Electronic Journal: Southwest Journal of Pure and Applied Mathematics Internet:
http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Vol. No. 02, 1996, pp. 11–28.
Submitted: April 11, 1996, Published: August 5, 1996.

Set-theoretical properties of extreme combinatorial problems

Anatoly D. Plotnikov

Abstract

Concepts of constructive combinatorial problem and, in particular, of extreme combinatorial problem (ECP) are defined. It is announced a sequential principle of construction of the constructive combinatorial problems' solution. This principle leads to necessity to consider ECP as system of independence with weighted elements. Auxiliary sets, which are connected to the solution of optimization problem, are defined. Concept of a normal form of the ECP is introduced. It is established that representation of a problem in normal form in some cases permits essentially to simplify an initial problem. It is introduced concepts of dual and self-dual problems. Two classes of combinatorial optimization problems, which are discerned by method of search of the admissible solution, are considered. Concept of the canonical form the ECP is introduced. It is shown that the problem in canonical form – new problem which is equivalent to initial, i.e. all optimum solutions of a problem in canonical form are the optimum solutions of an initial problem as well.

The fulfilled researches in some cases simplify the search of the optimization problem solution.

(1991) A.M.S. (MOS) Subject Classification codes: 05.
Keywords and Phrases. Extreme combinatorial problem, auxiliary sets,normal form, canonical form, optimum solutions

1 Statement of the problem

In combinatorics concept of *sample* is defined [6].

Let be k of final sets $W^{(1)}$, $W^{(2)}$,..., $W^{(m)}$.

The ordered or disorder set of elements

$$\pi = (r_1, r_2, \ldots, r_m),$$

where $r_1 \in W^{(1)}$, $r_2 \in W^{(2)}$,...,$r_m \in W^{(m)}$, called *ordered* or *disorder sample* respectively.

If $W^{(1)} = W^{(2)} = \cdots = W^{(m)} = W$, it is said to be an sample of elements from the set W.

One of main problems of combinatorics is construction of such sample π from elements of initial n-set W which satisfies to given restrictions. Any combinatorial problem on construction of sample we'll call *constructive*.

Department of Applied Mathematics and Computing Systems, Vinnitsa State Technical University, Vinnitsa, Ukraine.
E-mail: aplot@cec.vtsu.vinnica.ua

On each sample function f can be given: $f(\pi) \to R$, where R – set of real numbers. A problem, in which it is required to find sample π', which provides extremum of function f, we'll call *extremal*, and appropriate sample π' – *optimal.*

There is the plenty of theoretical and applied problems in various areas of knowledge which can be formulated as combinatorial optimization problems [3] – [11].

From definition of constructive combinatorial problems follows that each of them belongs to the class NP. Various definitions of this class of problems exist (see, for example, [1], [5], [9], [12]). We'll specify definition of the class NP.

Let $\sum$ – some final alphabet, x, y – line of symbols in this alphabet. Let, further, $|x|$ – number of symbols in the line x. Any problem Z belongs to *the class* NP, if:

- input of the problem is some final line x;
- output (solution) of the problem is the line y such, that $|y| = p(|x|)$, where $p(|x|)$ is some polynomial of $|x|$;
- there is an algorithm A which in polynomial time of $|x|$ checks that the line y satisfies to problem's restrictions.

Existing methods of the analysis and the solutions of constructive combinatorial problems are not too diverse. The poverty of used methods leads to difficulties of development of solving algorithms.

One of methods to facilitate the subsequent solving of formulated combinatorial problem and, in particular, an extremal combinatorial problem consists in following. Possibly large simplification must be done at analysis stage. Besides, one can appear useful to change an initial problem that the construction of solving algorithm will be sufficient simple.

Following questions arise during consideration of extremal combinatorial problems:

- Which form must have the mathematical model of the problem?
- Which general properties have extremal problems?
- Which ways of simplification of optimization problems exist?
- Which main classes of extremal problems exist?
- How we can change an initial problem that to reduce number of the admissible solutions without loss of the optimum solution?

The purpose of this article – to give the answers to delivered questions.

2 Sequential principle of the solving

Let be some combinatorial problem $Z \in NP$.

Experience of the solving of constructive combinatorial problems shows that any solution of such problem can be obtained step-by-step, element-by-element. Else, process of construction of some solution (sample) π expediently to consider extensive along time, during which some components, or subset of the required solution are obtained. Hence, during construction of solution π we'll obtain a sequence of subsets π_k $(k = \overline{1, l})$:

$$\pi_1 \subset \pi_2 \subset \cdots \subset \pi_l = \pi.$$

The solution π of the problem $Z \in NP$ we'll call *support*, if in problem Z has not another solution π' such that the solution π is an own part of the solution π'.

The solution π_1 of the problems $Z \in NP$ we'll call *partial*, if there is such solution π of the problem Z that $\pi_1 \subset \pi$.

Let π_1, π be the partial and support solutions of the problem Z respectively, where $\pi_1 \subset \pi$. Let, further, $t(\pi_1)$, $t(\pi)$ are the time of construction of the partial and support solutions of the problem Z. A method of the solving, in which for partial π_1 and support π ($\pi_1 \subset \pi$) solutions of the problem $Z \in NP$, the correlation take place:

$$t(\pi_1) < t(\pi)$$

we'll call *sequential.*

Theorem 1. *Any solution of problem $Z \in NP$ can be obtained by the sequential method.*

In fact, for example, let it is necessary to find independent set of vertices of an undirected graph, i.e. such set of its vertices, any pair from which is not connected by an edge (is not adjacent). Obviously, in common case the *simultaneous choice* even at least some of such vertices is impossible, if the graph structure was not stipulated beforehand. Therefore required independent set of the graph is possible to be constructed step-by-step, vertex-by-vertex. Clearly, each subsequent vertex can be chosen only if it is known which vertices was chosen earlier in independent set to be forming.□

3 Set-theoretical model of combinatorial problems

The sequential principle of construction of the solution permits to introduce set-theoretical model of a combinatorial problem.

Let us to consider two (W, Q) which has the set $W = \{r_1, \ldots, r_n\}$ $(n > 1)$ called *working*, and *the domain* Q is non-empty population of subsets π of set W which satisfies condition:

$$\mathbf{1}^0. \quad \textit{if } \pi \in Q \textit{ and } \pi_1 \subset \pi, \textit{ then } \pi_1 \in Q.$$

Two (W, Q), which satisfies the condition $\mathbf{1}^0$, called *system of independence*, and maximum relatively inclusion elements from domain Q called *bases of the independence system.*

Easily to see, that a set of all partial and support solutions of optimization problem $Z \in NP$ is domain Q. This domain and initial set W form a system of independence (W, Q).

Further, if during solving problem Z for each element of working set W we put in correspondence the number – weight of this element, then, we can believe, that set $M = \{\mu(r_1), \ldots, \mu(r_n)\}$ is given in combinatorial problem Z. The set M is population of whole non-negative numbers. If weight of elements of working set W was not stipulated in the problem then the weight of all elements equal to, for example, one.

At last, if function f conforms to each solution $\pi \in Q$ of the problem Z: $\pi \to \mathbf{R}$, where $\mathbf{R}$ is the set of real numbers, we come to conclusion that constructive combinatorial problem should be considered as four $Z = (W, Q, M, f) \in NP$.

