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WEAK CONDITIONS FOR THE CONVERGENCE OF ITERATIONS TO SOLUTIONS OF EQUATIONS ON PARTIALLY ORDERED TOPOLOGICAL SPACES

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Abstract. We provide sufficient conditions for the monotone convergence of Newton like methods to solutions of nonlinear operator equations on partially ordered topological spaces. We assume conditions weaker than standard Lipschitz-like conditions to produce two iterations, one increasing and one decreasing, that enclose and converge to a solution of an equation. Finally examples are given to show that our results apply where previous ones fail.

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I. INTRODUCTION

In this study we are concerned with approximating a solution of the nonlinear operator equation

$$F(x) = 0, \tag{1}$$

where F is an operator defined on a subset of partially ordered linear space $E_1 = (E_1, \leq)$ (POL-space) with values in a POL-space E_2 [1], [2], [5]–[8].

We suggest Newton-like iterations of the form

$$F(x_n) + A^1(x_{n-1}, x_n)(x_{n+1} - x_n) = 0 \tag{2}$$

$$F(y_n) + A^1(y_n, y_{n-1})(y_{n+1} - y_n) = 0 \tag{3}$$

for $[x_{-1}, y_{-1}] \subseteq D$, $x_{-1} \leq x_0, y_0 \leq y_{-1}$ and $A^1, A^2 \colon D^2 \to L(E_1, E_2)$ [2], [6]. Sufficient conditions for the monotone convergence of the sequences $\{x_n\}$, $\{y_n\}$

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($n \geq 1$) generated by (2) and (3) respectively to a solution of equation (1) have been given by various authors based mainly on the condition

$$F(y) - F(x) \leq A(w,z)(y-x) \quad \text{for } x \leq y \leq w \leq z, \tag{4}$$

$[x,z] \subseteq D$ and $A: D \to L(E_1, E_2)$ being linear operator having continuous nonsingular nonnegative left subinverse (see [2], [5], [6] and the references there). There are examples even in the scalar case (see the example after Theorem 1) where iterations $\{x_n\}$, $\{y_n\}$ converge to a solution of equation (1) but the sufficient conditions for the convergence fail to apply. That is why it is important to try to find weaker conditions under which these iterations converge to a solution. This is what we have done in this study.

Finally we note that for $A^1(x,y) = A^2(x,y) = F'(x)$ for all $x, y \in D$, where $F'(x)$ is the Fréchet-derivative of F we obtain Newton's method [3], [4], whereas for $A^1(x,y) = A^2(x,y) = [x,y;F]$ for all $x,y \in D$, where $[x,y;F]$ is the divided difference of order one of F at the points $x, y \in D$ we obtain the secant method [3], [4]. Several other choices for the operators A^1 and A^2 are also possible [2].

Theorem 1. *Consider an operator $F: D \subseteq E_1 \to E_2$ where E_1 is a regular POTL-space, E_2 is a POTL-space and D is an open convex subset of E_1. Let x_{-1}, x_0, y_{-1}, y_0 be points of D such that*

$$x_{-1} \leq x_0 \leq y_0 \leq y_{-1}, \quad [x_{-1}, y_{-1}] \subseteq D. \tag{5}$$

Assume:

(a) *there exists points $z_1, z_2 \in [x_0, y_0]$ such that*

$$z_1 \leq z_2 \quad \text{and} \quad F(z_1) = F(z_2) = 0.$$

(b) *$F(x) \leq 0$ for all $x \in [x_0, z_1]$ and $F(x) \geq 0$ for all $x \in [z_2, y_0]$.*

(c) *there exist operators $A^1, A^2: D^2 \to L(E_1, E_2)$ such that*

$$F(z_1) - F(x) \leq A^1(y,x)(z_1 - x) \quad \text{for all } y, x \text{ with } x_0 \leq y \leq x \leq z_1 \tag{6}$$

and

$$F(v) - F(z_2) \leq A^2(v,w)(v - z_2) \quad \text{for all } v, w \text{ with } z_2 \leq v \leq w \leq y_0. \tag{7}$$

(d) *The linear operators $A^1(y,x)$ and $A^2(v,w)$ have continuous nonsingular nonnegative left subinverses on $[x_{-1},\ y_{-1}]$.*

