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EIGENVALUE DISTRIBUTION OF INTEGRAL OPERATORS DEFINED BY BESOV–ORLICZ KERNELS

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Abstract. The real interpolation method has been generalized to the interpolation method with function parameter $(X_0, X_1)_{\rho,p}$ instead of $(X_0, X_1)_{\theta,p}$, replacing t^θ by some concave function ρ, see Gustavsson-Peetre [GP] and Fernandez-Garcia [FG]. The new method is particularly well-suited to interpolation of Besov-Orlicz spaces $B^s_{\Phi,q}$. The main aim of this paper is to combine the new interpolation method with König's eigenvalue theorem of integral operators defined by kernels in Besov and Sobolev spaces to find the optimal results for the asymptotic distribution of the eigenvalues of integral operators defined by kernel in Besov-Orlicz and Sobolev-Orlicz spaces. We derive some lemmas and theorems using the new interpolation method (e.g., the interpolation of L_p-spaces, Schatten classes and Besov spaces). Finally, we give a good estimate for the eigenvalues of integral operators defined by kernels belonging to vector-valued Besov-Orlicz spaces $B^s_{\Phi,q}(B^\sigma_{\Psi,v})$. We show that the eigenvalues of the integral operators defined by kernels $k \in B^s_{\Phi,q}(B^\sigma_{\Psi,v})$ belong to some Lorentz-Orlicz space $l_{\Phi,q}$. Moreover, we derive some special cases, e.g., $k \in W^s_\Phi(W^\sigma_\Psi)$ and $k \in B^s_{\Phi,q}(L_\Psi)$.

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1. The Fernandez-Garcia interpolation method.

We now introduce a variant of the Gustavsson-Peetre interpolation method. It will depend on function parameters. The function parameter will be taken in Peetre's class P^{+-} , i.e., the class of pseudo-concave positive functions ρ on $\mathbb{R}^+$ which satisfy

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$$\bar{\rho}(t) \quad = \quad \sup_{s>0} \frac{\rho(st)}{\rho(s)} \quad = \quad \circ(\max(1,t)). \tag{1}$$

For a function parameter $\rho \in P^{+-}$, we can replace (1) by

$$\bar{\rho}(t) \quad = \quad \circ(\max(t^{\varepsilon}, t^{1-\varepsilon})), \tag{2}$$

provided $\varepsilon > 0$ is small enough. Consequently, we have

$$\int_0^{\infty} \min(1, 1/t) \quad \bar{\rho}(t)\frac{dt}{t} < \infty,$$

which implies

$$\sum_{n=0}^{\infty} \frac{\rho(2^n)}{2^n} + \sum_{n=0}^{\infty} \rho(2^{-n}) < \infty. \tag{3}$$

We shall also consider the sequence $(\tilde{r_n}(.))_{n\in\mathbb{Z}}$ defined in $[0,1]$ by

$$\tilde{r_n} \quad = \quad \begin{cases} r_{2n} & \text{if} \quad n \geq 0 \\ \\ r_{2|n|-1} & \text{if} \quad n < 0, \end{cases}$$

where $(r_n(.))_{n\in\mathbb{Z}}$ is the sequence of Rademacher functions.

Definition 1.1. *Let (X_0, X_1) be a Banach couple. The space $\langle X_0, X_1\rangle_{\rho,p}$ is the linear space of all $x \in X_0 + X_1$ such that there is a sequence $(u_n)_{n\in\mathbb{Z}}$ in $X_0 \cap X_1$ which satisfies*

$$x = \sum_{n=-\infty}^{\infty} u_n \qquad (\textit{convergence in} \quad X_0 + X_1), \tag{4}$$

$$\sup_J \int_0^1 \left\| \sum_{n\in J} \tilde{r_n}(t) \frac{2^{kn}}{\rho(2^n)} u_n \right\|_{X_k}^p dt < \infty, \tag{5}$$

where the supremum is taken over all finite subsets J in $\mathbb{Z}$. Moreover, we shall also assume that, for $k = 0, 1$, the sequence $(\tilde{r_n}(.)\frac{2^{kn}}{\rho(2^n)} u_n)_{n\in\mathbb{Z}}$ is $L_p([0,1], X_k)$-summable.

We equip the space $\langle X_0, X_1\rangle_{\rho,p}$ with the norm

$$\|x\|_{\langle X_0,X_1\rangle_{\rho,p}} := \inf \max_{k=0,1} \sup_J \left(\left\| \sum_{n\in J} \tilde{r_n}(.) \frac{2^{kn}}{\rho(2^n)} u_n \right\|_{L_p([0,1],X_k)} \right), \tag{6}$$

where the infimum is taken over all admissible sequence $(u_n)_{n\in\mathbb{Z}}$.

Theorem 1.2. *Let (X_0, X_1) be a Banach couple, $\rho \in P^{+-}$ and $1 \leq p < \infty$. Then the space $\langle X_0, X_1\rangle_{\rho,p}$ is a Banach space.*

Proof : See [FG p. 25] □

To show that $\langle X_0, X_1 \rangle_{\rho,p}$ is an intermediate space between X_0 and X_1, we need the following two lemmas.

