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SUFFICIENT CONDITIONS FOR THE CONVERGENCE OF ITERATIONS TO POINTS OF ATTRACTION IN BANACH SPACE

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Abstract. In this study we provide weak sufficient conditions for the convergence of iterations to points of attraction in Banach space. Our results are used to show existence and uniqueness of solutions of nonlinear equations. We complete our study with an example that shows that our convergence results apply where earlier ones do not.

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I. INTRODUCTION

In this study, we are concerned with approximating a locally unique solution x^* of the nonlinear equation

$$F(x,h)=0 \quad \text{for all} \quad h\in E_0, \tag{1}$$

where $F\colon D\times E_0\subseteq E^{m+1}\to E$ is a continuous operator, E is a Banach space, D, E_0 are subsets of E and E^m respectively and $m\in N\cup\{0\}$.

We introduce the iteration

$$x_{n+1}=G(x_n,h_n) \quad (n\geq 0), \tag{2}$$

to approximate the solution x^* of equation (1), where

$$G(x,h)=x-F(x,h) \quad \text{for all} \quad (x,h)\in D\times E_0. \tag{3}$$

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For $m = 0$ equation (1) does not depend on h, setting $F(x,h) = P'(x)^{-1}P(x)$ for all $x \in D$, and using (2) and (3), we obtain Newton's method

$$x_{n+1} = x_n - P'(x_n)^{-1}P(x_n) \quad (n \geq 0), \tag{4}$$

where $P: D \to E$, $P'(x_n) \in L(E,E)$ denotes the Fréchet-derivative of P evaluated at $x = x_n$ $(n \geq 0)$ [1], [3], [4].

For $m = 1$ set $F(x,h) = [h,x;Q]^{-1}Q(x)$ for all $(x,h) \in D \times E_0$, where $Q: D \to E$, $[h,x;Q]$ is a divided difference of order one on E satisfying

$$[h,x;Q](x-h) = Q(x) - Q(h) \quad \text{for all} \quad (x,h) \in D \times E_0. \tag{5}$$

Then for $h_n = x_{n-1}$ $(n \geq 0)$ we obtain the secant method

$$x_{n+1} = x_n - [x_{n-1}, x_n; Q]^{-1}Q(x_n) \quad (n \geq 0), \tag{6}$$

[1], [3], [4]. Many other choices for the operator F are also possible. It follows immediately from (1) and (3) that any solution x^* of equation (1) is a fixed point of equation (3) and vice versa. If for some $x_0 \in D$ and any $h_n \in E^m$ iteration $\{x_n\}$ $(n \geq 0)$ is well defined and converges to some point $y^* \in D$ then y^* is called a point of attraction for this iteration. In this case by taking the limit as $n \to \infty$ in (2) we deduce

$$y^* = G(y^*, h) \quad \text{for all} \quad h \in E_0. \tag{7}$$

That is y^* is a fixed point of the operator $G(\cdot, h)$ for all $h \in E_0$.

The study of equations of the form (1) is very important since many real life problems can be reduced to solving such equations [1], [3], [4], via iterations of the form (2).

Sufficient conditions for the convergence of (2) for $m = 0$ (or not) have been given by various authors (see, e.g., [1]–[4] and the references there). Even under existing conditions one can construct examples where, say Newton's method (4) converges despite the fact that the conditions of existing theorems are violated. See Remark 2 for such an example. This important observation creates the need of the development of weaker sufficient conditions for the convergence of (2) than the ones already in the literature. This is what we have done in this study.

Finally we conclude our study with an example to show that our convergence results apply where previous ones fail.

II. CONVERGENCE ANALYSIS

Definition. Let $(E, \|\ \|)$ be a metric space. An operator $G: E^{m+1} \to E$ is called a generalized contractive operator if there exist $z \in E$ and a number $c \in [0,1)$ such that

$$\|G(x,h) - G(z,h)\| \leq c\|x - z\| \quad \text{for all} \quad x \in E,\ h \in E_0. \tag{8}$$

If z is such that $z = G(z,h)$ for all $h \in E$ and G satisfies (8) for $c = 1$, then G is called a generalized quasinonexpansive operator. We note that if $P: E \to E$ is such that $P(x) = G(x,h)$ for all $x \in E$, $h \in E_0$ then P is called a quasicontractive operator if (8) is satisfied [1], [3], [4].

Theorem 1. *Let* $(E, \| \ \|)$ *be a Banach space and* $G: E^{m+1} \to E$ *be a generalized quasicontractive operator with* x^* *such that* $G(x^*, h) = x^*$ *for all* $h \in E_0$.