Then there exist two sequences $\{x_n\}$, $\{y_n\}$ generated by (2) and (3) respectively ($n \geq 1$) and two points x^*, y^* of E_1 such that for all $n \geq 0$

$$F(x_n) \leq 0 \leq F(y_n), \tag{8}$$

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} \leq z_1 \leq z_2 \leq y_{n+1} \leq y_n \leq \cdots \leq y_1 \leq y_0, \tag{9}$$

(10) $$\lim_{n\to\infty} x_n = x^*, \quad \lim_{n\to\infty} y_n = y^*, \quad x^* \le z_1 \le z_2 \le y^*,$$

and

(11) $$x^* = y^* \Rightarrow x^* = z_1 = z_2 = y^*.$$

Moreover assume there exist operators $A^3, A^4: D^2 \to L(E_1, E_2)$ with nonnegative left superinverses satisfying

(12) $$F(z_1) - F(x) \ge A^3(x,y)(z_1 - x) \quad \textit{for all } x_0 \le x \le y \le z_1,$$

(13) $$F(v) - F(z_2) \ge A^4(w,v)(v - z_2) \quad \textit{for all } z_2 \le w \le v \le y_0,$$

$F(x^*) = F(z_1)$ and $F(y^*) = F(z_2)$.
Then we deduce

$$x^* = z_1 \quad \textit{and} \quad y^* = z_2.$$

Furthermore if $z_1 = z_2$ we get $x^* = y^* = z_1 = z_2$.

Proof. Let B_0^1 be a continuous nonsingular left subinverse of $A_0^1 = A^1(x_{-1},\ x_0)$ and consider the operator $P_1: [0, z_1 - x_0] \to E_1$ defined by

$$P_1(x) = x - B_0^1(F(x_0) + A_0^1(x)).$$

It is easy to see that P_1 is isotone and continuous. We also have

$$\begin{aligned} P_1(0) &= -B_0^1(F(x_0)) \ge 0 \\ P_1(z_1 - x_0) &= z_1 - x_0 - B_0^1(F(z_1)) + B_0(F(z_1) - F(x_0) - A_0^1(z, -x_0)) \\ &\le z_1 - x_0 - B_0^1(F(z_1)) \le z_1 - x_0 \end{aligned}$$

by hypotheses (a), (b) and (c).

According to Kantorovich's theorem on fixed points [2], [5], [8] the operator P_1 has a fixed point $q_1 \in [0, z_1 - x_0]$. Set $x_1 = x_0 + q_1$ then

$$F(x_0) + A_0^1(x_1 - x_0) = 0, \quad x_0 \le x_1 \le z_1.$$

By hypothesis (a) we also get

$$F(x_1) \le 0.$$

Define the operator $P_2: [0, y_0 - z_2] \to E_1$ by

$$P_2(x) = x + B_0^2(F(y_0) - A_0^2(x))$$

where B_0^2 is a continuous nonsingular nonnegative left subinverse of $A_0^2 = A^2(y_0, y_{-1})$. P_2 is clearly continuous, isotone with

$$P_2(0) = B_0^2(F(y_0)) \ge 0,$$

and

$$P_2(y_0 - z_2) = y_0 - z_2 + B_0^2(F(z_2)) + B_0^2(F(y_0) - F(z_2) - A_0^2(y_0 - z_2))$$
$$\leq y_0 - z_2 + B_0^2(F(z_2)) \leq y_0 - z_2$$

by hypotheses (a), (b) and (c).

As above we define the existence of a point $q_2 \in [0, y_0 - z_2]$ such that $P_2(q_2) = q_2$. Let $y_1 = y_0 - q_2$. It follows that

$$F(y_0) + A_0^2(y_1 - y_0) = 0, \quad z_2 \leq y_1 \leq y_0.$$

By hypothesis (b) we also have

$$F(y_1) \leq 0.$$

Using induction on $n \geq 0$ we can show that there exists two sequences $\{x_n\}$, $\{y_n\}$ $(n \geq 0)$ satisfying (2), (3), (8), (9). Since E_1 is regular from (9) it follows that there exist $x^*, y^* \in E_1$ such that $x^* = \lim_{n\to\infty} x_n$ and $y^* = \lim_{n\to\infty} y_n$. We obviously also have $x^* \leq z_1 \leq z_2 \leq y^*$.

