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COINCIDENCE POINTS OF COMPATIBLE MULTIVALUED MAPPINGS

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ABSTRACT. In this paper we establish a coincidence theorem for a generalized contractive type multivalued mappings in complete metric spaces generalizing some results of Constantin [1]. Two examples are given to support our generalization.

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1. INTRODUCTION

In [2,3,4,5,9,11,12,13,15,16], the authors proved some coincidence point theorems for multivalued mappings in complete metric spaces under various contractive conditions.

Recently Constantin [1] proved some coincidence point theorems for multivalued mappings in complete metric spaces generalizing some results in [4, 5, 11, 12] and others.

In this paper we consider a generalized contraction condition involving two multivalued mappings which are compatible with two single-valued mappings and we establish a coincidence theorem which improves, extends some results of Constantin [1].

Let (X, d) be a metric space. We denote by $CB(X)$ the set of all nonempty closed bounded subsets of (X, d) and by H the Hausdorff metric on $CB(X)$

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\},$$

where $A, B \in CB(X)$ and

$$d(x, A) = \inf_{y\in A} d(x, y).$$

Our main theorem is prefaced with the following lemma due to Nadler [10].

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Lemma 1.1. Let $A, B \in CB(X)$ and $k > 1$. Then for each $a \in A$, there is a point $b \in B$ such that

$$d(a, b) \leq kH(A, B).$$

For single-valued mappings the notion of the weak commutativity is introduced by S. Sessa [14].

Definition 1.1. The mappings $S, I : X \to X$ are said to be weakly commuting if

$$d(SIx, ISx) \leq d(Ix, Sx)$$

for all x in X.

It is clear that two commuting mappings are weakly commuting but the converse is not true as is shown in [14].

G.Jungck [6] extended the concept of weak commutativity in the following way:

Definition 1.2. Let S and I be mappings from a metric space (X, d) into itself. The mappings S and I are said to be *compatible* if

$$\lim_{n\to\infty} d(SIx_n, ISx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ix_n = z$$

for some $z \in X$.

It is obvious that two *weakly commuting mappings* are *compatible*, but the converse is not true as is shown in [6].

H. Kaneko and S. Sessa [8] extended the concept of compatibility for single-valued mappings to the setting of single-valued and multi-valued mappings in the following way:

Definition 1.3. The mappings $A : X \to CB(X)$ and $S : X \to X$ are *compatible* iff $SAx \in CB(X)$ for all $x \in X$ and $d(SAx_n, ASx_n) \to 0$ whenever $\{x_n\} \subset X$ such that $Ax_n \to M \in CB(X)$ and $Sx_n \to t \in M$.

Definition 1.3 extends Definition 1.2 above as well as the Definition 1.1 above, where one may replace $I : X \to X$ with $A : X \to CB(X)$, that is $S : X \to X$ and $A : X \to CB(X)$ are said to be weakly commuting if for all $x \in X, SAx \in CB(X)$ and $H(SAx, ASx) \leq d(Sx, Ax)$, where H is the Hausdorff metric defined on $CB(X)$.

On the other hand, O. Hadzic [4] introduced the following definition:

Definition 1.4. The mappings $A : X \to CB(X)$ and $S : X \to X$ are said to be *weakly commuting* iff for every $x \in X$ and $y \in X$ such that $y \in Ax$ we have

$$d(Sy, SAx) \leq d(Sx, Ax).$$

It is obvious that *a commutative* pair of mappings (S, A) is also *weakly commutative* in the sense of Definition 1.4 but the converse is not true as is shown by Constantin [1].

We now introduce the following new definition which is slight variant of Kaneko-Sessa's [8] definition in the light of Definition 1.4.