Lemma 1.3. *For all Banach spaces X, we have $\langle X, X \rangle_{\rho,p} \quad = \quad X$.*

Proof : The imbedding $X \hookrightarrow \langle X, X \rangle_{\rho,p}$ is immediate. On the other hand, we have $\langle X, X \rangle_{\rho,p} \subset X$. It remains to prove that the imbedding is bounded.
Let $x \in \langle X, X \rangle_{\rho,p}$ and let $(u_n)_{n\in\mathbb{Z}}$ be an admissible sequence in X for x, such that

$$\sup_J \left\| \sum_{n\in J} \tilde{r_n}(.) \frac{2^{kn}}{\rho(2^n)} u_n \right\|_{L_p([0,1],X)} \quad \leq \quad 2\|x\|_{\langle X,X\rangle_{\rho,p}}.$$

Hence, for each $n \in \mathbb{Z}$ and $k = 0, 1$, we have

$$\left\| \frac{2^{kn}}{\rho(2^n)} \quad u_n \right\|_X \quad \leq \quad 2\|x\|_{\langle X,X\rangle_{\rho,p}},$$

and recalling (3), it follows that

$$\begin{aligned}
\|x\|_X \quad &\leq \quad \sum_{n=-\infty}^{\infty} \|u_n\|_X \quad = \quad \sum_{n=0}^{\infty} \|u_n\|_X + \sum_{n=-\infty}^{-1} \|u_n\|_X \\
&\leq \quad 2\|x\|_{\langle X,X\rangle_{\rho,p}} \left(\sum_{n=0}^{\infty} \frac{\rho(2^n)}{2^n} + \sum_{n=-\infty}^{-1} \rho(2^n) \right) \\
&\leq \quad c\|x\|_{\langle X,X\rangle_{\rho,p}}.
\end{aligned}$$

□

Lemma 1.4. *Let (X_0, X_1) and (Y_0, Y_1) be Banach couples such that $X_k \subset Y_k, (k = 0, 1)$. Then*

$$\langle X_0, X_1 \rangle_{\rho,p} \hookrightarrow \langle Y_0, Y_1 \rangle_{\rho,p}.$$

Proof : Immediate. □

Theorem 1.5. *If (X_0, X_1) is a Banach couple we have*

$$X_0 \cap X_1 \hookrightarrow \langle X_0, X_1 \rangle_{\rho,p} \hookrightarrow X_0 + X_1,$$

i.e., $\langle X_0, X_1 \rangle_{\rho,p}$ is an intermediate space between X_0 and X_1,

Proof : The left imbedding is clear. Now since $X_k \hookrightarrow X_0 + X_1$ we have

$$\langle X_0, X_1 \rangle_{\rho,p} \hookrightarrow \langle X_0 + X_1, X_0 + X_1 \rangle_{\rho,p} \quad = \quad X_0 + X_1,$$

using the above two lemmas. □

One has the fact is that the functor $(X_0, X_1) \mapsto \langle X_0, X_1 \rangle_{\rho,p}$ is an interpolation functor.

Theorem 1.6. *Let (X_0, X_1) and (Y_0, Y_1) be Banach couples, $\rho \in P^{+-}$ and $1 \leq p < \infty$. Then, for all $T \in L(X_k, Y_k), (k = 0, 1)$, we have*

$$T \in L(\langle X_0, X_1\rangle_{\rho,p}, \langle Y_0, Y_1\rangle_{\rho,p}).$$

Moreover,

$$\|T\| \quad \leq \quad \max(\|T\|_0, \|T\|_1).$$

Proof : Let $x \in \langle X_0, X_1\rangle_{\rho,p}$, let $(u_n)_{n\in\mathbb{Z}}$ be an admissible sequence of x, and set $M_k = \|T\|_{L(X_k,Y_K)}, (k = 0, 1)$. Since the convergence in (4) is in $X_0 + X_1$, we have $Tx = \sum_{n\in\mathbb{Z}} Tu_n$.
Hence, setting $v_n = Tu_n$, we see that $\sum_{n\in\mathbb{Z}} v_n$ converges in $Y_0 + Y_1$ and is a representation of Tx.
Next, for $k = 0, 1$, we get

$$\begin{aligned}
\left\|\sum_{n\in J} \tilde{r_n}(.)\frac{2^{kn}}{\rho(2^n)}v_n\right\|_{L_p([0,1],Y_k)} &= \left\|T\left(\sum_{n\in J} \tilde{r_n}(.)\frac{2^{kn}}{\rho(2^n)}u_n\right)\right\|_{L_p([0,1],X_k)} \\
&\leq M_k \left\|\sum_{n\in J} \tilde{r_n}(.)\frac{2^{kn}}{\rho(2^n)}u_n\right\|_{L_p([0,1],X_k)},
\end{aligned}$$

and consequently $\|T\|_{\langle Y_0,Y_1\rangle_{\rho,p}} \quad \leq \quad c\|x\|_{\langle X_0,X_1\rangle_{\rho,p}}$.
The proof is complete. ☐

Theorem 1.7. *(Kahane, see [FG]) For every $1 < r < \infty$, for any Banach space X and every finite sequence $(u_j)_{1 \leq j \leq n}$ in X, we have*

$$\int_0^1 \left\|\sum_{j=1}^n r_j(t)u_j\right\|_X dt \approx \left(\int_0^1 \left\|\sum_{j=1}^n r_j(t)u_j\right\|_X^r dt\right)^{1/r}.$$

Theorem 1.8. *Let (X_0, X_1) and (Y_0, Y_1) be Banach couples, $\rho \in P^{+-}$ and $1 \leq p, q < \infty$. Then $\langle X_0, X_1\rangle_{\rho,p} = \langle X_0, X_1\rangle_{\rho,q}$, with norm equivalence.*

Proof : This follows at once from Kahane's theorem. ☐

2. Interpolation of $L_p(X)$ spaces

We shall show that L_p-spaces commute with the interpolation space $\langle X_0, X_1\rangle_{\rho,p}$.