Denote by $A^5(x^*, z_1)$ a nonnegative left superinverse of $A^3(x^*, z_1)$. Then by (12) we obtain

$$0 \leq z_1 - x^* \leq A^5(x^*, z_1)A^3(x^*, z_1)(z_1 - x^*) \leq A^5(x^*, z_1)(F(z_1) - F(x^*)) = 0$$

which implies $x^* = z_1$. Similarly by (13) we obtain $y^* = z_2$. By $z_1 = z_2$ we deduce $x^* = y^*$. Finally if $x^* = y^*$, (10) gives $z_1 = z_2$.

That completes the proof of the theorem.

Example. Let $E_1 = E_2 = R$, choose any $x_{-1} \leq 0$ and $y_{-1} \geq 0$ and set $D = [x_{-1}, y_{-1}]$. Define the function F on D by

$$F(x) = \begin{cases} x & x \leq 0 \\ 2x, & x \geq 0. \end{cases} \tag{14}$$

The only solution z of equation $F(x) = 0$ is $z = 0$. Let $z_1 = z_2 = z$. Choose also $A^1 = 2$ and $A^2 = 3$. Then hypotheses (a)–(d) of Theorem 1 are satisfied and iterations (2) and (3) give respectively

$$x_{n+1} = \left(\frac{1}{2}\right)^{n+1} x_0 \tag{15}$$

and

$$y_{n+1} = \left(\frac{1}{3}\right)^{n+1} y_0 \quad (n \geq 0) \tag{16}$$

which converge to 0 as $n \to \infty$ for any $x_0 \leq 0$ and $y_0 \geq 0$. However theorems on the convergence of Newton's method fail to apply since $F'(0)$ does not exist [2], [3], [4]. We note though that Newton's method converges to 0 since $x_1 = 0$ for any x_0. Similar observations can be made for the secant method by setting $A^1(y, x) = \frac{F(x)-F(y)}{x-y}$ and $A^2(v, w) = \frac{F(w)-F(v)}{w-v}$. By (14), approximations (2) and (3) become (15) and (16) respectively in this case which converge to 0. But theorems on the convergence of the Secant method require that $F'(0)$ exists [2], [3], [4], [6], [8]. It is also simple calculus to show that condition (4) is not satisfied for both Newton's and the Secant method.

Remark 1. We note that condition (4) implies (6) and (7) but not vice versa. (See the above example.) That is (6) and (7) are weaker conditions than (4).

The regularity of the space E_1 is a rather restrictive condition. It was used to prove that sequences $\{x_n\}$, $\{y_n\}$ $(n \geq 0)$ are well defined and convergent. To avoid this regularity condition we consider the explicit version of it:

$$x_{n+1} = x_n - B_n^1(F(x_n)) \quad (n \geq 0) \tag{17}$$

$$y_{n+1} = y_n - B_n^2(F(y_n)) \quad (n \geq 0) \tag{18}$$

where B_n^1, B_n^2 are nonnegative subinverses of A_n^1 and A_n^2 respectively.

We can now state and prove the following result:

Theorem 2. *Let $F\colon D \subseteq E_1 \to E_2$ be an operator between two POL-spaces and D be an open convex subset of E_1. Let x_{-1}, x_0, y_{-1}, y_0 be points of D satisfying* (5). *Assume that hypotheses* (a)–(c) *of Theorem* 1 *are true and linear operators $A^1(y, x)$ and $A^2(v, w)$ have nonnegative subinverses on $[x_{-1}, y_{-1}]$.*

Then approximations (17) *and* (18) *where B_n^1 and B_n^2 are nonnegative subinverses of A_n^1 and A_n^2 respectively define two sequences $\{x_n\}$, $\{y_n\}$ $(n \geq 0)$ having the properties* (8) *and* (9).

Moreover if $\lim_{n\to\infty} x_n = x^* = y^* = \lim_{n\to\infty} y_n$ then the following are true

$$x^* = z_1 = z_2 = y^* \quad \textit{and} \quad F(x^*) = F(y^*) = 0.$$

Furthermore, if F is continuous at x^* and y^* and $F(x^*) = F(y^*)$ we deduce $F(x^*) = F(y^*) = 0$.