Definition 1.5. The mappings $A : X \to CB(X)$ and $S : X \to X$ are said to be *compatible* if

$$\lim_{n\to\infty} d(Sy_n, ASx_n) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} y_n = z$$

for some z in X where $y_n \in Ax_n$ for $n = 1, 2, ...$

Clearly every weakly commuting pair of mappings (S, A) are compatible but the converse is not true. To see this, let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $y_n \in Ax_n$ for $n = 1, 2, ...$ and

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} y_n = z$$

for some z in X . Suppose S and A are weakly commuting. Since $d(Sx_n, Ax_n) \leq d(Sx_n, y_n)$, it follows that $\lim_{n\to\infty} d(Sx_n, Ax_n) = 0$. Since S and A are weakly commuting, we have

$$\lim_{n\to\infty} H(SAx_n, ASx_n) = 0.$$

Again, since $d(Sy_n, ASx_n) \leq H(SAx_n, ASx_n)$, we have

$$\lim_{n\to\infty} d(Sy_n, ASx_n) = 0,$$

proving that S and A are compatible.

Example 1.1. Let $X = [1, \infty)$ with the Euclidean metric d and define

$$Sx = 4x^4 - 3, \quad Ax = [1, x^2]$$

for all $x \geq 1$.

Clearly S and A are continuous and $A(X) = S(X) = X$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X defined by $x_n = 1 + \frac{1}{n}, y_n = 1$ for $n = 1, 2, ...$, we then have

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} y_n = 1 \in X, \quad y_n \in Ax_n.$$

Moreover,

$$\lim_{n\to\infty} H(SAx_n, ASx_n) = 0 \quad iff \quad x_n \to 1 \quad as \quad n \to \infty,$$

and since

$$d(Sy_n, ASx_n) \leq H(SAx_n, ASx_n),$$

we have

$$\lim_{n\to\infty} d(Sy_n, ASx_n) = 0.$$

Therefore S and A are compatible, but S and A are not weakly commuting at $x = 2$.

A point z in X is said to be a coincidence point of S and A if $Sz \in Az$ and is a fixed point of A if $z \in Az$.

2. A COINCIDENCE THEOREM

Following Constantin [1], we consider the set G of all continuous functions $g : [0, \infty)^5 \longrightarrow [0, \infty)$ satisfying the following conditions:

(i) $g(1, 1, 1, 2, 0) = g(1, 1, 1, 0, 2) = h \in (0, 1)$;

(ii) g is subhomogenuous nondecreasing in each variable;

(iii) $g(x, y, z, t, 0) < g(x_1, y_1, z_1, t, 0)$ and $g(x, y, z, 0, t) < g(x_1, y_1, z_1, 0, t)$ if $x \leq x_1, y \leq y_1, z \leq z_1, \max\{x, y, z\} < \max\{x_1, y_1, z_1\}$ and $t \leq \max\{x, y, z\}+\min\{x, y, z\}$;

(iv) if $u \leq g(v, v, u, u + v, 0)$ or $u \leq g(v, v, u, 0, u + v)$ or $u \leq g(v, u, v, u + v, 0)$ or $u \leq g(v, u, v, 0, u + v)$ then $u \leq hv$.

The following Lemma due to Constantin [1] is useful in the sequel.

Lemma 2.1. If $g \in G$ and $k > 1$ is such that $kh < 1$ we have that $kg \in G$.

Adrian Constantin [1] proved the following Theorem.

Theorem 2.1. Let (X, d) be a complete metric space, S and T continuous mappings from X into X, A and B H-continuous mappings from X into $CB(SX \cap TX)$ such that the pairs (A, S) and (B, T) are weakly commutative in the sense of Definition 1.4 and

$$H(Ax, By) \leq g(d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)), \tag{2.1}$$

for every $x, y \in X$, where $g \in G$. Then there exists a sequence $\{x_n\}_{n \in N}$ such that

1. For every $n \in N, Sx_{2n+1} \in Bx_{2n}, Tx_{2n} \in Ax_{2n-1}$;
2. There exists $z = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n+1}$;
3. $Sz \in Az, Tz \in Bz$.

Now replacing weakly commuting mappings with compatible mappings (Definition 1.5) in Theorem 2.1, we prove the following generalization of Theorem 2.1.