Then x^* *is the unique point in* E *such that* $G(x^*, h) = x^*$ *for all* $h \in E_0$ *and the iteration* $\{x_n\}$ $(n \geq 0)$ *generated by* (2) *converges to* x^* *for each* $x_0 \in E$ *and any* $h_0 \in E_0$ *with*

$$\|x_n - x^*\| \leq c^n \|x_0 - x^*\| \tag{9}$$

Moreover $x_n \in U(x^*, r) = \{x \in E \mid \|x - x^*\| \leq r\}$ $(n \geq 0)$ *for* $r \in [0, \|x_0 - x^*\|]$.

Proof. By hypothesie $G(x^*, h) = h$ for all $h \in E_0$, we obtain from (8) that

$$\|G(x, h) - x^*\| \leq c\|x - x^*\| \quad \text{for all} \quad x \in E, \ h \in E_0. \tag{10}$$

Let $x_0 \in E$, $h_0 \in E_0$ be arbitrary, then by (8), we get

$$\|x_{n+1} - x^*\| = \|G(x_n, h_n) - x^*\| \leq c\|x_n - x^*\| \leq \cdots \leq c^{n+1}\|x_0 - x^*\|. \tag{11}$$

But $c \in [0, 1)$ hence (11) gives $\lim_{n\to\infty} x_{n+1} = x^*$, and (9) holds for all $n \geq 0$. It also follows from (11) that $x_n \in U(x^*, r)$ for $r \in [0, \|x_0 - x^*\|]$. To show uniqueness let us assume that there exists z with $x \neq x^*$ and $G(z, h) = z$ for all $h \in E_0$. Then $\|x^* - z\| \neq 0$ and (8) gives $\|x^* - z\| = \|G(x^*, h) - G(z, h)\| \leq c\|x^* - z\|$ contradicting $c \in [0, 1)$.

That completes the proof of the theorem.

Remark 1. If E is a compact metric space then Theorem 1 remains true if we replace generalized quasicontractive by generalized quasinonexpansive. See also Edelstein's fixed point theorem [1], [3].

From now on we assume that E is a partially ordered Banach space [1], [3], [4], $D = [a, b]$, $a \leq b$ and $E_0 = E^m$.

Theorem 2. *Let* $(E, \leq)$ *be a partially ordered Banach space. Assume there exist:*

(a) *continuous operators* $F_1: E_1 \times E_0 \to E$, $F: [a, b] \times E_0 \to E$, *and* $F_2: E_2 \times E_0 \to E$ *with* $a \leq b$ *fixed in* E *satisfying for all* $h \in E_0$

$$x - F_1(x, h) \geq a \tag{12}$$

and

$$x - F_2(x, h) \leq b, \tag{13}$$

where

$$E_1 = \{x \in E \mid x \leq a\} \quad \textit{and} \quad E_2 = \{x \in E \mid b \leq x\};$$

(b) *a unique point* $x^* \in [a, b]$ *such that* $F(x^*, h) = 0$ *for all* $h \in E_0$;

(c) *an operator* $T \in L(E^{m+2}, E)$ *such that*

$$F(x, h) = T(x, x^*, h)(x - x^*) \quad \textit{for all} \quad x \in [a, b], \ h \in E_0; \tag{14}$$

and

(d) $d \in [0, 1)$, *where*

$$d = \sup_{\substack{x \in [a,b] \\ h \in E_0}} \|I - T(x, x^*, h)\|. \tag{15}$$

Then the iteration $\{x_n\}$ $(n \geq 0)$ *generated by* (2) *is well defined, remains in* $U(x^*, r_0)$ *for* $r_0 \in [0, \|x_0 - x^*\|]$ *and converges to* x^* *with* $G(x^*, h) = x^*$ *for all* $h \in E_0$ *and*

$$\|x_n - x^*\| \leq d^n \|x_0 - x^*\| \quad (n \geq 0). \tag{16}$$

Proof. We define operators $F, G\colon E \times E_0 \to E$ by

$$F(x,h) = \begin{cases} F_1(x,h) & x \leq a \\ F(x,h) & a \leq x \leq b \\ F_2(x,h) & b \leq x \end{cases} \tag{17}$$

That is, we use the same notation for the new F and

$$G(x,h) = x - F(x,h). \tag{18}$$

It follows immediately from hypothesis (b) and (18) that (7) is satisfied for $y^* = x^*$. Moreover from hypotheses (c), (d) and (14) we get

$$G(x,h) - x^* = x - x^* - F(x,h) = (I - T(x, x^*, h))(x - x^*),$$

and

$$\|G(x,h) - x^*\| \leq d\|x - x^*\| \tag{19}$$

for all $x \in [a,b]$ and $h \in E_0$. From the proof of Theorem 1 it follows that $\lim_{n\to\infty} x_n = x^*$ for each $x_0 \in [a,b]$ and all $h \in E_0$.

