Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Vol. No. 02, 1997, pp. 43–47.
Submitted: November 10, 1997. Published: December 30, 1997.

APPLICATIONS OF A SPECIAL REPRESENTATION OF ANALYTIC FUNCTIONS

I.K. Argyros, L.F. Mahrt, and F. Szidarovszky

Abstract. A special factored representation of analytic functions is applied to obtain iteration methods for solving nonlinear equations. As particular examples, the Newton method, and the Halley method are derived.

1991 A.M.S. (MOS) Subject Classification Codes. 26C15, 65, 47, 49.

Key Words and Phrases. Rational function, partial decomposition, fast algorithm, Newton's method.

1. INTRODUCTION.

In our earlier papers (Mahrt and Szidarovszky, 1997) we have introduced a special representation of analytic functions and have applied it to the efficient computation of the partial fraction decompositions of rational functions with poles of high multiplicity. The representation theorem can be stated as follows.

Theorem. *Let $n \geq 1$ be a positive integer and assume that functions P and Q are analytic on an open complex domain D. Let $\eta \in D$ be a given point such that for $k = 0, 1, \ldots, n-1$, $Q^{(k)}(\eta) \neq 0$. Then there exists a unique sequence A_0, $A_1, \ldots, A_{n-1}$ of complex numbers such that*

$$P(s) = Q(s)\big[A_0 + A_1 \cdot (s-\eta) + \cdots + A_{n-1} \cdot (s-\eta)^{n-1}\big] + P_1(s) \cdot (s-\eta)^n. \tag{1}$$

We have also shown that this representation can be obtained as the unique solution of the triangular system of linear algebraic equations

$$\underline{P} = \underline{T}\,\underline{a} \tag{2}$$

Department of Mathematics, Cameron University, Lawton, OK 73505, U.S.A.
E-mail Address: ioannisa@cameron.edu
Systems and Industrial Engineering Department, University of Arizona, Tucson, Az 85721, U.S.A.
E-mail address: lfmahrt@ccgate.hac.com
E-mail address: szidar@sie.arizona.edu

where

$$\underline{T} = \begin{pmatrix} t_{00} & 0 & \cdots & 0 \\ t_{10} & t_{11} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ t_{n-1,0} & t_{n-1,1} & \cdots & t_{n-1,n-1} \end{pmatrix}$$

with

$$t_{ki} = \frac{k!}{(k-i)!} Q^{(k-i)}(\eta),$$

furthermore

$$\underline{P} = \begin{pmatrix} P(\eta) \\ P'(\eta) \\ \vdots \\ P^{(n-1)}(\eta) \end{pmatrix} \quad \text{and} \quad \underline{a} = \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_{n-1} \end{pmatrix}.$$

In this short paper we will show how this representation can be used to construct iteration schemes for solving nonlinear equations.

2. CONSTRUCTION OF ITERATION METHODS.

In this paper we assume that we want to find a locally unique solution of equation

$$P(s) = 0 \tag{3}$$

where P is a function satisfying the conditions of the Theorem.

We will first show how the well known Newton method results from representation (1). Select $n = 2$ and $Q(x) \equiv 1$. In order to use equation (2) we need first to compute the corresponding matrix elements t_{ki}:

$$t_{00} = \frac{0!}{0!} Q^{(0)}(\eta) = 1,$$
$$t_{10} = \frac{1!}{1!} Q'(\eta) = 0,$$

and

$$t_{11} = \frac{1!}{0!} Q^{(0)}(\eta) = 1,$$

therefore equation (2) has the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} P(\eta) \\ P'(\eta) \end{pmatrix}$$

showing that

$$A_0 = P(\eta) \quad \text{and} \quad A_1 = P'(\eta).$$

Therefore the representation shows that

$$P(s) = P(\eta) + P'(\eta)(s-\eta) + P_1(s)(s-\eta)^2 = 0.$$

Let $\eta = x_m$, then from this equation

$$s = x_m - \frac{P(x_m)}{P'(x_m)} - \frac{P_1(s)}{P'(x_m)}(s - x_m)^2. \tag{4}$$

By omitting the last term we get the approximation

$$x_{m+1} = x_m - \frac{P(x_m)}{P'(m_m)} \tag{5}$$

which is the well known Newton method.