Theorem 2.2. Let (X, d) be a complete metric space, S and T continuous mappings from X into X, A and B H-continuous mappings from X into $CB(SX \cap TX)$ such that the pairs (A, S) and (B, T) are compatible in the sense of Definition 1.5 and condition (2.1) holds. Then there exists a sequence $\{x_n\}_{n \in N}$ such that

1. For every $n \in N, Sx_{2n+1} \in Bx_{2n}, Tx_{2n} \in Ax_{2n-1}$;
2. There exists $z = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n+1}$;
3. $Sz \in Az, Tz \in Bz$.

Proof. Let $k > 1$ be such that $kh < 1$. Let $x_0 \in X$. Since $Bx_0 \subset SX$ there exists $x_1 \in X$ such that $Sx_1 \in Bx_0$. Then there exists an element $u_1 \in Ax_1$ such that

$$d(u_1, Sx_1) \leq kH(Ax_1, Bx_0)$$

and $u_1 = Tx_2$ for some $x_2 \in X$ because $Ax_1 \subset TX$. Continuing in this fashion we define a sequence $\{x_n\}_{n \in N}$ in X where $Tx_{2n} \in Ax_{2n-1}$ and $Sx_{2n+1} \in Bx_{2n}$ are such that

$$d(Tx_{2n}, Sx_{2n-1}) \leq kH(Ax_{2n-1}, Bx_{2n-2}) \tag{2.2}$$

$$d(Sx_{2n+1}, Tx_{2n}) \leq kH(Bx_{2n}, Ax_{2n-1}) \tag{2.3}$$

for every $n \geq 1$.

We observe that by (2.1) and (2.3) we have that

$$d(Sx_{2n+1}, Tx_{2n}) \leq kH(Bx_{2n}, Ax_{2n-1}) \leq kg(d(Sx_{2n-1}, Tx_{2n}), d(Sx_{2n-1}, Ax_{2n-1}),$$
$$d(Tx_{2n}, Bx_{2n}), d(Sx_{2n-1}, Bx_{2n}), d(Tx_{2n}, Ax_{2n-1})) \leq kL,$$

where

$$L = g(d(Sx_{2n-1}, Tx_{2n}), d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n}, Sx_{2n+1}),$$
$$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, Sx_{2n+1}), 0).$$

By Lemma 2.1 we have that $kg \in G$ and so we deduce that

$$d(Tx_{2n}, Sx_{2n+1}) \leq khd(Sx_{2n-1}, Tx_{2n}).$$

In a similar way it follows that

$$d(Tx_{2n}, Sx_{2n-1}) \leq khd(Sx_{2n-1}, Tx_{2n-2}),$$

and so we deduce that

$$d(Tx_{2n}, Sx_{2n+1}) \leq (kh)^{2n} d(Sx_1, Tx_0). \tag{2.4}$$

and

$$d(Tx_{2n}, Sx_{2n-1}) \leq (kh)^{2n} d(Sx_1, Tx_0). \tag{2.5}$$

for every $n \geq 1$.

We deduce that $\{Sx_{2n-1}\}_{n \in N}$ is a Cauchy sequence in the complete metric space X and so $\lim_{n\to\infty} Sx_{2n-1} = z$ for some z in X. By (2.5), we also have $\lim_{n\to\infty} Tx_{2n} = z$.

