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ABSOLUTELY CONTINUOUS MEASURES AND COMPACT COMPOSITION OPERATOR ON SPACES OF CAUCHY TRANSFORMS

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Abstract. The analytic self map of the unit disk $\mathbf{D}$, φ is said to induce a composition operator C_φ from the Banach space X to the Banach Space Y if $C_\varphi(f) = f \circ \varphi \in Y$ for all $f \in X$. For $z \in \mathbf{D}$ and $\alpha > 0$ the families of weighted Cauchy transforms F_α are defined by $f(z) = \int_{\mathbf{T}} K_x^\alpha(z) d\mu(x)$ where $\mu(x)$ is complex Borel measures, x belongs to the unit circle $\mathbf{T}$ and the kernel $K_x(z) = (1 - \overline{x}z)^{-1}$. In this paper we will explore the relationship between the compactness of the composition operator C_φ acting on F_α and the complex Borel measures $\mu(x)$.

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1. Background

Let $\mathbf{T}$ be the unit circle and $\mathbf{M}$ be the set of all complex–valued Borel measures on $\mathbf{T}$. For $\alpha > 0$ and $z \in \mathbf{D}$, we define the space of weighted Cauchy transforms F_α to be the family of all functions $f(z)$ such that

$$f(z) = \int_{\mathbf{T}} K_x^\alpha(z) d\mu(x) \tag{1}$$

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where the Cauchy kernel $K_x(z)$ is given by

$$K_x(z) = \frac{1}{1 - \overline{x}z}$$

and where μ in (1) varies over all measures in $\mathbf{M}$. The class F_α is a Banach space with respect to the norm

$$\|f\|_{F_\alpha} = \inf \|\mu\|_{\mathbf{M}} \tag{2}$$

where the infimum is taken over all Borel measures μ satisfying (1). $\|\mu\|$ denotes the total variation norm of μ. The family F_1 has been studied extensively in the soviet literature. The generalizations for $\alpha > 0$, were defined by T. H. MacGregor [8]. The Banach spaces F_α have been well studied in [5, 8, 3, 4]. Among the properties of F_α we list the following:

- $F_\alpha \subset F_\beta$ whenever $0 < \alpha < \beta$.
- F_α is Möbius invariant.
- $f \in F_\alpha$ if and only if $f' \in F_{1+\alpha}$ and $\|f'\|_{F_{1+\alpha}} \leq \alpha \|f\|_{F_\alpha}$.
- If $g \in F_{\alpha+1}$ then $f(z) = \int_0^z g(w)dw \in F_\alpha$ and $\|f\|_{F_\alpha} \leq \frac{2}{\alpha}\|g\|_{F_{1+\alpha}}$.

The space F_α may be identified with $\mathbf{M}/\overline{H_0^1}$ the quotient of the Banach space $\mathbf{M}$ of Borel measures by $\overline{H_0^1}$ the subspace of L^1 consisting of functions with mean value zero whose conjugate belongs the Hardy space H^1. Hence F_α is isometrically isomorphic to $\mathbf{M}/\overline{H_0^1}$. Furthermore, $\mathbf{M}$ admits a decomposition $\mathbf{M} = L^1 \oplus \mathbf{M}_s$, where $\mathbf{M}_s$ is the space of Borel measures which are singular with respect to Lebesgue measure, and $\overline{H_0^1} \subset L^1$. According to the Lebesgue decomposition theorem any $\mu \in \mathbf{M}$ can be written as $\mu = \mu_a + \mu_s$, where μ_a is absolutely continuous with respect to the Lebesgue measure and μ_s is singular with respect to the Lebesgue measure $(\mu_a \perp \mu_s)$. Furthermore the supports $S(\mu_a)$ and $S(\mu_s)$ are disjoint. Since $|x| = 1$ in (1) , if we let $x = e^{it}$ then $d\mu(e^{it}) = g_x(e^{it})dt + d\mu_s(e^{it})$ where $g_x(e^{it}) \in \overline{H_0^1}$. Consequently F_α is isomorphic to $L^1/\overline{H_0^1} \oplus \mathbf{M}_s$. Hence, F_α can be written as $F_\alpha = F_{\alpha a} \oplus F_{\alpha s}$, where $F_{\alpha a}$ is isomorphic to $L^1/\overline{H_0^1}$ the closed subspace of $\mathbf{M}$ of absolutely continuous measures, and $F_{\alpha s}$ is isomorphic to $\mathbf{M}_s$ the subspace of $\mathbf{M}$ of singular measures. If $f \in F_{\alpha a}$, then the singular part is nul and the measure μ for which (1) holds is such that $d\mu(x) = d\mu(e^{it}) = g_x(e^{it})dt$ where $g_x(e^{it}) \in L^1$ and dt is the Lebesgue measure on $\mathbf{T}$, see [1]. Hence the functions in $F_{\alpha a}$ may be written as,