Theorem 2.1 (FG)**.** *Let (X_0, X_1) be a Banach couple of U.M.D. spaces (see FG p. 23).Then the interpolation space $\langle X_0, X_1\rangle_{\rho,p}$ is also a U.M.D. space.*

Theorem 2.2. *Let L_Φ be an Orlicz space such that the Orlicz function Φ and the conjugate function Φ^* satisfy the Δ_2-condition. Then L_Φ has the U.M.D. property.*

Corollary 2.3. *The space $Y(X)$ is U.M.D. if and only if Y and X are U.M.D. spaces.*

Lemma 2.4 (FG)**.** *Let X be a Banach space, $\rho \in P^{+-}$ and $1 \leq p < \infty$. Then if $(u_j)_{1\leq j\leq n}$ is a finite sequence in X, then*

$$\left\|\sum_{j=1}^{n} |u_j|\right\|_X \quad \leq \quad cR\left(\left\|(\sum_{j=1}^{n}(\frac{|u_j|}{\rho(2^j)})^p)^{1/p}\right\|_X, \left\|(\sum_{j=1}^{n}(\frac{2^j|u_j|}{\rho(2^j)})^p)^{1/p}\right\|_X\right),$$

where $R(s,t) \quad = \quad s\rho(t/s)$.

Theorem 2.5. *Let (X_0, X_1) be a U.M.D Banach couple. Then for all $p, 1 \leq p < \infty$, we have*

$$\langle L_p(X_0), L_p(X_1)\rangle_{\rho,p} \quad = \quad L_p(\langle X_0, X_1\rangle_{\rho,p}), \tag{7}$$

with equivalent norms.

Proof : See [FG]

Remark 2.6. *Using the fact that l_q^s-spaces are special L_q-spaces and the previous theorem, we have*

$$\langle l_q^s(X_0), l_q^s(X_1)\rangle_{\rho,p} \quad = \quad l_q^s(\langle X_0, X_1\rangle_{\rho,p}), \tag{8}$$

with the same assumption as in theorem (2.5).

The following theorem shows that reflexive Orlicz spaces can be characterized as interpolation spaces between L_p-spaces. This is a converse of Gustavsson-Peetre's theorem [GP theorem (7.3)].
By an Orlicz function we mean a convex and continuous function
$\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$.

Theorem 2.7 (FG)**.** *Let Φ be an Orlicz function and let $L_\Phi(E)$ be the corresponding E-valued Orlicz space on a σ-finite measure space, where E is a U.M.D. Banach space. Suppose that Φ and the conjugate Φ^* satisfy the Δ_2-condition. Then there exist p_0 and p_1, with $1 < p_1 < p_0 < \infty$,*

$$\langle L_{p_0}(E), L_{p_1}(E)\rangle_{\rho,p} \quad = \quad L_\Phi(E) \qquad ,\textit{where} \tag{9}$$

$$\rho(t) \quad = \quad \begin{cases} t^{\frac{p_1}{p_1-p_0}} \quad \Phi^{-1}(t^{\frac{p_0p_1}{p_0-p_1}}) & ,t>0 \\ \\ 0 & t=0 \end{cases} \tag{10}$$

and $1 < p < \infty$.

3. Besov-Orlicz and Sobolev-Orlicz spaces

Definition 3.1. *Suppose $s \in \mathbb{R}$ and let (ϕ_n) be a system of functions(see T p. 171). Let X be a Banach space and Φ be an Orlicz function satisfy the Δ_2-condition,we define for $1 \leq q \leq \infty$, the Besov-Orlicz spaces*

$$B^s_{\Phi q}(\mathbb{R}^N; X) = \{f \in S' \quad : (\phi_n * f)_n \in l^s_q(L_\Phi(\mathbb{R}^N; X))\} \tag{11}$$

We equip the spaces $B^s_{\Phi q}(\mathbb{R}^N; X)$ with the norm

$$\|f\|_{B^s_{\Phi q}} = \|(\phi_n * f)_n\|_{l^s_q(L_\Phi(\mathbb{R}^N;X))} = \|(\|\phi_n * f\|_{L_\Phi(X)})_n\|_{l^s_q}, \tag{12}$$

Remark 3.2 (FG)**.** *The spaces $B^s_{\Phi q}$ do not depend on the particular system of functions used to define them.*

We shall prove that the Besov-Orlicz space $B^s_{\Phi q}(\mathbb{R}^N; X)$ is a retract of the space $l^s_q(L_\Phi(\mathbb{R}^N; X))$ using the composition

$$B^s_{\Phi q}(\mathbb{R}^N; X) \xrightarrow{I} l^s_q(L_\Phi(\mathbb{R}^N; X)) \xrightarrow{P} B^s_{\Phi q}(\mathbb{R}^N; X).$$

Theorem 3.3. *Let Φ be an Orlicz function and let $L_\Phi(X)$ be the corresponding X-valued Orlicz space. Then the space $B^s_{\Phi q}(\mathbb{R}^N; X)$ is a retract of $l^s_q(L_\Phi(\mathbb{R}^N; X))$ $\quad(s \in \mathbb{R}, 1 \leq q \leq \infty)$.*

Proof : Consider the mappings I and P are defined by

$$(If)_j = \phi_j * f \qquad j = 1, 2, \ldots \qquad , (If)_0 = \psi * f,$$

$$P\alpha = \sum_{j=0}^{\infty} \tilde{\phi}_j * \alpha_j \qquad (\text{convergence in} S^*) \qquad \text{where} \quad \alpha = (\alpha_j)_{j=0}^{\infty},$$

$$\alpha_j \in S^*, \quad j = 1, 2, \ldots \qquad \text{and} \qquad \tilde{\phi}_0 = \psi + \phi_1$$

$$\tilde{\phi}_j = \sum_{l=-1}^{1} \phi_{j+l} \quad j = 1, 2, \ldots \qquad .$$

Clearly $PIf = f \quad , f \in S^* \quad$, since $\tilde{\phi}_j * \phi_j = \phi_j \quad j = 1, 2, \ldots$ and $\tilde{\phi}_0 * \psi = \psi$.