Select next $n = 2$ and $Q(x) = [P'(x)]^{1/2}$. Then we have

$$t_{00} = \frac{0!}{0!} \cdot P'(\eta)^{1/2},$$
$$t_{10} = \frac{1!}{1!} \cdot \frac{1}{2} P'(\eta)^{-1/2} P''(\eta),$$

and

$$t_{11} = \frac{1!}{0!} \cdot P'(\eta)^{1/2},$$

therefore equation (2) can be written as

$$\begin{pmatrix} P'(\eta)^{1/2} & 0 \\ \frac{1}{2} P'(\eta)^{-1/2} P''(\eta) & P'(\eta)^{1/2} \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} P(\eta) \\ P'(\eta) \end{pmatrix}$$

which implies that

$$A_0 = P(\eta) P'(\eta)^{-1/2}$$

and

$$A_1 = \left(P'(\eta) - \frac{1}{2} P(\eta) P'(\eta)^{-1} P''(\eta) \right) P'(\eta)^{-1/2}.$$

Therefore representation (1) has the form

$$\begin{aligned} P(s) = P(\eta) &+ \left(P'(\eta) - \frac{1}{2} P(\eta) P'(\eta)^{-1} P''(\eta) \right) (s - \eta) \\ &+ \ P_1(s)(s - \eta)^2 = 0. \end{aligned}$$

Let $\eta = x_m$, then from this equation

$$\begin{aligned} s = x_m &- \frac{P(x_m) P'(x_m)^{-1}}{1 - \frac{1}{2} P(x_m) P'(x_m)^{-2} P''(x_m)} \\ &- \ \frac{P_1(s)}{P'(x_m) - \frac{1}{2} P(x_m) P'(x_m)^{-1} P''(x_m)} (s - x_m)^2. \end{aligned} \tag{6}$$

By omitting the last term we get the approximation

$$x_{m+1} = x_m - \frac{P(x_m) P'(x_m)^{-1}}{1 - \frac{1}{2} P(x_m) P'(x_m)^{-2} P''(x_m)}, \tag{7}$$

which is the method of tangent hyperbolas or the Halley method.

Select finally $n = 2$ and $P(x) = [P'(x)]^\alpha$ where $\alpha > 0$ is a real number. Then

$$t_{00} = \frac{0!}{0!} P'(\eta)^\alpha,$$
$$t_{10} = \frac{1!}{1!} \alpha P'(\eta)^{\alpha-1} P''(\eta),$$

and

$$t_{11} = \frac{1!}{0!} P'(\eta)^\alpha,$$

therefore equation (2) has the form

$$\begin{pmatrix} P'(\eta)^\alpha & 0 \\ \alpha P'(\eta)^{\alpha-1} P''(\eta) & P'(\eta)^\alpha \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \begin{pmatrix} P(\eta) \\ P'(\eta) \end{pmatrix}$$

from which we have

$$A_0 = P(\eta)P'(\eta)^{-\alpha}$$

and

$$A_1 = \big(P'(\eta) - \alpha P(\eta)P'(\eta)^{-1}P''(\eta)\big)P'(\eta)^{-\alpha}.$$

Therefore representation (1) can be written as

$$\begin{aligned} P(s) = P(\eta) &+ \big(P'(\eta) - \alpha P(\eta)P'(\eta)^{-1}P''(\eta)\big)(s-\eta) \\ &+ P_1(s)(s-\eta)^2 = 0. \end{aligned}$$

By selecting $\eta = x_m$ we obtain

$$\begin{aligned} s = x_m &- \frac{P(x_m)P'(x_m)^{-1}}{1 - \alpha P(x_m)P'(x_m)^{-2}P''(x_m)} \\ &- \frac{P_1(s)}{P'(x_m) - \alpha P(x_m)P'(x_m)^{-1}P''(x_m)}(s - x_m)^2. \end{aligned} \tag{8}$$

By omitting the last term we obtain the generalized Halley method:

$$x_{m+1} = x_m - \frac{P(x_m)P'(x_m)^{-1}}{1 - \alpha P(x_m)P'(x_m)^{-2}P''(x_m)}. \tag{9}$$

The convergence rate of these methods can be analyzed by using equations (4), (6), and (8). The details are not given here, since they are standard techniques. Note however that as it can immediately be seen from (4), (6) and (8), methods (5), (7) and (9) are of order at least 2. Moreover if η is a zero of multiplicity β for function P_1 then the order of convergence of these methods becomes at least $\beta + 2$. In particular in Argyros (1998) it was shown that method (9) for $\alpha = \frac{1}{2}$, is of order at least 3 and for quadratic equations of order at least 4. Furthermore the results mentioned here hold in the more general setting of a Banach space. We mention finally that by different selection of $Q(x)$ different methods can be obtained. A summary of the most popular iteration methods can be found, for example, in Argyros and Szidarovszky (1993).

REFERENCES

[1] Argyros, I.K., Improving the order and rates of convergence for the Super-Halley method in Banach spaces (to appear, 1998).

[2] Argyros, I.K. and Szidarovszky, F., *The Theory and Applications of Iteration Methods*, CRC Press, Inc. Boca Raton, Florida/London, 1993.

[3] Mahrt, L.F. and Szidarovszky, F., A Factored Form of Complex Analytic Functions and Its Application to finding Partial Fraction Decompositions, *Southwest Journal of Pure and Applied Mathematics*, **2** (1997).