$$f(z) = \int_{-\pi}^{\pi} K_x^\alpha(z) g_x(e^{it})dt$$

Furthermore if $g_x(e^{it})$ is nonnegative then

$$\|f\|_{F_\alpha} = \inf_M \|\mu\| = \left\|g_x(e^{it})\right\|_{L^1}$$

Remark: For simplicity, we will adopt the following notation throughout the article. We will reserve μ for the Borel measures of $\mathbf{M}$, and since in (1) $|x| = 1$, we can write $x = e^{it}$ where $t \in [-\pi, \pi)$. We will reserve dt for the normalized Lebesgue of the unit circle $\mathbf{T}$, and $d\sigma$ for the singular part of $d\mu$. Hence instead of writing $d\mu(x) = d\mu(e^{it}) = d\mu_a(e^{it}) + d\mu_s(e^{it}) = g_x(e^{it})dt + d\mu_s(e^{it})$ we may simply write $d\mu(x) = g_x dt + d\sigma(t)$.

2. Introduction

If X and Y are Banach spaces, and L is a linear operator from X to Y, we say that L is bounded if there exists a positive constant A such that $\|L(f)\|_Y \leq A\|f\|_X$ for all f in X. We denote by $C(X,Y)$ the set of all bounded linear operators from X to Y. If $L \in C(X,Y)$, we say that L is a compact operator from X to Y if the image of every bounded set of X is relatively compact (i.e. has compact closure) in Y. Equivalently a linear operator L is a compact operator from X to Y if and only if for every bounded sequence $\{f_n\}$ of X, $\{L(f_n)\}$ has a convergent subsequence in Y. We will denote by $K(X,Y)$ the subset of $C(X,Y)$ of compact linear operators from X into Y.

Let $H(\mathbf{D})$ denote the set of all analytic functions on the unit disk $\mathbf{D}$ and map $\mathbf{D}$ into $\mathbf{D}$. If X and Y are Banach spaces of functions on the unit disk $\mathbf{D}$, we say that $\varphi \in H(\mathbf{D})$ induces a bounded composition operator $C_\varphi(f) = f(\varphi)$ from X to Y, if $C_\varphi \in C(X,Y)$ or equivalently $C_\varphi(X) \subseteq Y$ and there exists a positive constant A such that for all $f \in X$ and $\|C_\varphi(f)\|_Y \leq A\|f\|_X$. In case $X = Y$ then we say φ induces a composition operator C_φ on X. If $f \in X$, then $C_\varphi(f) = f(\varphi) \in X$. Similarly, we say that $\varphi \in H(\mathbf{D})$ induces a compact composition operator if $C_\varphi \in K(X,Y)$.

A fundamental problem that has been studied concerning composition operators is to relate function theoretic properties of φ to operator theoretic properties of the restriction of C_φ to various Banach spaces of analytic functions. However since the spaces of Cauchy transforms are defined in terms of Borel measures, it seems natural to investigate the relation between the behavior of the composition operator and the measure. The work in this article was motivated by the work of J. Cima and A. Matheson in [1], who showed that C_φ is compact on F_1 if and only if $C_\varphi(F_1) \subset F_{1a}$. In our work we will generalize this result for $\alpha > 1$.