In constructed model any element π of the domain Q we'll call *an admissible solution* of the problem Z. So, further, speaking about admissible solution of the problem Z, we'll mean partial and support solutions of the problem Z and, conversely, speaking about partial or support solution of the problem, we'll mean only its admissible solutions.

Let it is required to find such support solution $\widehat{\pi} \in Q$ that $f(\widehat{\pi})$ is extremal.

We'll call the formulated above problem as *extremal combinatorial problem (ECP).* Then for problem maximization it is required to find such support solution $\widehat{\pi} \in Q$ that

$$f(\widehat{\pi}) \geq f(\pi)$$

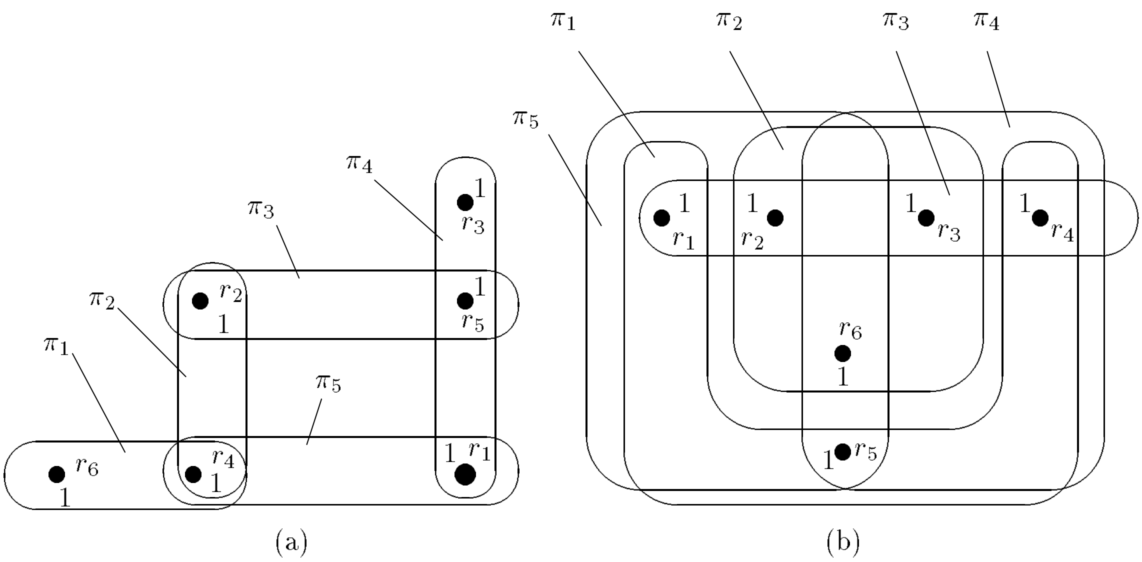


Figure 1: Representation ECP by hypergraph

and for problem of minimization

$$f(\widehat{\pi}) \leq f(\pi),$$

where $\pi \in Q$ is any support solution of the problem $Z \in NP$.

We'll further believe that function $f(\pi)$ is additive, i.e.

$$f(\pi) = \sum_{\forall \; r_i \; \in \; \pi} \mu(r_i).$$

4 Methods of the ECPs' specification

During set-theoretical research of the problem $Z \in NP$ it is conveniently to use two methods of its specification: graphic and matrix.

In graphic (more evident) method of the specification to each element $r \in W$ we'll put in correspondence a point (circle) on plane with record near each point of a name (designation) of element and number – weight of the given element. All elements of the working set W, which form the support solution $\pi \in Q$, are outlined by a closed line.

Obviously, that as a result of the described procedure we'll obtain the graphic image of hypergraph (without the enclosed one into another of edges). Each hyperedge corresponds to the support solution of the problem $Z \in NP$.

On the fig. 1 (a) and (b) problems Z_1 and Z_2 are given respectively.

The problem $Z \in NP$ can be also given by a boolean matrix $\parallel \alpha_{ij} \parallel$, in which $\alpha_{ij} = 1$, if the element $r_i \in W$ $(i = \overline{1,n})$ belongs to the support solution $\pi_j \in \mathcal{B}$ $(j = \overline{1,p})$, where $\mathcal{B} \subset \mathcal{Q}$ – set of all support solutions of the problem Z, and $\alpha_{ij} = 0$ otherwise. We'll call such matrix as *matrix of the support solutions*

$$\| \alpha_{ij} \| = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{array} \begin{array}{c} \begin{array}{ccccc} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \end{array} \\ \left\| \begin{array}{ccccc} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array} \right\| \\ (a) \end{array} \begin{array}{c} M \\ \left| \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \right| \\ \end{array} \qquad \| \alpha^*_{ij} \| = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{array} \begin{array}{c} \begin{array}{ccccc} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \end{array} \\ \left\| \begin{array}{ccccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{array} \right\| \\ (b) \end{array} \begin{array}{c} M \\ \left| \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \right| \\ \end{array}$$

Figure 2: Support solutions matrixes

(SSM). Besides, the vector $M = (\mu(r_1), \dots, \mu(r_n))$ of elements' weights of working set W should be given in this case.

Clearly, that SSM is the matrix analogue of a graphic method of the problem's specification. It is obvious also, that SSM determines bases of system of independence (W, Q).

SSM and column vector of elements' weights are shown on the fig. 2 (a) and (b) for problems Z_1 and Z_2 respectively.

5 Auxiliary sets of the solutions of ECP

Let there be the problem $Z \in NP$.

Let, further, there be also the union $T(\pi)$ of all such admissible solutions $\pi_l \in Q$ of the optimization problem $Z \in NP$, in which solution $\pi \in Q$ is included, i.e.

$$T(\pi) = \bigcup_{\forall \pi_l \supset \pi} \pi_l.$$

Clearly, that $T(\pi) \subset W$.

The set

$$S(\pi) = W \setminus T(\pi)$$

we'll call *adjoint* to solution $\pi \in Q$.

Theorem 2. *If $\pi_1, \pi_2 \in Q$ and $\pi_1 \subset \pi_2$ then $S(\pi_1) \subset S(\pi_2)$.*

Obviously, that if $\pi_1 \subset \pi_2$ then $T(\pi_2) \subset T(\pi_1)$. Therefore, $S(\pi_1) = W \setminus T(\pi_1) \subset W \setminus T(\pi_2) = S(\pi_2)$. □

The set

$$W(\pi) = W \setminus [\pi \cup S(\pi)]$$

we'll call *residual* for solution $\pi \in Q$.

Theorem 3. *If $\pi \in Q$ and $r \in W(\pi) \neq \oslash$ then $\pi \cup \{r\} \in Q$.*

Really,

$$W(\pi) = W \setminus [\pi \cup S(\pi)] = (W \setminus \pi) \cap [W \setminus S(\pi)] = (W \setminus \pi) \cap T(\pi) = [W \cap T(\pi)] \setminus \pi = T(\pi) \setminus \pi.$$

Therefore, if $W(\pi) \neq \oslash$ then the set $\pi \cup \{r\}$ is included at least into one admissible solution of domain Q and according to property $\mathbf{1^0}$ it is also solution.□

Theorem 4. *If $\pi \in Q$ is the support solution then $W(\pi) = \oslash$ and $\pi \cup S(\pi) = W$.*

If we'll assume that $W(p) \neq \oslash$ then we'll come to conclusion that domain Q includes solution $\pi = \pi \cup \{r\}$ $(r \in W(\pi))$, an own part of which is the solution $\pi \in Q$, and it contradicts with definition of the support solution.