(i) First,we note that

$$\|f\|_{B^s_{\Phi q}(\mathbb{R}^N;X)} \sim \left(\sum_{j=0}^{\infty} (2^{js} \quad \|\phi_j * f\|_{L_\Phi(\mathbb{R}^N;X)})^q \right)^{1/q}$$

$$= \quad \left(\sum_{j=0}^{\infty} (2^{js} \quad \|(If)_j\|_{L_\Phi(\mathbb{R}^N;X)})^q \right)^{1/q}$$

$$= \quad \|If\|_{l_q^s(L_\Phi(\mathbb{R}^N;X))}.$$

Thus $I : B^s_{\Phi q}(\mathbb{R}^N;X) \to l_q^s(L_\Phi(\mathbb{R}^N;X))$.

(ii) Since $\phi_j * P\alpha \quad = \quad \phi_j * \alpha_j \quad (j \geq 1)$, we have

$$\|P\alpha\|_{B^s_{\Phi q}(\mathbb{R}^N;X)} = \left(\sum_{j=1}^{\infty}(2^{js} \quad \|\phi_j * \alpha_j\|_{L_\Phi(\mathbb{R}^N;X)})^q\right)^{1/q}$$

$$\leq \quad c \quad \left(\sum_{j=1}^{\infty}(2^{js} \quad \|\alpha_j\|_{L_\Phi(\mathbb{R}^N;X)})^q\right)^{1/q}$$

$$\leq \quad \|\alpha\|_{l_q^s(L_\Phi(\mathbb{R}^N;X))}.$$

Thus $B^s_{\Phi q}(\mathbb{R}^N;X)$ is a retract of $l_q^s(L_\Phi(\mathbb{R}^N;X))$. □

Definition 3.4. *Assume that $s \in \mathbb{R}$ and let Φ be an Orlicz function satisfy the Δ_2-condition. We define the Sobolev-Orlicz spaces $W^s_\Phi(\mathbb{R}^N;X)$ by*

$$W^s_\Phi(\mathbb{R}^N;X) \quad = \quad \{f \in S' \quad : J^s f \in L_\Phi(\mathbb{R}^N;X)\}. \tag{13}$$

Here J^s is the Bessel operator .

We equip the space $W^s_\Phi(\mathbb{R}^N;X)$ with the norm

$$\|f\|_{W^s_\Phi(\mathbb{R}^N;X)} \quad = \quad \|J^s f\|_{L_\Phi(\mathbb{R}^N;X)}.$$

Definition 3.5. *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary domain.Further,let $-\infty < s < \infty, 1 < p < \infty$ and $1 \leq q \leq \infty$. Then $B^s_{pq}(\Omega;X)$ is the restriction of $B^s_{pq}(\mathbb{R}^N;X)$ to Ω,*

$$\|f\|_{B^s_{pq}(\Omega;X)} = \inf_{g|_\Omega = f} \|g\|_{B^s_{pq}(\mathbb{R}^N;X)}, \quad g \in B^s_{pq}(\mathbb{R}^N;X).$$

And $W^s_p(\Omega;X)$ is the restriction of $W^s_p(\mathbb{R}^N;X)$ to Ω,

$$\|f\|_{W^s_p(\Omega;X)} = \inf_{g|_\Omega = f} \|g\|_{W^s_p(\mathbb{R}^N;X)}, \quad g \in W^s_p(\mathbb{R}^N;X).$$

We shall now introduce two mappings S and R.The mapping S (coretraction) maps $B^s_{pq}(\Omega;X)$ to $B^s_{pq}(\mathbb{R}^N;X)$.It is defined by

$$S(f) = \lim_j f\psi_j, \qquad f \in B^s_{pq}(\Omega;X)$$

The mapping R (retraction) is given by

$$R(\sum_j \beta_j) = \sum_j \psi_j * f_j,$$

where the mapping I (coretraction) maps $B^s_{pq}(\mathbb{R}^N;X)$ to $l_q^s(L_p(\mathbb{R}^N;X))$ is defined by

$$I(g) = \{g * \phi_j\}_{j=1}^{\infty} = (g_j)_{j=1}^{\infty}, \qquad g \in B^s_{pq}(\mathbb{R}^N;X),$$

and the mapping P (retraction) is given by

$$P(\alpha) = \sum_j \psi_j * g_j, \qquad \alpha = (g_j)_{j=1}^{\infty} = \sum_j \beta_j.$$

Therefore, the mapping IS (coretraction) maps $B^s_{pq}(\Omega; X)$ to $l^s_q(L_p(\mathbb{R}^N; X))$ and the mapping RP is the corresponding retraction.

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain.*
If $-\infty < s < \infty, 1 < p < \infty$ and $1 \leq q \leq \infty$, then the restriction operator from $B^s_{pq}(\mathbb{R}^N; X)$ onto $B^s_{pq}(\Omega; X)$ is a retraction.
If $r \in \mathbb{N}$, then there exists a corresponding coretraction independent of $1 < p < \infty, 1 \leq q \leq \infty$ and $|s| < r$.

Proof : See H.Trieble [T p. 311] □

Remark 3.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain, $s \in \mathbb{R}$,*
$1 \leq q \leq \infty$ and Φ be an Orlicz function satisfy the Δ_2-condition. Then $B^s_{\Phi q}(\Omega; X)$ is a retract of $l^s_q(L_\Phi(\Omega; X))$

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain, Φ be an Orlicz function which satisfies, together with its conjugate Φ^*, the Δ_2-condition. Then there exist p_0 and p_1, with $1 < p_0, p_1 < \infty$, such that*

$$\langle B^s_{p_0 q}(\Omega; X), B^s_{p_1 q}(\Omega; X)\rangle_{\rho,p} \quad = \quad B^s_{\Phi q}(\Omega; X), \tag{14}$$

where ρ is given by (10) .