We shall now prove that $Sz \in Az$, i.e., z is a coincidence point of S and A. For every $n \in N$, we have

$$d(STx_{2n}, Az) \leq d(STx_{2n}, ASx_{2n-1}) + H(ASx_{2n-1}, Az). \tag{2.6}$$

By H-continuity of A and the fact that $\lim_{n\to\infty} Sx_{2n-1} = z$, we have

$$\lim_{n\to\infty} H(ASx_{2n-1}, Az) = 0.$$

Since S and A are compatible mappings and $\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} y_n = z$, where $y_n = Tx_{2n} \in Ax_{2n-1}$ and $z_n = x_{2n-1}$, we have

$$\lim_{n\to\infty} d(STx_{2n}, ASx_{2n-1}) = \lim_{n\to\infty} d(Sy_n, ASz_n) = 0.$$

Thus the inequality (2.6) yields $\lim_{n\to\infty} d(STx_{2n}, Az) = 0$ and so from

$$d(Sz, Az) \leq d(Sz, STx_{2n}) + d(STx_{2n}, Az),$$

as $n \to \infty$ and the continuity of S, it follows that $d(Sz, Az) = 0$. Hence $Sz \in Az$, Since Az is a closed subset of $SX \cap TX$. We can similarly prove that $Tz \in Bz$, that is z is a coincidence point of T and B. This completes the proof.

Now , we give two examples to support our generalization.

Example 2.1. Let $X = [1, \infty)$ be with the Euclidean metric d. Define $A, B : X \to CB(SX \cap TX)$ and $S, T : X \to X$ by

$$Ax = [1, x^2], \quad Bx = [1, x^4]$$
$$Sx = 2x^4 - 1, \quad Tx = 2x^8 - 1$$

for all $x \in X, n \geq 1$.

Clearly A, B, S and T are continuous and $A(X) = B(X) = S(X) = T(X) = X$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X defined by $x_n = 1 + \frac{1}{n}, y_n = 1$ for $n = 1, 2, \ldots$, we then have

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} y_n = 1 \in X, y_n \in Ax_n.$$

Moreover,

$$\lim_{n\to\infty} H(SAx_n, ASx_n) = \lim_{n\to\infty} |(2x_n^4 - 1)^2 - 2x_n^8 + 1|$$
$$= 0 \quad iff \quad x_n \to 1 \quad as \quad n \to \infty,$$

and since $d(Sy_n, ASx_n) \leq H(SAx_n, ASx_n)$, we have

$$\lim_{n\to\infty} d(Sy_n, ASx_n) = 0.$$

Therefore S and A are compatible . Similarly, one can show that B and T are compatible.

Assuming $g \in G$ as

$$g(t_1, t_2, t_3, t_4, t_5) = \frac{t_1}{2}$$

for every $t_i \in R^+, i = 1, 2, 3, 4, 5$.

We have for any $x, y \in X$

$$\begin{aligned}
H(Ax, By) &= |x^2 - y^4| \\
&\leq |x^2 - y^4||x^2 + y^4| \\
&= |x^4 - y^8| \\
&= \frac{1}{2}|(2x^4 - 1) - (2y^8 - 1)| \\
&= \frac{d(Sx, Ty)}{2} \\
&= g(d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)).
\end{aligned}$$

All the assumptions of Theorem 2.2 are verified and 1 is the coincidence point of S and A and of T and B.

Note that Theorem 2.1 is not applicable even if $A = B$ and $S = T$ because A and S are not weakly commuting mappings at $x = 2$.

Example 2.2. Let $X = [0, \infty)$ be with the Euclidean metric d. Define $A, B : X \to CB(SX \cap TX)$ and $S, T : X \to X$ by

$$Ax = [0, x^2], \quad Bx = [0, 3x^2]$$
$$Sx = 4x^4, \quad Tx = 36x^4$$

for all $x \in X$.

Clearly A, B, S and T are continuous and $A(X) = B(X) = S(X) = T(X) = X$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X defined by $x_n = \frac{1}{n^2}, y_n = \frac{1}{n^3}$ for $n = 1, 2, ...$, we then have

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} y_n = 0 \in X, y_n \in Ax_n.$$

Moreover,

$$\begin{aligned}\lim_{n\to\infty} H(SAx_n, ASx_n) &= \lim_{n\to\infty} |4x_n^4 - 16x_n^4| \\ &= 0 \quad iff \quad x_n \to 0 \quad as \quad n \to \infty,\end{aligned}$$

and since $d(Sy_n, ASx_n) \leq H(SAx_n, ASx_n)$, we have

$$\lim_{n\to\infty} d(Sy_n, ASx_n) = 0.$$

Therefore S and A are compatible . Similarly, one can show that B and T are compatible.