Further, if $W(\pi) = \oslash$ then, according to definition of residual set, the correlation $\pi \cup S(\pi) = W$ is fulfilled for any support solution $\pi \in Q.Box$

6 Constituents

Let $\mathcal{B} = \{\pi_1, \ldots, \pi_p\}$ is the set of all support solutions of the problem $Z \in NP$.

We'll designate $\overline{\pi} = W \setminus \pi$ and

$$\pi^{\sigma} = \begin{cases} \pi, & if\ \sigma = 1, \\ \overline{\pi}, & if\ \sigma = 0. \end{cases}$$

Following [7], each set of a type

$$K_{I_1} = \pi_1^{\sigma_1} \cap \pi_2^{\sigma_2} \cap \cdots \cap \pi_p^{\sigma_p}$$

$(\sigma_j \in \{0,1\}; j = \overline{1,p})$ we'll call *constituent of the support solutions* of the problem Z, or simply – *constituent*, and set I_1 – *calling set of constituent*, where $I_1 = \{\sigma_1, 2\sigma_2, \ldots, p\sigma_p\}$.

The set of all non-empty constituents of the problem Z we'll designate: $H(Z)$.

It is not difficult to see [7] that the following properties are satisfied:

K1. $K_{I_1} \cap K_{I_2} = \oslash,\ if\ I_1 \neq I_2;$

K2. $\bigcup\limits_{\forall\ I_q\ \subset\ \{1,\ 2,\ldots,\ p\}} K_{I_q} = W;$

K3. $\pi_j = \bigcup\limits_{\forall\ I_{\pi_j}\ \in\ H(Z)} K_{I(\pi_j)},$

where $I(\pi_j)$ is calling set such that $\sigma_j = 1$ $(j = \overline{1,p}; \pi_j \in \mathcal{B})$.

Theorem 5. *Any element of the working set W belongs to one and only one constituent.*

Opposite will lead to contradiction to property **K1**.□

From definition of the SSM of the problem Z follows that for its i-th row we can put in correspondence constituent K_{I_i}, where

$$I_i = \{\sigma_{i_1}, 2\sigma_{i_2}, \ldots, p\sigma_{i_p}\}$$

$(i = \overline{1,n})$. Thus, SSM also can be considered as incident system for calling sets of constituents from $H(Z)$.

7 The normal form of ECPs

Let be $W_1 \subset W$. By $\gamma(W_1)$ we'll designate intersection of calling sets of all constituents from $H(Z)$ which include at least one element of W_1, and by $\overline{\gamma(W_1)}$ – *complement* $\gamma(W_1)$ to the set $I = \{1, 2, \ldots, p\}$, i.e. $\overline{\gamma(W_1)} = I \setminus \gamma(W_1)$.

Theorem 6. *$W_1 \subset W$ is the admissible solution of the problem Z iff $\gamma(W_1) \neq \oslash$.*

According to property $\mathbf{1}^0$, any admissible solution of the problem Z can be considered as a subset at least one support solutions $\pi_j \in \mathcal{B}$ $(j = \overline{1,p})$. Therefore, according to property **K3**, the calling sets of constituents, which include elements of the solution $W_1 \subset W$, have at least one common element, i.e. $\gamma(W_1) \neq \oslash$.

The converse is evident.□

Consequence 7. *Let be $\pi_1, \pi_2 \in Q$. Then $\pi_1 \cup \pi_2 \in Q$, iff $\gamma(\pi_1) \cap \gamma(\pi_2) \neq \oslash$.*

Theorem 8. *If $\pi \in Q$ then*

$$S(\pi) = \bigcup_{\forall \ I_t \ \in \ \mathcal{P}_0[\overline{\gamma(\pi)}]} K_{I_t},$$

where $\mathcal{P}_0[\overline{\gamma(\pi)}]$ is the set of all subsets of set $\overline{\gamma(\pi)}$.

Obviously, that the set $S(\pi)$ consists of all elements $r \in W$, each of which belongs to constituent K_{I_t} such that $\gamma(\pi) \cap I_t = \oslash$.

The validity of a Theorem 8 follows from latter.□

Consequence 9. *If $\pi_1, \pi_2 \in Q$ and $\gamma(\pi_1) = \gamma(\pi_2)$ then $S(\pi_1) = S(\pi_2)$.*

Consequence 10. *If $I_1 \subset I_2$ then $S(K_{I_2}) \subset S(K_{I_1})$.*

Theorems 6 and 8 permit to formulate the simple rules of a finding of elements of the problem Z solution by its SSM. So, the elements of W will form the admissible solution $\pi \in Q$, if corresponding to them the rows of SSM have single components in same (at least one) column. The set of all such columns determines, obviously, the set $\gamma(\pi)$. Then the adjoint set for solution $\pi \in Q$ consists from those and only of those elements of working set W for which rows having zero components in columns of $\gamma(\pi)$ correspond.

Theorem 11. *If $r_1 \in K_{I_1}$ and $r_1 \in \pi$, where $\pi \in \mathcal{B}$ is the support solution of the problem Z, then $K_{I_1} \subset \pi$.*

Let conditions of a Theorem 11 are fulfilled and we allow that there is an element $r_2 \in K_{I_1}$ such that $r_2 \notin \pi$. However, $r_2 \notin S(\pi)$, as $\gamma(\pi) \cap I_1 \neq \oslash$, and $\pi \cup S(\pi) = W$. It is contradiction.□

We'll call the problem Z *represented in normal form*, or *normalized*, if two of any elements of working set W belong to various constituents, i.e. for any $r_{i_1}, r_{i_2} \in W$ $(i_1 \neq i_2)$ we have: $r_{i_1} \in K_{I_1}$, $r_{i_2} \in K_{I_2}$ and $I_1 \neq I_2$ $(K_{I_1}, K_{I_2} \in H(Z))$.

Obviously, that the problem Z is normalized if its SSM has the various rows.

The essence of problem normalization consists in transition from consideration of an initial problem $Z \in NP$ to consideration (other) similar problem $Z_n \in NP$, in which the elements of working set W_n correspond to various classes of equivalence of working set elements W – elements, belonging to same constituent of the problem Z. It is not difficult to see, that weight of each element of working set W_n is equal to total weight of elements of working set W, belonging to respective constituent of the problem Z, and the optimum solution of a problem Z_n determines the optimum solution of the problem Z.

Thus, if we'll replace each group of identical rows of SSM $\| \alpha_{ij} \|$ of the problems Z by one and to confer weight of respective element of new working set W_n, equal to total weight of group of replaced elements of set W, then we'll obtain, obviously, SSM $\| \alpha^n_{ij} \|$ and weight vector M_n of a new normalized problem Z_n.

Example 1.

