \subset S^s_{s_i,p}. \tag{17}$$

By Theorems (1.2), (1.4), (1.11) and Corollary (1.5) we have

$$\langle l_p(l_{q_0}), l_p(l_{q_1})\rangle_{\rho,q} = l_p(l_\Phi) \subset \langle S^s_{s_0p}, S^s_{s_1p}\rangle_{\rho,q} = S^s_{\Psi p}, \tag{18}$$

with,

$$\rho(\tau) = \tau^{\frac{q_1}{q_1-q_0}} \quad \Phi^{-1}(\tau^{\frac{q_0q_1}{q_0-q_1}}). \tag{19}$$

and

$$\rho(\tau) = \tau^{\frac{s_1}{s_1-s_0}} \quad \Psi^{-1}(\tau^{\frac{s_0s_1}{s_0-t_1}}). \tag{20}$$

(i) The case $p \geq 2$ or $p < 2$, $q_i \leq 2$:
In this case we have $s_0 = s_1 = p$, hence $l_p(l_\Phi) \subset S_p$.
(ii) The case $p < 2, q_i > 2$:
In this case we have

$$\frac{1}{s_0} - \frac{1}{s_1} = \frac{1}{q_0} - \frac{1}{q_1} \Rightarrow \qquad \frac{s_0s_1}{s_0-s_1} = \frac{q_0q_1}{q_0-q_1} = m$$

$$\text{and} \quad \frac{s_1}{s_1-s_0} = -\frac{m}{t_0} \quad , \quad \frac{q_1}{q_1-q_0} = -\frac{m}{q_0}.$$

Substituting in (19) and (20) we have

$$\Phi^{-1}(\tau^m) = \tau^{m(\frac{1}{q_0}-\frac{1}{s_0}} \quad \Phi^{-1}(\tau^m) = \Phi^{-1}(\tau^m) = \tau^{m(\frac{1}{2}-\frac{1}{p}} \quad \Phi^{-1}(\tau^m).$$

Then, we readily get the desired results.

Using the same method as above we can prove the following theorems.

Theorem 2.5. *Let $1 \leq p, q, u_0, u_1, v \leq \infty$ and $\alpha, \beta > 0$ with $\alpha + \beta > 1 - \frac{1}{p} - \frac{1}{u_i}$. Define s_i by*

$$\frac{1}{s_i} = \alpha + \beta + \begin{cases} \frac{1}{p} + \frac{1}{u_i} - \frac{1}{2}, & 2 \leq u_i \leq \infty \\ \frac{1}{p} & 1 \leq u_i \leq 2 \end{cases} \tag{21}$$

Then $K \in b^\alpha_{pq}(b^\beta_{\Phi v})$ implies $(\lambda_n(K))_{n\in\mathbb{N}} \in l_{\Psi,q}$ with

$$\Phi^{-1}(t) = t^m \Psi^{-1}(t) \quad , \quad m = \frac{1}{2} - (\alpha + \beta + \frac{1}{p})$$

Theorem 2.6. *Let $1 \leq p_0, p_1, q, u, v \leq \infty$ and $\alpha, \beta > 0$ with $\alpha + \beta > 1 - \frac{1}{p_i} - \frac{1}{u}$. Define s_i by*

$$\frac{1}{s_i} = \alpha + \beta + \begin{cases} \frac{1}{p_i} + \frac{1}{u} - \frac{1}{2}, & 2 \leq u \leq \infty \\ \frac{1}{p_i} & 1 \leq u \leq 2 \end{cases} \tag{22}$$

Then $K \in b^{\alpha}_{\Phi q}(b^{\beta}_{uv})$ *implies* $(\lambda_n(K))_{n\in\mathbb{N}} \in l_{\Psi,q}$ *with*

$$\Phi^{-1}(t) = t^m \Psi^{-1}(t) \quad , \quad m = \frac{1}{2} - (\alpha + \beta + \frac{1}{u})$$

Theorem 2.7. *Let* $\frac{1}{p} + \frac{1}{q_i} \leq 1 \quad (i = 0, 1)$. *Then the eigenvalues of any kernel* $K \in L_p(L_\Phi(X))$ *belong to the Lorentz sequence space* l_Ψ *with*

$$\Phi^{-1}(t) = t\Psi^{-1}(t^{-1}) \quad .$$

References

C: B. Carl; On a characterization of operators from l_q into a Banach space of type p with some applications to eigenvalue problems. J. Funct. Anal. 48 394–407.

EF: E. Elshobaky, M.Faragallah; Eigenvalue distribution of integral operators defined by Besov–Orlicz kernels. J. Southwest of Pure and Applied Math. 2 (1997) 29– 42.

FG: D. L. Fernandez, J. B. Garcia; Interpolation of Orlicz-valued spaces, Stud. Math. 99 (1991) 23– 39.

GP: J. Gustavsson, J. Peetre; Interpolation of Orlicz spaces,Stud. Math. 60 (1977) 33– 59.

KW: H. König, L. Weis; On the eigenvalues of order bounded integral operators. Integral equation and operator theory, 6 (1983) 706– 729.

P1: A. Pietsch; Eigenvalues of integral operators, Math. Ann. 247 169– 178 .

P2: A. Pietsch; Eigenvalues and s- numbers; 1987.