Proof : Using remark (3.7), remark (2.6) and theorem (2.7), we have

$$\begin{aligned}
\langle B^s_{p_0 q}(\Omega; X), B^s_{p_1 q}(\Omega; X)\rangle_{\rho,p} &= \langle RP(l^s_q(L_{p_0}(\Omega; X))), RP(l^s_q(L_{p_1}(\Omega; X)))\rangle_{\rho,p} \\
&= RP(l^s_q(\langle L_{p_0}(\mathbb{R}^N; X), L_{p_1}(\Omega; X)\rangle_{\rho,p})) \\
&= RP(l^s_q(L_\Phi(\Omega; X))) \quad = \quad B^s_{\Phi q}(\Omega; X).
\end{aligned}$$
□

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain ,*
$1 < p, q < \infty$, and $s \in \mathbb{R}$. Then we have, for Banach spaces X_0, X_1

$$\langle B^s_{pq}(X_0), B^s_{pq}(X_1)\rangle_{\rho,p} \quad = \quad B^s_{pq}(\langle X_0, X_1\rangle_{\rho,p}). \tag{15}$$

Proof : Using remark (3.7) , remark (2.6) and theorem (2.5), we have

$$\begin{aligned}
\langle B^s_{pq}(X_0), B^s_{pq}(X_1)\rangle_{\rho,p} &= \langle RP(l^s_q(L_p(X_0))), RP(l^s_q(L_p(X_1)))\rangle_{\rho,p}) \\
&= RP(l^s_q(\langle L_p(X_0), L_p(X_1)\rangle_{\rho,p}) \\
&= RP(l^s_q(L_p(\langle X_0, X_1\rangle_{\rho,p})))
\end{aligned}$$

$$= \quad B^s_{pq}(\langle X_0, X_1\rangle_{\rho,p}).$$

$\square$

Theorem 3.10. *Let $1 < p, q, u_0, u_1, v < \infty$ and $s \in \mathbb{R}$. Then we have*

$$\langle B^s_{pq}(B^\sigma_{u_0v}(\Omega; X)), B^s_{pq}(B^\sigma_{u_1v}(\Omega; X))\rangle_{\rho,q} \quad = \quad B^s_{pq}(B^\sigma_{\Phi v}(\Omega; X)), \tag{16}$$

with,

$$\rho(\tau) \quad = \quad \tau^{\frac{u_1}{u_1-u_0}} \quad \Phi^{-1}(\tau^{\frac{u_0u_1}{u_0-u_1}}).$$

Proof : Using theorem (3.9) and theorem (3.8) ,we have

$$\langle B^s_{pq}(B^\sigma_{u_0v}(\Omega; X)), B^s_{pq}(B^\sigma_{u_1v}(\Omega; X))\rangle_{\rho,q}$$

$$= \quad B^s_{pq}(\langle B^\sigma_{u_0v}(\Omega; X), B^\sigma_{u_1v}(\Omega; X))\rangle_{\rho,q}) \quad = \quad B^s_{pq}(B^\sigma_{\Phi v}(\Omega; X)).$$

$\square$

4. Schatten-Orlicz classes

Definition 4.1. *Let Φ be an Orlicz function as before. We define for $1 \leq q < \infty$, the Lorentz-Orlicz spaces $l_{\Phi q}$ as follows*

$$l_{\Phi q} = \left\{ x = (x_n)_n : \|x\|_{l_{\Phi q}} := \inf \left(c : (\sum_{n=1}^{\infty} \Phi(\frac{|x_n|}{c})^q)^{1/q} \leq 1 \right) < \infty \right\}, \tag{17}$$

and,

$$l_{\Phi\infty} = \left\{ x = (x_n)_n \quad : \|x\|_{l_{\Phi\infty}} := \inf \left(c : \sup_n \Phi(\frac{|x_n|}{c}) \leq 1 \right) < \infty \right\}. \tag{18}$$

Definition 4.2. *Let Φ be an Orlicz function . We define for $1 \leq q \leq \infty$, the Schatten-Orlicz classes $S^s_{\Phi q}$ as follows*

$$S^s_{\Phi q}(X) \quad := \quad \{T \in L(X) \quad : (s_n(T))_n \in l_{\Phi q}\}. \tag{19}$$

Theorem 4.3. *Let $1 < p_0, p_1 < \infty$. Then we have*

$$\langle S^s_{p_0q}(X), S^s_{p_1q}(X)\rangle_{\rho,p} \subset S^s_{\Phi q}(X), \tag{20}$$

where ρ is given by (10).

Proof : Let $T \in \langle S^s_{p_0q}(X), S^s_{p_1q}(X)\rangle_{\rho,p} \subset \langle S^s_{p_0\infty}(X), S^s_{p_1\infty}(X)\rangle_{\rho,p}$ with $\|T\|_{\langle S^s_{p_0\infty}(X), S^s_{p_1\infty}(X)\rangle_{\rho,p}} \leq 1$ and let $(T_n)_n$ be an admissible sequence for T such that

$$\max_{k=0,1} \sup_J \left\| \sum_{n\in J} \tilde{r_n}(.) \frac{2^{kn}}{\rho(2^n)} T_n \right\|_{L_p([0,1], S^s_{p_k\infty}(X))} < 1.$$