One of important examples of ECP is a covering problem, which traditionally formulated as follows [3], [5].

Let there be the set $A = \{a_1, \ldots, a_m\}$ $(m > 1)$. Let, further, $W = \{r_1, \ldots, r_n\}$ is the family of subsets r_i $(i = \overline{1, n})$ of the set A. We put in correspondence a number $\mu(r_i) > 0$ to each subset $r_i \in R$ – weight of the subset r_i.

The subfamily $\pi \subset W$ is called *cover* of the sets A if it satisfies to a condition

$$\bigcup_{\forall\, r_i\, \in\, \pi} r_i = A.$$

The cover $\pi \subset W$ is irreducible (unredundant), if

$$\bigcup_{r_i\, \in\, \pi\, \setminus\, \{r_q\}} r_i \neq A$$

for any $r_q \in \pi$.

It is required to find such cover $\pi \subset W$ for which the quantity

$$f(\pi) = \sum_{\forall\, r_i\, \in\, \pi} \mu(r_i)$$

is minimum.

Clearly, that some irreducible cover of the set A will satisfy to an indicated request.

Easily to see, that the formulated problem is ECP $Z = (W, Q, M, f) \in NP$.

The covering problem is usually given by *a covering matrix (CM)* $\| \beta_{ik} \|$ $(i = \overline{1, n};\ k = \overline{1, m})$, where $\beta_{ik} = 1$, if $a_k \in r_i$, and $\beta_{ik} = 0$ otherwise.

There are some methods of simplification of a covering matrix [3], that permits to reduce the size of a initial problem. The normalization of the problem Z permits to formulate one (new) rule of CM simplification.

It is known, that each column of SSM determines irreducible cover $\pi_j \in \mathcal{B}$ $(j = \overline{1, p})$ of the set A (see, for example, S.Petrik's method [2]). Obviously, that due to dual principle each column of CM will determine cover of the set B by elements of working set W (strictly speaking, the element $r_i \in W$ will be in this case "to present" a subset $\mathcal{B}_i$ sets of all support solutions $\mathcal{B}$).

From latter easily to establish that the elements $r_{i_1}, r_{i_2} \in W$ $(i_1 \neq i_2)$ belong to same constituent $K_{I_1} \in H(Z)$ problems Z, if:

- in CM $\| \beta_{ik} \|$ there are the rows i_1, i_2, having same number of unities, and such, that $\beta_{i_1 k}\ \& \beta_{i_2 k} = 0$ for all $k = \overline{1, m}$
- for each k_1-th column, for which $\beta_{i_1 k_1} = 1$, will be found k_2-th column, for which $\beta_{i_2 k_2} = 1$ such, that for all $i = \overline{1, n}$ $(i \neq i_1,\ i \neq i_2)$ correlation is fulfilled: $\beta_{i k_1} = \beta_{i k_2}$.

Clearly, that in this case from CM $\| \beta_{ik} \|$ it is necessary to delete i_1-th (i_2-th) row and all columns $k = \overline{1, m}$, for which $\beta_{i_1 k} = 1$ $(\beta_{i_2 k} = 1)$ and to assign $\mu(r^n_{i_2}) = \mu(r_{i_1}) + \mu(r_{i_2})$ (respectively $\mu(r^n_{i_1}) = \mu(r_{i^n_2})$).

So, if CM of the problem looks like submitted on the fig. 3 (a), after normalization we we'll obtain new CM, represented on the fig. 3 (b), in which

$$r^n_1 = \{r_1, r_5\},$$

$$r^n_2 = \{r_2, r_6\},$$

$$r^n_3 = \{r_3, r_7\},$$

$$r^n_4 = \{r_4, r_8\}.$$

$$
\| \beta_{ik} \| = \begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \end{array}
\begin{array}{c} \\ \left\| \begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array} \right\| \end{array}
\begin{array}{c} M \\ \left| \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \right| \end{array}
$$

(a)

$$
\| \beta^n_{ij} \| = \begin{array}{c} \\ r^n_1 \\ r^n_2 \\ r^n_3 \\ r^n_4 \end{array}
\begin{array}{c} \\ \left\| \begin{array}{cccc}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array} \right\| \end{array}
\begin{array}{c} M_n \\ \left| \begin{array}{c} 2 \\ 2 \\ 2 \\ 2 \end{array} \right| \end{array}
$$

(b)

Figure 3: Initial and normalized covering problem

Example 2.

Let there be graph $G = (W, S)$, where W is the set of graph vertices and S is mapping W into W.

The set $U \subset W$, satisfying to a condition

$$U \cap SU = \oslash,$$

we'll call *independent set of vertices* of the graph G.

Independent set of vertices U is called *maximal (MIS)*, if

$$U \cup SU = W.$$

MIS $\widehat{U} \subset W$ is called *maximum (MMIS)*, if $Card(\widehat{U}) \geq Card(U)$ for all MIS $U \subset W$ of the graph G.

Clearly, that in this case we have ECP $Z = (R, Q, M, f) \in NP$, where $\mu(r_i) = 1$ for all $\mu(r_i) \in M$ $(i = \overline{1, n})$.

The vertices $r_{i_1}, r_{i_2} \in W$ $(i_1 \neq i_2)$ belong to same constituent $K_{I_1} \in H(Z)$ problems Z, if $S(\{r_{i_1}\}) = S(\{r_{i_2}\})$, which can be find easily in adjacency matrix of the graph G.

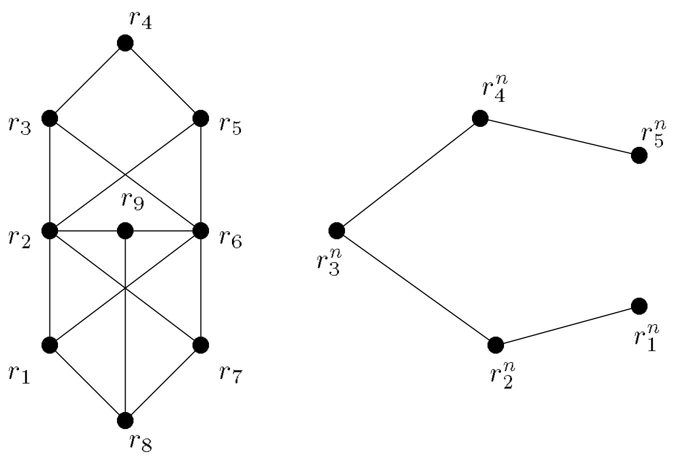
The normalization of a problem Z leads to consideration of the graph with weighted vertices and it's carried out by obvious way [10].

Initial and normalized problems of search of MMIS of the graph are represented on the fig. 4 (a) and (b) respectively. Here:

$$r^n_1 = \{r_8\},$$

$$r^n_2 = \{r_1, r_7, r_9\},$$

$$r^n_3 = \{r_2, r_6\},$$



(a) (b)

Figure 4: Initial and normalized problem of search of MMIS

$$r_4^n = \{r_3, r_5\},$$
$$r_5^n = \{r_4\}.$$

8 Dual ECP

Let $Z = (W, Q, M, f) \in NP$ is the problem maximization (minimization). The problem of minimization (maximization) $Z^* = (W, Q^*, M, f) \in NP$ we'll call *dual* to the problem Z, if each support solution $\Pi_j \in \mathcal{B}$ $(j = \overline{1,p})$ of the problem Z^* is connected to support solution $\pi \in \mathcal{B}$ $(j = \overline{1,p})$ of the problem Z by correlation:

$$\Pi = W \setminus \pi = \overline{\pi}_j. \qquad (*)$$

From definition of the dual problem Z^* follows that $(Z^*)^* = Z$, i.e. initial (direct) problem Z is dual to problem Z^*.

