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A FACTORED FORM OF COMPLEX ANALYTIC FUNCTIONS AND ITS APPLICATION TO FINDING PARTIAL FRACTION DECOMPOSITIONS

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Abstract. A fast algorithm is introduced to find the coefficients of the partial fraction decomposition of proper rational functions. Instead of solving a large-scale system of linear algebraic equations with full coefficient matrix or differentiating repeatedly rational functions, a special system of linear algebraic equations is solved, where the coefficient matrix is lower triangular. Therefore, there is no need for elimination, the solutions are obtained by forward substitution making the procedure very effective. The method is based on a special representation of analytic functions which has further applications in constructing iteration methods for solving nonlinear equations.

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1. INTRODUCTION.

Special function representations have been shown to be very useful in all kinds of applications. This paper introduces a new factored form of analytic complex functions, which can be applied to derive a fast algorithm to find partial fraction decompositions for rational functions with high multiplicity poles as well as in constructing iteration schemes for solving nonlinear equations. Integrating rational functions and solving time-invariant linear systems by the transformation method require the determination of the partial fraction decomposition of proper rational functions ([2],[3],[4]). In addition, several computer methods for solving polynomial equations are based on manipulations of rational functions ([1]). Partial fraction decomposition make the application of such methods easier to apply.

In the literature two methods are prevalent for performing partial fraction decompositions. In the case of first method, known as the algebraic method, a large system of linear equations is solved, which can be very difficult, unless the coefficient matrix has a special structure. In the second method, called the residue method, repeated derivatives of rational functions have to be determined, which is sometimes very complicated. The algorithm presented in this paper also requires the solution of a large system of linear algebraic equations; however, the coefficient matrix is of lower-triangular type and therefore there is no need for elimination, the solution can be easily obtained by forward substitution.

This paper develops as follows. In section 2 a unique representation theorem will be proved for analytic functions. This will be the basis for determining the partial fraction coefficients. The detailed algorithm will be presented in section 3, and section 4 concludes the paper with an illustrative example.

2. A REPRESENTATION THEOREM.

Let $n \geq 1$ be an integer, and assume that functions P and Q are analytic on an open complex domain D. Let $\eta \in D$ be a given point and assume that for $k = 0, 1, \ldots, n-1, Q^{(k)}(\eta) \neq 0$.

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Theorem 1. *There exists a unique sequence $a_0, A_1, \ldots, A_{n-1}$ of complex numbers and a unique analytic complex function P_1 such that*

$$P(s) = Q(s) \cdot [A_0 + A_1(s-\eta) + \cdots + A_{n-1}(s-\eta)^{n-1}] + P_1(s) \cdot (s-\eta)^n. \tag{1}$$

Proof. Define $R(s) = A_0 + A_1(s-\eta) + \cdots + A_{n-1}(s-\eta)^{n-1}$, and assume that decomposition (1) exists. For $k = 0, 1, \ldots, n-1$, differentiate both sides of equation (1)$k-$ times and substitute $s = \eta$ to have

$$\begin{aligned} P^{(k)}(\eta) &= \sum_{i=0}^{k} \binom{k}{i} Q^{(k-i)}(\eta) \cdot \frac{d^i}{ds^i}[A_0 + A_1(s-\eta) + \cdots + A_{n-1}(s-\eta)^{n-1}]_{s=\eta} \\ &+ \sum_{i=0}^{k} \binom{k}{i} P_1^{k-i}(\eta) \cdot \frac{d^i}{ds^i}[(s-\eta)^n]_{s=\eta} \\ &= \sum_{i=0}^{k} \binom{k}{i} Q^{k-i}(\eta) \cdot A_i \cdot i!. \end{aligned}$$