Assuming $g \in G$ as

$$g(t_1, t_2, t_3, t_4, t_5) = \frac{t_1}{4}$$

for every $t_i \in R^+, i = 1, 2, 3, 4, 5$.

We have for any $x, y \in X$

$$\begin{aligned}H(Ax, By) &= |x^2 - 3y^2| \\ &\leq |x^2 - 3y^2||x^2 + 3y^2| \\ &= |x^4 - 9y^4| \\ &= \frac{1}{4}|4x^4 - 36y^4| \\ &= \frac{d(Sx, Ty)}{4} \\ &= g(d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)).\end{aligned}$$

All the assumptions of Theorem 2.2 are verified and 0 is the coincidence point of S and A and of T and B.

Note that Theorem 2.1 is not applicable even if $A = B$ and $S = T$ because A and S are not weakly commuting mappings at $x = 2$.

Corollary 2.1 ([9]). Let (X, d) be a complete metric space, S and T continuous mappings from X into X, A and B closed mappings from X into $CB(SX \cap TX)$ such that $ASx = SAx, BTx = TBx$ for every $x \in X$ and

$$H(Ax, By) \leq q \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}[d(Sx, By) + d(Ty, Ax)]\},$$

for every $x, y \in X$, where $q \in (0, 1)$. Then there exists a sequence $\{x_n\}_{n \in N}$ in X such that

1. For every $n \in N, Sx_{2n+1} \in Bx_{2n}, Tx_{2n} \in Ax_{2n-1}$;
2. There exists $z = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n+1}$;
3. $Sz \in Az, Tz \in Bz$.

Corollary 2.2 ([3]). Let (X, d) be a complete metric space, S and T continuous mappings from X into X, A a closed mapping from X into $CB(SX \cap TX)$ such that $ASx = SAx, ATx = TAx$ for every $x \in X$ and

$$H(Ax, Ay) \leq qd(Sx, Ty),$$

for every $x, y \in X$, where $q \in (0, 1)$. Then there exists a sequence $\{x_n\}_{n \in N}$ in X such that

1. For every $n \in N, Sx_{2n+1} \in Ax_{2n}, Tx_{2n} \in Ax_{2n-1}$;
2. There exists $z = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n+1}$;
3. $Sz \in Az, Tz \in Az$.

Corollary 2.3 ([7]). Let (X, d) be a complete metric space, $A : X \to CB(X)$ and S a continuous mapping from X into X, and

$$H(Ax, Ay) \leq hd(Sx, Sy),$$

for every $x, y \in X$, where $0 \leq h < 1$ and $ASx = SAx$ for every $x \in X$. If $A(X) \subseteq S(X)$, then there exists $t \in X$ such that $St \in At$.

Corollary 2.4 ([11]). Let (X, d) be a complete metric space, S and T continuous mappings from X into X, A and B closed mappings from X into $CB(SX \cap TX)$ such that $ASx = SAx, BTx = TBx$ for every $x \in X$ and

$$\begin{aligned} &H^2(Ax, By) \leq q \max\{d(Sx, Ax)d(Ty, By), d(Sx, By)d(Ty, Ax), \\ &\frac{1}{2}[d(Sx, Ax)d(Sx, By) + d(Ty, Ax)d(Ty, By)], d^2(Sx, Ty)\}, \end{aligned}$$

for every $x, y \in X$, where $q \in (0, 1)$. Then there exists a sequence $\{x_n\}_{n \in N}$ in X such that

1. For every $n \in N, Sx_{2n+1} \in Bx_{2n}, Tx_{2n} \in Ax_{2n-1}$;
2. There exists $z = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n+1}$;
3. $Sz \in Az, Tz \in Bz$.

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