Theorem 12. *The solution $\widehat{\pi} \in \mathcal{B}$ is optimum for problem Z iff the corresponding solution $\widehat{\Pi} = W \setminus \widehat{\pi} \in \mathcal{B}$ is optimum for dual problem Z^*, and $f(\widehat{\pi}) = f(W) - f(\widehat{\Pi})$.*

Taking into account Theorem 4 and correlation $(*)$, we have:

$$\Pi_j = S(\pi_j), \qquad \pi_j = S(\Pi_j)$$

and, besides, it is obvious, that

$$f(W) = f(\pi_j) + f(\Pi_j),$$

where π_j and Π_j are support solutions of problems Z and Z^* respectively.

Therefore, if $\widehat{\pi} \in Q$ has the optimum solution of a maximization problem Z then

$$f(\widehat{\pi}) = f(W) - f(\widehat{\Pi}) \geq f(W) - f(\Pi_j) = f(p),$$

where π_j – any support solution of a problem Z. Whence we have:

$$f(\widehat{\Pi}) \geq f(\Pi_j),$$

where $\widehat{\Pi} \in Q^*$ – optimum solution, and Π_j – any support solution of a minimization problem Z^*.

Clearly, that we'll have similar correlations if Z is the minimization problem.

Thus, the sufficiency of conditions of Theorem 12 is proved. Necessity of them is evident.$\square$

Let $H(Z)$ is the set of all (non-empty) constituents of the problem Z, and $H(Z^*)$ is the same set of the dual problem Z^*.

Theorem 13. *Constituent $K_{I_1} \in H(Z)$ iff constituent $K_{\overline{I}_1} \in H(Z^*)$, where $\overline{I}_1 = I \setminus I_1$.*

In fact, let

$$K_{I_1} = \pi_1^{\sigma_1} \cap \pi_2^{\sigma_2} \cap \dots \cap \pi_p^{\sigma_p} \in H(Z).$$

Then for dual problem Z^* we'll have:

$$K_{\overline{I}_1}^* = \Pi_1^{\overline{\sigma}_1} \cap \Pi_2^{\overline{\sigma}_2} \cap \dots \cap \Pi_p^{\overline{\sigma}_p} \in H(Z)^*.$$

Taking into account $(*)$, we'll obtain:

$$K_{\overline{I}_1}^* = \pi_1^{\overline{\sigma}_1} \cap \pi_2^{\overline{\sigma}_2} \cap \dots \cap \pi_p^{\overline{\sigma}_p}.$$

The reverse is proved analogously.$\square$

From Theorem 13 follows, that if $r_{i_1} \in K_{I_1}$ and $r_{i_2} \in K_{\overline{I}_1}$ then rows i_1, i_2 of SSM $\parallel \alpha_{ij} \parallel$ of the problems Z satisfy to a condition

$$\alpha_{i_1 j} \ \& \ \overline{\alpha}_{i_2 j} = \overline{\alpha}_{i_1 j} \ \& \ \alpha_{i_2 j}$$

for all $j = \overline{1,p}$.

Constituents $K_{I_1} \in H(Z)$ and $K_{\overline{I}_1} \in H(Z^*)$ we'll call *orthogonal.*

We'll designate as $I(H(Z))$ and $I(H(Z^*))$ sets of calling sets of constituents, which belong to sets $H(Z)$ and $H(Z^*)$ respectively.

We'll call a problem Z *self-dual*, if $I(H(Z)) = I(H(Z^*))$. Or else, Z is the self-dual problem, if any constituent and orthogonal to it constituent belong to the set $H(Z)$.

Obviously, that if Z is a self-dual problem, then $Card(H(Z))$ is the even number.

Theorem 14. *If π is support solution of the problem Z and orthogonal constituents $K_{I_1}, K_{\overline{I}_1} \in H(Z)$ exist then either $K_{I_1} \subset \pi$ or $K_{\overline{I}_1} \subset \pi$.*

For proof it is enough to indicate, that for any support solution $\pi \in Q$ of the problem Z we have $Card[\gamma(p)] = 1$.$\square$

Consequence 15. *If Z is a self-dual problem then each its support solution contains the same number constituents.*

Consequence 16. *If Z is a normalized self-dual problem then each its support solution has the same number of elements.*

We'll call a self-dual problem *trivial*, if for any its orthogonal constituents K_{I_1}, $K_{\overline{I}_1}$ a correlation is fulfilled:

$$\mu(K_{I_1}) = \mu(K_{\overline{I}_1}).$$

Theorem 17. *If Z is a trivial self-dual problem then any its support solution is optimum.*

Obviously.$\square$

Example 3.

Let $Z = (W, Q, M, f)$ is the covering problem which given by CM $\parallel \beta_{ik} \parallel$ $(i = \overline{1,n}; k = \overline{1,m}; m > 1)$.

Easily to establish, that elements $r_{i_1}, r_{i_2} \in W$ $(i_1 \neq i_2)$ belong to orthogonal constituents of the problem Z, if in CM $\parallel \beta_{ik} \parallel$ rows i_1, i_2 correspond to it and the following conditions satisfy:

1. there is such column $k_1 \in J = \{1, 2, \dots, m\}$, that $\beta_{i_1 k_1} = \beta_{i_2 k_2} = 1$ and $\beta_{i k_1} = 0$ for all $i = \overline{1, n}$ $(i \neq i_1, i \neq i_2)$;
2. for all columns $k = \overline{1, m}$ $(k \neq k_1)$ $\beta_{i_1 k} \& \beta_{i_2 k} = 0$; and, besides, one of below-mentioned conditions is fulfilled:
   (a) $Card(J_{i_1}) = 1$ $(Card(J_{i_2}) = 1)$;
   (b) if $Card(J_{i_1}) > 1$ and $Card(J_{i_2}) > 1$ and there exist at least one column $k_2 \in J$ such that $\beta_{i_1 k_1} = \beta_{i_2 k_2} = 0$ and for each component-by-component disjunction of column vectors $\tilde{\beta}_{k'} \vee \tilde{\beta}_{k''}$ $(k' \in J_{i_1}, k'' \in J_{i_2}, k' \neq k_1, k'' \neq k_1)$ there will be the column vector $\| \tilde{\beta}_k \|$ $(k \in J \setminus (J_{i_1} \cup J_{i_2}))$ such that $\tilde{\beta}_k \leq \tilde{\beta}_{k'} \vee \tilde{\beta}_{k''}$.