Since the sequence $(\tilde{r_n}(.)\frac{2^{kn}}{\rho(2^n)}T_n)_{n\in\mathbb{Z}}$ is $L_p([0,1],S^s_{p_k\infty})$-summable , given $\varepsilon>0$, there exist a finite subset $J_\varepsilon\subset\mathbb{N}$ such that for all $J'\subset\mathbb{N}$ with $J'\cap J_\varepsilon = \emptyset$ we have

$$\left\|\left(\sum_{n\in J'}(\frac{2^{kn}}{\rho(2^n)}T_n)^2\right)^{1/2}\right\|_{S^s_{p_k\infty}(X)} \leq \varepsilon^{1/p_k},$$

Now setting

$$A_k(m) = \max_n n^{1/p_k}\left\|\left(\sum_{n\in J'}(\frac{2^{kn}}{\rho(2^n)} \quad s_m(T_n))^2\right)^{1/2}\right\|_X \leq \varepsilon^{1/p_k} \quad \text{and}$$

$$A(m) = \max_{k=0,1}\left(\frac{A_k(m)}{2\varepsilon}\right)^{p_k}.$$

From the fact, $\frac{\rho(t)}{t}$ is non-increasing and lemma (2.4)

$$\frac{1}{2\varepsilon}\|\sum_{n\in J'}s_m(T_n)\|_X \leq c\frac{A_0(m)}{2\varepsilon} \quad \rho\left(\frac{A_1(m)}{A_0(m)}\right)$$

$$\leq cA(m)^{\frac{1}{p_0}} \quad \rho(A(m)^{\frac{1}{p_1}-\frac{1}{p_0}}) \leq c\Phi^{-1}(A(m)).$$

Therefore,

$$\Phi\left(\frac{1}{2c\varepsilon}\|\sum_{n\in J'}s_m(T_n)\|_X\right) \leq A(m)$$

$$\leq \frac{1}{2}\left((\frac{A_0(m)}{\varepsilon})^{p_0}+(\frac{A_1(m)}{\varepsilon})^{p_1}\right) \leq 1.$$

Therefore,$\|\sum_{n\in J'}T_n\|_{S_{\Phi\infty}(X)} \leq 2c\varepsilon$, the sequence $(T_n)_n$ satisfies the Cauchy condition in $S_{\Phi\infty}(X)$.
Then, we have $T\in S_{\Phi\infty}(X)\Rightarrow T\in S_{\Phi q}(X)$. Hence,

$$\langle S^s_{p_0q}(X),S^s_{p_1q}(X)\rangle_{\rho,q}\subset S^s_{\Phi q}(X) \quad \text{with} \quad \Phi^{-1}(t) = t^{\frac{1}{p_0}} \quad \rho(t^{\frac{1}{p_1}-\frac{1}{p_0}})$$

which equivalent to the relation (10). □

5. Eigenvalue distribution of integral operators

Now, we derive our main results using the eigenvalue theorem for Besov spaces and the new interpolation results (interpolation of S^s_{pq}-spaces, and B^s_{pq}-spaces). We will prove that the eigenvalues of integral operators defined by kernels $k\in B^s_{\Phi q}(B^\sigma_{\Psi v})$ belong to some Lorentz-Orlicz space $l_{\Phi q}$.

Theorem 5.1. *(Eigenvalue theorem I) [K] Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain, $1 \leq p,q,u,v \leq \infty$ and $s,\sigma > 0$ with $s+\sigma > N(\frac{1}{p}+\frac{1}{u}-1)$. Define t by $\frac{1}{t} := \frac{s+\sigma}{N} + \frac{1}{\max(2,u')}$. Then the eigenvalues of any kernel $k \in B^s_{pq}(\Omega; B^\sigma_{uv}(\Omega))$ belong to the Lorentz sequence space $l_{t,q}$ with*

$$\|(\lambda_n(k))_{n\in\mathbb{N}}\|_{l_{t,q}} \leq c\|k\|_{B^s_{pq}(B^\sigma_{uv})}. \tag{21}$$

The constant c depend only on the indices and the domain Ω.

Theorem 5.2 (K). *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain, $1 \leq p,q,u \leq \infty$ and $s > 0$ with $s > N(\frac{1}{p}-\frac{1}{u'})$. Define t by $\frac{1}{t} := \frac{s}{N} + \frac{1}{\max(2,u')}$. Then the eigenvalues sequence of any kernel $k \in B^s_{pq}(\Omega; L_u(\Omega))$ belong to the Lorentz sequence space $l_{t,q}$ with*

$$\|(\lambda_n(k))_{n\in\mathbb{N}}\|_{l_{t,q}} \leq c_{p,q,u,s,\Omega}\|k\|_{B^s_{pq}(L_u)}. \tag{22}$$

Theorem 5.3 (K). *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain , $1 \leq p,u,v \leq \infty$ and $\sigma > 0$ with $\sigma > N(\frac{1}{u} - \frac{1}{\max(p',v')})$. Define t by $\frac{1}{t} := \frac{\sigma}{N} + \frac{1}{\max(2,u')}$. Then the eigenvalues of any kernel $k \in L_p(\Omega; B^\sigma_{uv}(\Omega))$ belong to the Lorentz sequence space $l_{t,v}$ with*

$$\|(\lambda_n(k))_{n\in\mathbb{N}}\|_{l_{t,v}} \leq c_{p,u,v,\sigma,\Omega}\|k\|_{L_p(B^\sigma_{uv})}. \tag{23}$$

Remark 5.4. *(i) As shown by Besov [B] , the Sobolev spaces $W^s_u(\Omega)$ of integer order $s \in \mathbb{N}$ satisfy the inclusion $W^s_u(\Omega) \subset B^s_{uv}(\Omega)$ with $v := \max(2,u)$. Hence the eigenvalues of the kernel operator defined in $L_p(\Omega)$ by any kernel $k \in L_p(\Omega; W^s_u(\Omega))$ belong to l_{tv} for $\frac{1}{t} := \frac{s}{N} + \frac{1}{\max(2,u')}$ provided that $s > N(\frac{1}{u} - \frac{1}{\max(p',v')})$.*
(ii) For $v \geq p, r \in \mathbb{N}$,we have

$$W^r_p(\Omega; X_{\theta,v}) \subset (W^r_p(\Omega; X_0), W^r_p(\Omega; X_1))_{\theta,v}.$$