Now if $C_\varphi \in C(F_\alpha, F_\alpha)$ then $C_\varphi(f) = (f \circ \varphi) = f(\varphi) \in F_\alpha$ for all $f \in F_\alpha$ and there exists a positive constant A such that

$$\|C_\varphi(f)\|_{F_\alpha} = \|f(\varphi)\|_{F_\alpha} = \|[\mu]\| \leq A\|f\|_{F_\alpha}$$

Since F_α can be identified with the quotient space $\mathbf{M}/\overline{H_0^1}$ we can view C_φ as a map:

$$\begin{aligned} C_\varphi : \mathbf{M}/\overline{H_0^1} &\to \mathbf{M}/\overline{H_0^1} \\ f &\mapsto f(\varphi) \end{aligned}$$

The equivalence class of a complex measure μ will be written as:

$$[\mu] = \mu + \overline{H_0^1} = \{\mu + \overline{h} : h \in H_0^1\}$$

and

$$\|[\mu]\| = \inf_h \left\|\mu + \overline{h}\right\|$$

The space $C(F_\alpha, F_\alpha)$ has been studied by [6] where the author showed that:

(1) If $\alpha \geq 1$, then $C_\varphi \in C(F_\alpha, F_\alpha)$ for any analytic self map φ of the unit disc.
(2) $C_\varphi \in C(F_\alpha, F_\alpha)$ if and only if $\{K_x^\alpha(\varphi) : |x| = 1\}$ is a norm bounded subset of F_α.
(3) If $C_\varphi \in C(F_\alpha, F_\alpha)$ then $C_\varphi \in C(F_\beta, F_\beta)$ for $0 < \alpha < \beta$.
(4) If $C_\varphi \in C(F_\alpha, F_\alpha)$ then the operator $\varphi' C_\varphi \in C(F_{\alpha+1}, F_{\alpha+1})$.

In this article we will investigate necessary and sufficient conditions for C_φ to be compact on F_α for $\alpha \geq 1$ if and only if $C_\varphi(F_1) \subset F_{1a}$. Since F_α is Mobius invariant, then there is no loss of generality in assuming that $\varphi(0) = 0$.

3. Compactness and absolutely continuous measures

In this section we will show that compactness of the composition operator C_φ on F_α is strongly tied with the absolute continuity of the measure that supports it. First we state this Lemma due to [7].

Lemma 1. *If $0 < \alpha < \beta$ then $F_\alpha \subset F_{\beta a}$ and the inclusion map is a compact operator of norm one.*

Next we use the above result and the known fact that $H^\infty \subset F_{1a}$ to show that bounded function of F_α belong to $F_{\alpha a}$.

Proposition 1. $H^\infty \cap F_\alpha \subset F_{\alpha a}$ *for* $\alpha \geq 1$.

Proof. Let $f \in H^\infty \cap F_\alpha$, then using the previous lemma we get that for $\alpha \geq 1$ and any $z \in \mathbf{D}$, $f(z) \in H^\infty \subset H^1 \subset F_{1a} \subseteq F_{\alpha a}$, then $f(z) \in F_{\alpha a}$ for all $\alpha \geq 1$, which gives us the desired result. □

Theorem 1. *For a holomorphic self-map φ of the unit disc $\mathbf{D}$ and $\alpha \geq 1$, if C_φ is compact on F_α then $(C_\varphi \circ K_x^\alpha)(z) \in F_{\alpha a}$ and*

$$(C_\varphi \circ K_x^\alpha)(z) = \int_{-\pi}^{\pi} g_x\left(e^{it}\right) K_x^\alpha(z) dt \tag{3}$$

where $\|g_x\|_{L^1} \leq a < \infty$, g_x is nonnegative and L^1 continuous function of x .