Here J_{i_1}, J_{i_2} is the set of all columns of CM $\| \tilde{\beta}_k \|$ such that

$$\beta_{i_1 k} = 1 \ (\beta_{i_2 k} = 1)$$

for $k \in J$;

$$\tilde{\beta}_k = (\beta_{1k}, \beta_{2k}, \dots, \beta_{nk})$$

– column vector of CM $\| \beta_{ik} \|$;

$$\tilde{\beta}_{k'} \vee \tilde{\beta}_{k''} = (\beta_{1k'} \vee \beta_{1k''}, \dots, \beta_{nk'} \vee \beta_{nk''}).$$

So, if CM of the problem Z looks like on the fig. 5 (a), 1-th and 4-th of its row satisfy to the above-stated conditions and, hence, the elements r_1 and r_4 belong to orthogonal constituents of the problem.

The fastest method of construction of a self-dual covering problem, the optimum solution of which coincides optimum solution of the problem $Z = (W, Q, M, f)$ consists in following.

Let $\| \beta_{ik} \|$ $(i = \overline{1, n}; k = \overline{1, m})$ is CM of the problem Z.

We'll construct CM $\| \beta'_{ik} \|$ $(i = \overline{1, 2n};\ k = \overline{1, m+n})$ of new problem Z', by adding n rows and n columns to CM $\| \beta_{ik} \|$ of the initial problem Z.

Let's assume:

$$\beta'_{ik} = \beta_{ik} \text{ for all } i = \overline{1, n},\ k = \overline{1, m};$$
$$\beta'_{ik} = 0 \text{ for all } i = \overline{n+1, 2n},\ k = \overline{1, m};$$
$$\beta'_{ik} = 1, \text{ if } k = m + i \ and \ i = \overline{1, n};$$
$$\beta'_{ik} = 1,\ if\ k = m - n + i \text{ and } i = \overline{n+1, m+n};$$
$$\beta'_{ik} = 0 \text{ in all other cases (it is evident, that here } k = \overline{m+1, m+n}).$$

Besides, let's assume $\mu(r'_i) = \mu(r_i)$, if $i = \overline{1, n}$ and $\mu(r'_i) = 0$, if $i = \overline{n+1, 2n}$.

Obviously, that as a result a self-dual covering problem $Z' = (W', Q', M', f')$ will be constructed.

So, if the initial problem is given by CM, which represented on the fig. 5 (a), then respective self-dual problem will have the covering matrix, represented on the fig. 5 (b).

Problem Z^*, which is dual to Z', we'll obtain, if let's assume $\mu(r^*_i) = 0$ when $i = \overline{1, n}$ and $\mu(r^*_i) = \mu(r'_{2n+1-i})$ when $i = \overline{n+1, 2n}$. The covering matrixes of the problems Z' and Z^*, obviously, coincide, if we'll establish correspondence: $r'_i \leftrightarrow r^*_i$ $(i = \overline{1, 2n})$.

Example 4.

Example of dual and self-dual problems for problem about finding MMIS of the graph is stated in detail in work [10].

Direct and dual problems of search MMIS of the graph are represented on the fig. 6 (a) and (b) respectively.

$$\| \beta_{ik} \| = \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{array} \left\| \begin{array}{cccccccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right\|$$

(a)

$$\| \beta'_{ij} \| = \begin{array}{c} r'_1 \\ r'_2 \\ r'_3 \\ r'_4 \\ r'_5 \\ r'_6 \\ r'_7 \\ r'_8 \\ r'_9 \\ r'_{10} \\ r'_{11} \\ r'_{12} \end{array} \left\| \begin{array}{cccccccccccccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right\|$$

(b)

Figure 5: Direct and dual covering problems

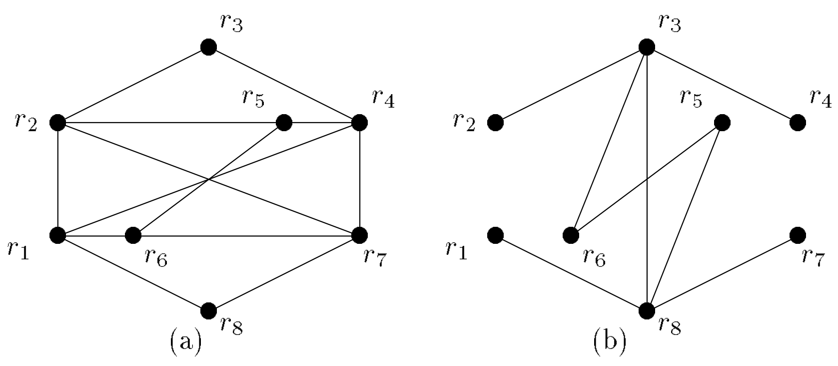


Figure 6: Direct and dual problems of search of MMIS

9 Cyclic and acyclic ECPs

We'll designate: $\mathcal{P}(C) = \mathcal{P}_0(C) \setminus \{\oslash\}$, where $\mathcal{P}_0(C)$ is the set of all subsets of the set C.

Let $D = \{D_1, \ldots, D_{t_1}\}$, $E = \{E_1, \ldots, E_{t_2}\}$ is the family of subsets of some set C. We'll define operation of *direct union* $\#$ on families:

$$D\#E = \{D_i \cup E_k : \ \forall D_i \in D, \forall E_k \in E\}.$$

Let be $C = C_1 \cup C_2$ and $C_1 \cap C_2 = \oslash$. Then, obviously, we'll have:

$$\mathcal{P}_0(C) = \mathcal{P}_0(C_1) \cup \mathcal{P}(C_2) \cup [\mathcal{P}(C_1) \ \# \ \mathcal{P}(C_2)]. \qquad (**)$$

Let be the problem $Z = (W, Q, M, f) \in NP$.

Theorem 18. *If $\pi_1, \pi_2 \in Q$ and $\gamma(\pi_1) \cap \gamma(\pi_2) \neq \oslash$,* then

$$S(\pi_1 \cup \pi_2) = S(\pi_1) \cup [S(\pi_2) \setminus S(\pi_1)] \cup S(\pi_1, \pi_2), \qquad (***)$$

where

$$S(\pi_2) \setminus S(\pi_1) = \bigcup_{\forall \ I_q \ \in \ \mathcal{P}[\gamma(\pi_1) \setminus \gamma(\pi_2)]} K_{I_q},$$

$$S(\pi_1, \pi_2) = \bigcup_{\forall \ I_l \ \in \ \mathcal{P}[\overline{\gamma(\pi_1)}] \# \mathcal{P}[\gamma(\pi_1) \setminus \gamma(\pi_2)]} K_{I_q}.$$

In fact, according to Theorem 8

$$S(\pi_1 \cup \pi_2) = \bigcup_{\forall \ I_g \ \in \ \mathcal{P}_0[\overline{\gamma(\pi_1) \cap \gamma(\pi_2)}]} K_{I_g}.$$

As $\overline{\gamma(\pi_1) \cap \gamma(\pi_2)} = \overline{\gamma(\pi_1)} \cup [\gamma(\pi_1) \setminus \gamma(\pi_2)]$, then taking into account $(**)$, we'll obtain

$$\mathcal{P}_0[\overline{\gamma(\pi_1) \cap \gamma(\pi_2)}] = \mathcal{P}_0[\overline{\gamma(\pi_1)}] \cup \mathcal{P}[\gamma(\pi_1) \setminus \gamma(\pi_2)] \cup \{\mathcal{P}[\overline{\gamma(\pi_1)}] \# \mathcal{P}[\gamma(\pi_1) \setminus \gamma(\pi_2)]\}.$$

The correlation $(***)$ follows from latter.$\square$

We'll call a problem Z *acyclic*, if

$$S(\pi_1 \cup \pi_2) = S(\pi_1) \cup [S(\pi_2) \setminus S(\pi_1)] = S(\pi_1) \cup S(\pi_2).$$

Otherwise we'll call a problem Z *cyclic*.