This yields that the eigenvalues of kernels $k \in W^r_p(\Omega; B^\sigma_{uv}(\Omega))$ belong to l_t, v, $\frac{1}{t} := \frac{\sigma+r}{N} + \frac{1}{\max(2,u')}$, provided that $\sigma + r$ satisfies $\sigma + r > N(\frac{1}{u} - \frac{1}{\max(p',v')})$.
Using the above inclusion $W^s_p(\Omega) \subset B^s_{uv}(\Omega)$ for $v := \max(u,2)$, one finds:

If $r,s \in \mathbb{N}$ with $\quad r+s > N(\dfrac{1}{u} - \dfrac{1}{\max(p',v')}) \quad$ *and* $\; k \in W^r_p(\Omega; W^s_u(\Omega))$,

the eigenvalues of T_k (e.g., in $L_p(\Omega)$ if $p \geq u'$) belong to $l_{t,v}$ for

$$\frac{1}{t} := \frac{r+s}{N} + \frac{1}{\max(2,u')} \quad \text{and} \quad v := \max(2,u).$$

Theorem 5.5. *(The eigenvalue theorem II.) Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain, $1 \leq p,q,u_0,u_1,v \leq \infty$, $u_1 < u_0$ and $s+\sigma > 0$ with $s+\sigma > N(\frac{1}{p}+\frac{1}{u_1}-1)$. Define*

$$\Psi^{-1}(\tau) = \tau^{\frac{s+\sigma}{N}+\frac{1}{u_1'}-\frac{1}{\min(2,u_0)}} \quad \Phi^{-1}(\tau). \tag{24}$$

Then the eigenvalues of any kernel $k \in B^s_{pq}(\Omega; B^\sigma_{\Phi v}(\Omega))$ *belong to the Lorentz-Orlicz space* $l_{\Psi q}$ *with*

$$\|(\lambda_n(k))_{n\in\mathbb{N}}\|_{l_{\Psi,q}} \leq c\|k\|_{B^s_{pq}(B^\sigma_{\Phi v})}. \tag{25}$$

Proof : Using the eigenvalue theorem (5.1), we have, for $s+\sigma > N(\frac{1}{p}+\frac{1}{u_1}-1), i=0,1$

$$B^s_{pq}(B^\sigma_{u_i v}) \subset S^s_{t_i,q} \quad \text{with} \quad \frac{1}{t_i} := \frac{s+\sigma}{N} + \frac{1}{\max(2,u_i')}. \tag{26}$$

By theorem (3.10), we have

$$\langle B^s_{pq}(B^\sigma_{u_0 v}(X)), B^s_{pq}(B^\sigma_{u_1 v}(X))\rangle_{\rho,q} = B^s_{pq}(B^s_{\Phi v}(X)), \tag{27}$$

with,

$$\rho(\tau) = \tau^{\frac{u_1}{u_1-u_0}} \quad \Phi^{-1}(\tau^{\frac{u_0 u_1}{u_0-u_1}}). \tag{28}$$

(i) The case $u_1 < u_0 < 2 \quad (2 < u_0' < u_1')$:

Using theorem (4.3),for $\quad \dfrac{1}{t_i} := \dfrac{s+\sigma}{N} + \dfrac{1}{u_i'}, \quad$ we have,

$$\langle S^s_{t_0 q}(X), S^s_{t_1 q}(X)\rangle_{\rho,q} = S^s_{\Psi q}(X), \tag{29}$$

with,

$$\rho(\tau) = \tau^{\frac{t_0}{t_0-t_1}} \quad \Psi^{-1}(\tau^{\frac{t_0 t_1}{t_1-t_0}}). \tag{30}$$

Let $\quad \beta = \dfrac{t_0 t_1}{t_1-t_0} \Rightarrow \dfrac{1}{\beta} = \dfrac{1}{t_0} - \dfrac{1}{t_1} = \dfrac{u_1'-u_0'}{u_0' u_1'} \Rightarrow \qquad \beta = \dfrac{u_0' u_1'}{u_1'-u_0'},$

and $\quad \gamma = \dfrac{t_0}{t_0-t_1} = -\dfrac{1}{t_1}\beta = (\dfrac{s+\sigma}{N} + \dfrac{1}{u_1'})(\dfrac{u_0' u_1'}{u_0'-u_1'}).$

Consequently, we have

$$\rho(\tau) = \tau^\gamma \quad \Psi^{-1}(\tau^\beta). \tag{31}$$

From (27) , (28) , (29) , (31) and the fact, that if $X_k \subset Y_k, (k=0,1) \Rightarrow \langle X_0, X_1\rangle_{\rho,p} \subset \langle Y_0, Y_1\rangle_{\rho,p}$, we have

$$B^s_{pq}(B^s_{\Phi v}(X)) \subset S^s_{\Psi q}(X) \quad \text{with} \quad \Psi^{-1}(\tau^\beta) = \tau^{\frac{u_1}{u_1-u_0}-\gamma} \quad \Phi^{-1}(\tau^{\frac{u_0 u_1}{u_0-u_1}}).$$

But,$\beta = \frac{u_0' u_1'}{u_1'-u_0'} = \frac{u_0 u_1}{u_0-u_1} \qquad$ and

$$\frac{u_1}{u_1-u_0} - \gamma = \frac{-\beta}{u_0} + \beta(\frac{s+\sigma}{N} + \frac{1}{u_1'}) = \beta(\frac{s+\sigma}{N} + \frac{1}{u_1'} - \frac{1}{u_0}).$$