Introducing the quantities $t_{ki} = \frac{k!}{(k-i)!} Q^{(k-i)}(\eta)$ the above equations can be rewritten as a system of linear algebraic equations

$$\underline{p} = \underline{T} \cdot \underline{a} \tag{2}$$

where

$$\underline{T} = \begin{bmatrix} t_{00} & 0 & \ldots & 0 \\ t_{10} & t_{11} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ t_{n-10} & t_{n-11} & \ldots & t_{n-1n-1} \end{bmatrix}, \underline{p} = \begin{bmatrix} P(\eta) \\ P^{(1)}(\eta) \\ \vdots \\ P^{(n-1)}(\eta) \end{bmatrix}$$

and

$$\underline{a} = \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_{n-1} \end{bmatrix}$$

The assumption $Q^{(k)}(\eta) \neq 0 \mathrm{for} k = 0.1. \ldots, n-1$ quarantees that the diagonal elements of $\underline{T}$ are nonzero, therefore system (2) has a unique solution which can be obtained by simple forward substitution.

For a given pair P, Q of analytic functions consider the coefficients $A_0, A_1, \ldots, A_{n-1}$ obtained by solving system (2). The construction of these constants implies that for $k = 0, 1, \ldots, n-1$

$$P^{(k)}(\eta) = \frac{d^k}{ds^k}[Q(s) \cdot (A_0 + A_1(s-\eta) + \cdots + A_{n-1}(s-\eta)^{n-1})]_{s=\eta}$$

Therefore function $P(s) - Q(s) \cdot R(s)$ is also analytic and η is its root with multiplicity at least n. Hence, with some analytic function P_1,

$$P(s) - Q(s) \cdot R(s) = P_1(s) \cdot (s-\eta)^n, \tag{3}$$

which completes the proof.♠

Remark 1. From equation (2) we have

$$A_0 = \frac{P(\eta)}{t_{00}}, \tag{4}$$

and for $k = 0, 1, \ldots, n-1$

$$A_k = \frac{1}{t_{kk}} \left(P^{(k)}(\eta) - \sum_{i=0}^{k-1} t_{ki} \cdot A_i \right). \tag{5}$$

In addition, from equation (3) we conclude that

$$p_1(s) = \frac{P(s) - Q(s) \cdot R(s)}{(s-\eta)^n}. \tag{6}$$

Remark 2.. Assume now that P and Q are polynomials of degree p and q, respectively. Then the degree of $P - Q \cdot R$ is at most $m = max\{p, q+n-1\}$, and η is its root with multiplicity of at least n. Therefore P_1 is also a polynomial with degree at most $m - n$. Furthermore, if we introduce the notation $f_n = P - Q \cdot R$, then

$$P_1(s) = \sum_{k=0}^{p_1} \frac{f_n^{(n+k)}(\eta)}{(n+k)}! \cdot (s-\eta)^k, \tag{7}$$

where p_1 is the degree of P_1.

Notice that in Theorem 1, functions P, Q, and P_1 do not need to be polynomials. Representation (1) has many applications, and one of them will be elaborated in the next section. Other applications in constructing Newton- like methods and other iteration schemes of fast convergence rates will be discussed in a future paper.

3. PARTIAL FRACTION DECOMPOSITION.

Any proper rational function can be written as

$$\frac{P(s)}{(s-\eta)^n Q(s)} = \frac{B_0}{(s-\eta)} + \frac{B_1}{(s-\eta)^2} + \cdots + \frac{B_{n-1}}{(s-\eta)^n} + \frac{P_1(s)}{Q(s)} \tag{8}$$

where $Q(\eta) \neq 0$ and P_1 is a polynomial of degree less than that of Q. Multiply both sides of equation (8) by the common denominator to have

$$P(s) = Q(s) \cdot \left(A_0 + A_1(s-\eta) + \cdots + A_{n-1}(s-\eta)^{n-1}\right) + P_1(s) \cdot (s-\eta)^n,$$

which is exactly the same as (1) with $A_i = B_{n-l-i} (i = 0, 1, \ldots, n-1)$. If function (8) is proper rational, then $p < n + q$. Hence, the algorithm of the previous section is applicable to find $A_0, A_1, \ldots, A_{n-1}$ and P_1. The algorithm has to be then repeated for a pole of the decompostion remainder rational function $\frac{P_1(s)}{Q(s)}$, and so on, until all poles of the original rational function have been exhausted.