Theorem 19. *Following assertions are equivalent:*

1. the problem Z is cyclic;

2. there are the calling sets I_1, I_2, I_3 of constituent from $H(Z)$ such that $I_q \cap I_l \neq \oslash$ $(q \neq l;\ q, l \in \{1, 2, 3\})$ and $I_1 \cap I_2 \cap I_3 = \oslash$;

3. there are constituents $K_{I_1}, K_{I_2}, K_{I_3} \in H(Z)$ such that $K_{I_q} \cup K_{I_l} \in Q$ $(q \neq l;\ l \in \{1, 2, 3\})$ and $K_{I_1} \cup K_{I_2} \cup K_{I_3} \notin Q$.

Let Z is a cyclic problem. Then, according to definition, there are such solutions $\pi_1, \pi_2 \in Q$, that $\pi_1 \cup \pi_2 \in Q$ and the set $S(\pi_1, \pi_2) \neq \oslash$ in the correlation $(***)$. Therefore, there is at least one constituent $K_{I_3} \in H(Z)$ such that $K_{I_3} \subset S(\pi_1, \pi_2)$. As $K_{I_3} \subset S(\pi_1)$ and $K_{I_3} \subset S(\pi_2)$ then, according to Consequence 7, correlation are fulfilled: $I_3 \cap \gamma(\pi_1) \neq \oslash$ and $I_3 \cap \gamma(\pi_2) \neq \oslash$, i.e. the set I_3 has at least one common element with calling sets of constituents, elements of which belong to the solution π_1 and solution π_2 as well. Hence, for any $r_1 \in K_{I_1}, r_2 \in K_{I_2}$ $(I_1 \neq I_2)$ such that $r_1 \in \pi_1$, $r_2 \in \pi_2$ we'll have:

$$I_1 \cap I_2 \neq \oslash;\ I_1 \cap I_3 \neq \oslash; I_2 \cap I_3 \neq \oslash.$$

However, from correlation $(***)$ follows, that

$$I_3 \cap \gamma(\pi_1 \cup \pi_2) = I_3 \cap \gamma(\pi_1) \cap \gamma(\pi_2) = \oslash.$$

Obviously, that it will be if and only if two such elements $r_1 \in K_{I_1}, r_2 \in K_{I_2}$ $(I_1 \neq I_2)$ will found, that $r_1 \in \pi_1$, $r_2 \in \pi_2$ and $I_1 \cap I_2 \cap I_3 \neq \oslash$.

The implication $(1) \rightarrow (2)$ is proved.

The implication $(2) \rightarrow (3)$ follows from definition of a constituent and Theorem 6.

The implication $(3) \rightarrow (1)$ follows from definition of a cyclic problem.□

Consequence 20. *The problem Z is cyclic iff in SSM $\| \alpha_{ij} \|$ $(i = \overline{1,n};\ j = \overline{1,p})$ of the problem will be found rows $i_1, i_2, i_3 \in \{\overline{1,n}\}$ $(i_k \neq i_l;\ k,l \in \{1,2,3\})$ and columns $j_1, j_2, j_3 \in \{\overline{1,p}\}$ $(j_k \neq j_l;$ $k,l \in \{1,2,3\})$ such that respective square submatrix in each row and each column has only one element equal to zero, and matrix $\| \alpha_{ij} \|$ have no column $j \in \{\overline{1,p}\} \setminus \{j_1, j_2, j_3\}$ such, that $\alpha_{i_1 j} = \alpha_{i_2 j} = \alpha_{i_3 j} = 1$.*

Any population of three elements $r_1, r_2, r_3 \in W$ of the problems Z such that $r_1 \in K_{I_1}, r_2 \in K_{I_2}$ and $r_3 \in K_{I_3}$ and constituent $K_{I_1}, K_{I_2}, K_{I_3}$ satisfy (3) of Theorem 19 we'll call *cycle*.

Obviously, that the problem Z, which represented on the fig. 1 (a), is acyclic, and the problem Z, which represented on the fig. 1 (b), is cyclic, since elements $r_2, r_3, r_5 \in W$ will form the cycle.

Example 5.

Typical example of a cyclic problem is, obviously, covering problem, and acyclic – problem about finding of the maximum independent set of the graph vertices. It is possible to show, that if CM of a covering problem in each column contains two and only two units, then problem dual to it is acyclic (in this case, given CM can be interpreted as a incidence matrix of undirected graph) [3].

10 Canonical form of ECP

Let there be ECP $Z = (W, Q, M, f) \in NP$.

L-bunch of constituent $K_{I_1} \in H(Z)$ (in problem Z) we'll call set

$$L(K_{I_1}) = \bigcup_{\forall\ I_q\ \supset\ I_1} K_{I_q}.$$

P-bunch of constituent $K_{I_1} \in H(Z)$ (in problem Z) we'll call set

$$P(K_{I_1}) = \bigcup_{\forall\ I_q\ \subset\ I_1} K_{I_q}.$$

Clearly, that $K_{I_1} \subset L(K_{I_1})$ and $K_{I_1} \subset P(K_{I_1})$.

It is not difficult to see, that $L(K_{I_1}) \in Q$ and $P(K_{I_1}) \in Q$.

Theorem 21. *If $\pi \in Q$ is the support solution of a problem Z and $K_{I_1} \subset \pi$ then $L(K_{I_1}) \subset \pi$.*

Really, due to Consequence 10 we have $S(K_{I_q}) \subset S(K_{I_1})$, if $I_q \supset I_1$. Then, taking into account $(***)$, we'll obtain:

$$S[L(K_{I_1}) = S(K_{I_1}).$$

According to condition of Theorem 21 $K_{I_1} \subset \pi$ and consequently $\gamma(\pi) \cap I_1 \neq \oslash$. However from definition L-bunch $L(K_{I_1})$ follows, that $I_1 \cap I_q = I_1$ for all $I_q \supset I_1$. Then we have $K_{I_q} \subset S(\pi)$, i.e. $\gamma(\pi) \cap I_q \neq \oslash$. Taking into account, that π is support solution, i.e. $\pi \cup S(\pi) = W$, we'll obtain: $K_{I_q} \subset \pi$ for all $I_q \supset I_1$. Consequently, $L(K_{I_1}) \subset \pi$, that must be proven.□

Theorem 22. *If π is a support solution of the problem Z and $K_{I_1} \not\subset \pi$ then $P(K_{I_1}) \not\subset \pi$.*

The proof of this Theorem is similar to the proof of Theorem 21.□

Problem $Z = (W, Q, M, f) \in NP$ we'll call *canonical*, if for any its constituents $K_{I_1}, K_{I_2} \in H(Z)$ $(I_1 \neq I_2)$ the correlation take place: $I_1 \not\subset I_2$.