Then, $\qquad \Psi^{-1}(\tau^\beta) = \tau^{\beta(\frac{s+\sigma}{N}+\frac{1}{u_1'}-\frac{1}{u_0})} \quad \Phi^{-1}(\tau^\beta).$

Take $\tau^\beta = \xi$, we find

$$\Psi^{-1}(\xi) = \xi^{\frac{s+\sigma}{N}+\frac{1}{u_1'}-\frac{1}{u_0}} \quad \Phi^{-1}(\xi). \tag{32}$$

(ii) The case $u_1 < 2 < u_0$ $(u_0' < 2 < u_1')$:
Using the same method as above, we can find

$$\Psi^{-1}(\xi) = \xi^{\frac{s+\sigma}{N}+\frac{1}{u_1'}-\frac{1}{2}} \quad \Phi^{-1}(\xi). \tag{33}$$

Combining (32) and (33), we have

$$\Psi^{-1}(\tau) = \tau^{\frac{s+\sigma}{N}+\frac{1}{u_1'}-\frac{1}{\min(2,u_0)}} \quad \Phi^{-1}(\tau).$$

Then, we readily get the desired results. □

Theorem 5.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain, $1 \le p, q, u_0, u_1 \le \infty, u_1 < u_0$ and $s > 0$ with $s > N(\frac{1}{p} + \frac{1}{u_1} - 1)$. Define*

$$\Psi^{-1}(\tau) = \tau^{\frac{s}{N}+\frac{1}{u_1'}-\frac{1}{\min(2,u_0)}} \quad \Phi^{-1}(\tau). \tag{34}$$

Then the eigenvalues of any kernel $k \in B^s_{pq}(\Omega; L_\Phi(\Omega))$ belong to the Lorentz-Orlicz space l_{Ψ_q} with

$$\|(\lambda_n(k))_{n\in\mathbb{N}}\|_{l_{\Psi,q}} \le c\|k\|_{B^s_{pq}(L_\Phi)}. \tag{35}$$

Proof : Replacing $s + \sigma$ by s in theorem (5.5) and using the same method, we get directly the desired results □

Theorem 5.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^∞-domain, $1 \le p, u_0, u_1, v \le \infty, u_1 < u_0$ and $\sigma > 0$ with $\sigma > N(\frac{1}{u_1} - \frac{1}{\max(p',v')})$. Define*

$$\Psi^{-1}(\tau) = \tau^{\frac{\sigma}{N}+\frac{1}{u_1'}-\frac{1}{\min(2,u_0)}} \quad \Phi^{-1}(\tau). \tag{36}$$

Then the eigenvalues of any kernel $k \in L_p(\Omega; B^\sigma_{\Phi v}(\Omega))$ belong to the Lorentz-Orlicz space l_{Ψ_v} with

$$\|(\lambda_n(k))_{n\in\mathbb{N}}\|_{l_{\Psi,v}} \le c\|k\|_{L_p(B^\sigma_{\Phi v})}. \tag{37}$$

Proof : Using theorem (5.3) and the proof of theorem (5.5) . □

Remark 5.8. *(i) As shown by Besov [B], The Sobolev spaces W^σ_u of integer order $\sigma \in \mathbb{N}$ satisfy the inclusion $W^\sigma_u \subset B^\sigma_{uv}$ with $v := \max(2, u)$.*
Hence, using theorem (5.5) together with remark (5.4), we can prove that, the eigenvalues of any kernel $k \in W^s_p(\Omega; W^\sigma_\Phi(\Omega))$ belong to the Lorentz-Orlicz space $l_{\Psi,v}$ with

$$\Psi^{-1}(\tau) = \tau^{\frac{s+\sigma}{N}+\frac{1}{u_1'}-\frac{1}{\min(2,u_0)}} \quad \Phi^{-1}(\tau),$$

provided that $s+\sigma > N(\frac{1}{u_1} - \frac{1}{\max(p',v')})$.
(ii) Also, we can prove that, $k \in L_p(\Omega; W^{\sigma}_{\Phi}(\Omega))$ *belong to the Lorentz-Orlicz space* $l_{\Psi,v}$ *with*

$$\Psi^{-1}(\tau) = \tau^{\frac{s+\sigma}{N}+\frac{1}{u_1'}-\frac{1}{\min(2,u_0)}} \quad \Phi^{-1}(\tau),$$

provided that $s > N(\frac{1}{u_1} - \frac{1}{\max(p',v')})$.

Corollary 5.9. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded* C^{∞}*-domain,* $1 \le p, q, u_0, u_1, v \le \infty, u_1 < u_0$ *and* $s+\sigma > 0$ *with* $s+\sigma > N(\frac{1}{p}+\frac{1}{u_1}-1)$*. Define* t *by*

$$\frac{1}{t} := \frac{s+\sigma}{N} + \frac{1}{u_1'} - \frac{1}{\min(2,u_0)} + \frac{1}{u} \quad \textit{with} \quad \frac{1}{u} = \frac{1-\theta}{u_0} + \frac{\theta}{u_1},$$

$0 < \theta < 1$. *Then the eigenvalues of any kernel* $k \in B^s_{pq}(\Omega; B^{\sigma}_{uv}(\Omega))$ *belong to the Lorentz-Orlicz space* l_{tq} *with*

$$\|(\lambda_j(k))_{j\in\mathbb{N}}\|_{l_{tq}} \le c \quad \|k\|_{B^s_{pq}(B^{\sigma}_{uv})}. \tag{38}$$

Proof : Consider $\rho(t) = t^{\theta}$.Then using the prove of theorem (5.5),we get the desired result. □

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