4. AN ILLUSTRATIVE EXAMPLE.

Consider the rational function

$$\frac{4s^8 - 33s^7 + 101s^6 - 146s^5 + 94s4 + 94s^3 - 178s^2 - 166s + 66}{(s-3)^4 \cdot (s^5 + 2s^3 + 4s^2 + 2s + 3)}$$

where we have

$$P(s) = 4s^8 - 33s^7 + 101s^6 - 146s^5 + 94s^3 - 178s^2 - 166s + 66,$$
$$Q(s) = s^5 + 2s^3 + 4s^2 + 2s + 3,$$

$\eta = 3, n = 4, q = 5$ and $p = 8$. Simple differentiation shows that $Q^{(0)}(3) = 342, Q^{(1)}(3) = 485, Q^{(2)}(3) = 584, Q^{(3)}(3) = 552$, and $P^{(0)}(3) = 342, P^{(1)}(3) = 1169, P^{(2)}(3) = 4576, P^{(3)}(3) = 20994$. Therefore

$$\underline{T} = \begin{pmatrix} 342 & o & o & o \\ 485 & 342 & 0 & 0 \\ 584 & 970 & 684 & 0 \\ 552 & 1752 & 2910 & 2052 \end{pmatrix}, and \underline{p} = \begin{bmatrix} 342 \\ 1169 \\ 4576 \\ 20994 \end{bmatrix}$$

which implies that the solution of equation (2) is given by

$$\underline{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

so $A_0 = 1, A_1 = 2, A_2 = 3$, and $A_3 = 4$. Hence,

$$R(s) = 1 + 2(s-3) + 3(s-3)^2 + 4(s-3)^3 = 4s^3 - 33s^2 + 92s - 86,$$

and therefore

$$f_4(s) = P(s) - Q(s) \cdot R(s) = s^6 - 10s^5 - 48s^3 + 81s^2 - 270s + 324.$$

The degree of P_1 is $6 - 4 = 2$, and simple differentiation shows that

$$f_4^{(4)}(3) = 456, f_4^{(5)}(3) = 960, f_4^{(6)}(3) = 720.$$

From equation (7) we conclude that

$$\begin{aligned} P_1(s) &= \frac{456}{4!}(s-3)^0 + \frac{960}{5!}(s-3)^1 + \frac{720}{6!}(s-3)^2 \\ &= 19 + 8(s-3) + (s-3)^2 = s^2 + 2s + 4, \end{aligned}$$

that is, the partial fraction decomposition (8) is as follows:

$$\frac{4}{(s-3)} + \frac{3}{(s-3)^2} + \frac{2}{(s-3)^3} + \frac{1}{(s-3)^4} + \frac{s^2 + 2s + 4}{s^5 + 2s^3 + 4s^2 + 2s + 3},$$

since $B_0 = A_3 = 4, B_1 = A_2 = 3, B_2 = A_1 = 2$, and $B_3 = A_0 = 1$.

We finally mention that a subsequent paper we will provide more numerical examples and applications to validate our results and compare them to those of earlier methods.

References

1. Argyros, I.K., *An Iterative Solution of the Polynomial Equation in Banach Space*, Bulletin of the Institute of Mathematics Academia Sinica **15 NO.4** (1987), 403-410.
2. Davis, H.T., *Introduction to Nonlinear Differential and Integral Equations* (1962), Dover Publications, New York.
3. La Page, W.R., *Complex Variables and the Laplace Transform for Engineers* (1961), McGraw-Hill, New York.
4. Szidarovszky, F. and A.T. Bahill, *Linear Systems Theory* (1992), CRC Press, Boca Raton.