Easily to see, that the solution of any problem $Z = (W, Q, M, f) \in NP$ can be reduced to solution of an respective canonical problem $Z_c = (W_c, Q_c, M_c, f_c)$ and such, that optimum solution of it is also optimum solution of the problem Z.

Such reduction of a problem can be carried out by fulfillment of a standard procedure, which we'll call *canonization* of a problems $Z \in NP$, and which consists in following.

For each element $r \in W$ of a normalized problem $Z = (W, Q, M, f)$ to find L-bunch $L(r)$ and to construct a new problem $Z_1 = (W_1, Q_1, M_1, f_1)$. L-bunches, which have been found, must correspond to elements of problem Z_1, i.e. $r'_i \in W$, if $r'_i \leftrightarrow L(r_i)$ $(i = \overline{1, n})$, and each subset $\{r'_{i_1}, \ldots, r'_{i_t}\} \in Q$, when $\bigcup L(r_{i_q}) \in Q$. Besides, we believe $\mu(r'_i) = \mu[L(r_i)]$. Execution of one step of canonization finished by construction of the problem Z_1. It is clear, that as result we will not lose neither support solution of the problem Z.

Canonization of the problem Z is finished after construction of the problem Z_c, in which L-bunch of each element of working set consists only from one (considered) element. It must be noted, that in common case as a result of canonization of the problem Z, the resulting problem Z_c can have as support solution such one, which corresponds to admissible (not support) solution of the problem Z.

The technique of execution of a step canonization depends from form of the initial specification of a problem Z.

Example 6.

The simplest problem is canonization of problem of finding of the maximum independent set of graph $G = (W, S)$ vertices.

L-bunch of vertex $r \in W$ contains those vertices of graph G for which the adjacency list is a subset of the adjacency list of vertex r.

The graph $G_1 = (W_1, S_1)$, which obtained as a result of canonization step, has equal number of vertices with initial graph $G = (W, S)$, and, the subset of vertices graph G, forming L-bunch of vertex $r_i \in W$ $(i = \overline{1, n})$ put in correspondences to any vertex $r'_i \in W_1$, i.e.

$$r'_i \leftrightarrow \bigcup_{\forall\, r_q:\ S(\{r_q\}) \,\subset\, S(\{r_i\})} \{r_q\},$$

$$\mu(r'_i) = \sum_{\forall\, r_q:\ S(\{r_q\}) \,\subset\, S(\{r_i\})} \mu(r_q)$$

and $r'_k \in S_1(r'_i)$, if $r_k \in S(\{r_i\})$ or $S(\{r_k\}) \subset S(\{r_i\})$.

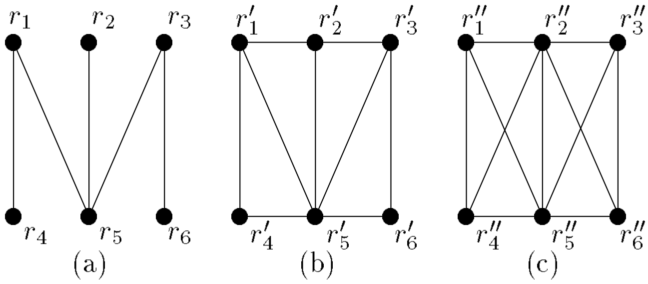


Figure 7: Procedure canonization of the graph

Normalized graph is represented on the fig. 7 (a) and graphs, obtained after execution of two steps of normalized graph canonization, on the fig. 7 (b) and 7 (c).

Easily to see, that

$$r'_1 \leftrightarrow \{r_1, r_2\},$$

$$r'_2 \leftrightarrow \{r_2\},$$

$$r'_3 \leftrightarrow \{r_2, r_3\},$$

$$r'_4 \leftrightarrow \{r_4\},$$

$$r'_5 \leftrightarrow \{r_4, r_5, r_6\}$$

$$r'_6 \leftrightarrow \{r_6\},$$

further:

$$r''_1 \leftrightarrow \{r'_1\} \leftrightarrow \{r_1, r_2\},$$

$$r''_2 \leftrightarrow \{r'_2, r'_4, r'_6\} \leftrightarrow \{r_2, r_4, r_6\},$$

$$r''_3 \leftrightarrow \{r'_3\} \leftrightarrow \{r_2, r_3\},$$

$$r''_4 \leftrightarrow \{r'_4\} \leftrightarrow \{r_4\},$$

$$r''_5 \leftrightarrow \{r'_5\} \leftrightarrow \{r_4, r_5, r_6\},$$

$$r''_6 \leftrightarrow \{r'_6\} \leftrightarrow \{r_6\}.$$

Assuming, that $\mu(r_i) = 1$ for all $i = \overline{1, 6}$, obviously, we'll have: $\mu(r''_1) = 2$, $\mu(r''_2) = 3$, $mu(r''_3) = 1$, $\mu(r''_4) = 1$, $\mu(r''_5) = 3$, $\mu(r''_6) = 1$.

References

[1] Aho A.V., Hopcroft J.E., Ullman J.D. The design and analysis of computer algorithms. Addison-Wesley Publishing Company, Reading, Massachusetts ◦ Menlo Park, California ◦ London ◦ Amsterdam ◦ Don Mills, Ontario ◦ Sydney, 1976.

[2] Birkhoff G., Bartee T.C. Modern applied algebra. McGraw-Hill book company, New York ◦ St. Louis ◦ San Francisco ◦ Dusseldorf ◦ London ◦ Mexico ◦ Panama ◦ Sydney ◦ Toronto.

[3] Christofides N. Graph theory (an algorithmic approach). Academic Press, New York ◦ London ◦ San Francisco, 1975.

[4] Ford L.R., Fulkerson D.R. Flows in networks. Princeton University Press, Princeton, New Jersey, 1962.

[5] Garey M.R., Johnson D.S. Computers and intractability (a guide to the theory of NP-completeness). W.H. Freeman and Company, San Francisco, 1979.

[6] Kaufmann A. Introduction a la combinatorique en vue des applications. Dunod, Paris, 1968.

[7] Kuratowski K., Mostowski A. Set theory. North-Holland publishing company, msterdam ∘ PWN-Polish scientific publishers, Warsawa, 1967.

[8] Minieka E. Optimization algorithms for networks and graphs. Marcel Dekker, Inc., New York and Basel, 1978.

[9] Papadimitriou C.H. Combinatorial optimization (algorithms and complexity). Prentice-Hall, Inc., New Jersey, 1982.

[10] Plotnikov A.D. On the problem of finding of an independent set of graph vertices.//Kibernetika. – 1989. – N1. – 119 – 121 (russian).

[11] Swamy M.N.S., Thulasiraman K. Graphs, Networks, and algorithms. A Wiley Interscience Publication, New York ∘ Chichester ∘ Brisbane ∘ Toronto.

[12] Reingold E.M., Nievergelt J., Deo N. Combinatorial algorithms (theory and practice). Prentice-Hall, Inc., New Jersey, 1977.

[13] Tarjan R.E. Complexity of combinatorial algorithms, SIAM, Rew., **20**, N 3, 1978, 457 – 491.