If some iterate $x_k \notin [a,b]$ then $x_k \in E - [a,b]$. It then follows from (12), (13) and (2) that

$$x_{k+1} = G(x_k, h_k) = x_k - F_1(x_k, h_k) \geq a \;\text{ for } x_k \leq a$$

or

$$x_{k+1} = G(x_k, h_k) = x_k - F_2(x_k, h_k) \leq b \;\text{ for } x_k \geq b.$$

Since iteration $\{x_n\}$ $(n \geq 0)$ converges to x^* and $a \leq x^* \leq b$ there exists $v \geq 0$, $v \in N$ such that

$$x_{n+1} = G(x_n, h_n) = G^{n+1}(x_0, h) \in [a,b] \quad \text{for } n \geq v.$$

That is $G^{v+1}([a,b] \times E_0) \subseteq [a,b] \times E_0$. Furthermore by (19) we have

$$\|G^n(x,h) - x^*\| \leq d^n \|x - x^*\| \quad \text{for all} \quad x \in [a,b],\ h \in E_0. \tag{20}$$

It follows that operator G^{v+1} satisfies all requirements of Theorem 1. Thus x^* is a fixed point of G^{v+1}, hence a fixed point of G and $\lim_{n\to\infty} x_n = x^*$ for each $x_0 \in [a,b]$. The rest of the conclusions of the theorem follow immediately from (20). We also note that by reordering iteration $\{x_n\}$ $(n \geq 0)$ we can take $v = 0$.

That completes the proof of the Theorem.

Remark 2. For Newton's method (4) (i.e., $m = 0$) operators F_1 and F_2 appearing in Theorem 2 can be defined by

$$F_1(x, h) = P'(a)(x - a) + P(a) \quad \text{for all} \quad x \le a, \ h \in E_0,$$

and

$$F_2(x, h) = P'(b)(x - b) + P(b) \quad \text{for all} \quad b \le x, \ h \in E_0,$$

assuming that P is Fréchet-differentiable on $U(x^*, r)$, $x^* \in [a, b]$, $r \ge 0$ fixed and $P(x^*) = 0$. Convergence theorems for Newton's method have been given by various authors using the in general expansive information $(x_0, P(x_0), P'(x_0), P''(x_0))$ or the more relaxed $(x_0, P(x_0), P'(x_0))$ [1]–[4]. We compare our Theorem 2 with corresponding results involving the latter information.

Let for simplicity $E = R$, then operator T can be defined by

$$T(x, x^*, h) = \frac{P'(x^* + t(x - x^*))}{P'(x)} \qquad t \in (0,1)\text{all } x \in [a, b].$$

Moreover define $P\colon [0,1] \to R$ by

$$P(x) = x^2 + 2x.$$

Let $a = 0$ and $b = 1$. Note that $x^* = 0$ is the only zero of equation $P(x) = 0$, on $[0, 1]$. Ostrowski's theorem on fixed points requires that $q = \max_{x \in [0,1]} |P'(0)| \in [0, 1)$ [1], [4] which is not true since $q = 2$. Newton's theorem on fixed points [3, pp. 128] (uses information on the second derivative also) or more recently Berinde's theorem [2, thm. 5, pp. 20] cannot apply because their common condition $f(a)f(b) < 0$ is violated since $f(0)f(1) = 0$. However it can easily be seen that all hypotheses of Theorem 2 are satisfied and $d = 1$ (see also Remark 1). Hence the conclusions for the scalar equation $P(x) = 0$ and Newton's method (4) follow. Finally for $x_0 = .1$ (4) gives in turn

$$x_1 = .0090909, \quad x_2 = .0000819, \quad x_3 = .000000006 \quad \text{and} \quad x_4 = 0.$$

Finally we note that Theorem 2 can be reduced to Berinde's Theorem 5 since hypotheses (a)–(d) will be true if replaced by the stronger (f_1), (f_2') and (f_3') of Berinde's and $E = R^k$, $k \in N$.

Remark 3. For the secant method (6) operators F_1 and F_2 appearing in Theorem 2 can be given by

$$F_1(x, h) = [h, a; Q](x - a) + Q(a) \quad \text{for all} \quad a \le x, \ h \in E_0$$

and

$$F_2(x, h) = [h, b; Q](x - b) + Q(b) \quad \text{for all} \quad b \le x, \ h \in E_0.$$

Several other choices are also possible.

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