Proof. For $n = 0$ we obtain in turn from approximations (17) and (18)

$$x_1 - x_0 = -B_0^1(F(x_0)) \geq 0,$$
$$z_1 - x_1 = z_1 - x_0 + B_0^1 F(x_0) = (z_1 - x_0) - B_0^1(F(z_1) - F(x_0)) \geq 0 \quad \text{by (6)}$$

and

$$F(x_1) \leq 0 \quad \text{by (b)},$$
$$y_0 - y_1 = B_0^2(F(y_0)) \geq 0$$
$$y_1 - z_2 = y_0 - B_0^2(F(y_0)) - z_2 = y_0 - z_2 - B_0^2(F(y_0) - F(z_2)) \geq 0 \quad \text{by (7)}$$

and

$$F(y_1) \geq 0 \quad \text{by (6)}.$$

Thus, we have proved $x_0 \leq x_1 \leq z_1 \leq z_2 \leq y_1 \leq y_0$. Proceeding by induction on $n \geq 0$ we obtain (8) and (9). If $x^* = \lim_{n\to\infty} x_n$, $y^* = \lim_{n\to\infty} y_n$ and $x^* = y^*$ it follows that $x^* = z_1 = z_2 = y^*$ and since $F(z_1) = F(z_2) = 0$ we get $F(x^*) = F(y^*) = 0$.

Finally if F is continuous at x^* and y^* by letting $n \to \infty$ in (8) we get $F(x^*) \leq 0 \leq F(y^*)$ and since $F(x^*) = F(y^*)$ we deduce $F(x^*) = F(y^*) = 0$.

That completes the proof of the theorem.

Remark 2. Several natural conditions (other than the ones in Theorems 1 and 2) under which x^* and y^* are solutions of equation (1) as well as other variations of approximations (2)–(3) and (17)–(18) can be found in [2, Ch. 6].

Remark 3. The conclusions of Theorems 1 and 2 remain true if estimates (6), (7), (12) and (13) are valid on $[x_0, y_0]$, since $x_n \leq z_1$ and $z_2 \leq y_n$ $(n \geq 0)$.

The following theorem is also available on fixed points.

Theorem 3. *Let $P\colon D \subseteq E_1 \to E_1$ where E_1 is a regular POTL-space and D is a subset of E_1. Define the iterations*

$$x_{n+1} = P(x_n) \quad (n \geq 0)$$

and

$$y_{n+1} = P(y_n) \quad (n \geq 0)$$

for some $[x_0, y_0] \in D$. Assume:

(a) *there exists $z \in D$ such that*

$$x_0 \leq P(x_0) \leq z \leq P(y_0) \leq y_0;$$

(b) *$P(x) \leq z$ for $x \leq z$, $x \in D$ and $P(x) \geq z$ for $x \geq z$, $x \in D$;*

(c) *$P(x) \leq P(y)$ for $x \leq y$, $x, y \in D$ and $P(x) \geq P(y)$.*

Then there exist two points $x^*, y^* \in D$ such that

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} \leq x^* \leq z \leq y^* \leq y_{n+1} \leq y_n \leq \cdots \leq y_1 \leq y_0, \tag{18}$$

$$\lim_{n\to\infty} x_n = x^* \quad \textit{and} \quad \lim_{n\to\infty} y_n = y^*. \tag{19}$$

Proof. By hypothesis $x_0 \leq x_1 = P(x_0) \leq z$, $y_{-1} \geq y_1 = P(y_0) \geq z$. Assume $x_{k-1} \leq x_k \leq z$ and $y_{n-1} \geq y_k \geq z$ for $k = 1, 2, \ldots, n$. Then $x_{k+1} = P(x_k) \leq z$, $y_{k+1} = P(y_k) \geq z$, $x_{k+1} - x_k = P(x_k) - P(x_{k-1}) \geq 0$ and $y_{k+1} - y_k = P(y_k) - P(y_{k-1}) \leq 0$. Moreover since E_1 is regular there exist x^*, y^* such that (18) and (19) are satisfied.

That completes the proof of the Theorem.

Remark 4. (a) If $x^* = y^*$ then $x^* = z = y^*$. Hence if z is a fixed point of P on $[x_0, y_0]$ then $P(x^*) = x^* = y^* = P(y^*)$.

(b) If z is the unique fixed point of P on $[x_0, y_0]$ and P is continuous at x^* and y^* then since $P(x^*) = x^*$ and $P(y^*) = y^*$ we get $x^* = z = y^*$.

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