Proof. Assume that C_φ is compact and let $\{f_j\}_{j=1}^\infty$ be a sequence of functions such that

$$f_j(z) = K_x^\alpha(\rho_j z) = \frac{1}{(1-\rho_j \overline{x} z)^\alpha}$$

where $0 < \rho_j < 1$ and $\lim_{j\to\infty} \rho_j = 1$. Then it is known from [4] that $f_j(z) \in F_\alpha$ for every j, and $\|f_j(z)\|_{F_\alpha} = 1$. Furthermore there exist $\mu_j \in \mathbf{M}$, such that $\|\mu_j\| = 1$, $d\mu_j >> 0$ and

$$\begin{aligned} f_j(z) &= \frac{1}{(1-\rho_j \overline{x} z)^\alpha} \\ &= \int_{\mathbf{T}} K_x^\alpha(z)\, d\mu_j(x) \\ &= \int_{\mathbf{T}} \frac{1}{(1-\overline{x} z)^\alpha} d\mu_j(x). \end{aligned}$$

Since C_φ is compact on F_α then $(C_\varphi \circ f_j) \in F_\alpha$ and $\|C_\varphi(f_j)\| \leq \|C_\varphi\|\, \|f_j\|_{F_\alpha} = \|C_\varphi\|$ for all j. Furthermore $C_\varphi \circ f_j \in H^\infty$, thus using the previous result, we get that $(C_\varphi \circ f_j) \in H^\infty \cap F_\alpha \subset F_{\alpha a}$ for every j. Therefore there exist L^1 nonnegative function g_x^j such that $d\mu_j(x) = g_x^j dt$, $\|g_x^j\|_{L^1} \leq \|C_\varphi\|$ and

$$\begin{aligned} (f_j \circ \varphi)(z) &= (K_x^\alpha \circ \varphi)(\rho_j z)) \\ &= \int_{-\pi}^{\pi} g_x^j\left(e^{it}\right) K_x^\alpha(\rho_j z)\, dt. \end{aligned}$$

Now because $F_{\alpha a}$ is closed and C_φ is compact, the sequence $\{f_j \circ \varphi\}_{j=1}^\infty$ has a convergent subsequence $\{f_{j_k} \circ \varphi\}$ that converges to $(K_x^\alpha \circ \varphi)(z) \in$

$F_{\alpha a}$. Therefore,

$$\begin{aligned} \lim_{k\to\infty}\left(f_{j_k}\circ\varphi\right)(z) &= \lim_{k\to\infty}\left(K_x^\alpha\circ\varphi\right)\left(\rho_{j_k}z\right) \\ &= \lim_{k\to\infty}\int_0^{2\pi} g_x^{j_k}\left(e^{it}\right)K_x^\alpha\left(\rho_{j_k}z\right)dt \\ &= \int_0^{2\pi} g_x\left(e^{it}\right)K_x^\alpha(z)\,dt \\ = \left(K_x^\alpha\circ\varphi\right)(z) &= \frac{1}{(1-\overline{x}\varphi(z))^\alpha}\in F_{\alpha a} \end{aligned}$$

where the function g_x is an L^1 nonnegative continuous function of x, and $\|g_x\|_{L^1}\le\|C_\varphi\|$. For the continuity of g_x in L^1 with respect to x where $\|x\|=1$, we take a sequence $\{x_k\}$, such that $\|x_k\|=1$ and $x_k\to x$. Now since C_φ is compact then

$$\lim_{k\to\infty}\left(K_\alpha\circ\varphi\right)\left(\overline{x}_k z\right)=\left(K_\alpha\circ\varphi\right)\left(\overline{x}z\right)$$

which concludes the proof. □

Corollary 1. *Let $g_x(e^{it})$ be as in the last theorem then the operator $\int g_x\left(e^{it}\right)h(x)dx=u(e^{it})\in\overline{H_0^1}$, for $h(x)\in\overline{H_0^1}$ is bounded on $\overline{H_0^1}$.*

Proof. For the operator to be well defined, $\int\dfrac{h(x)dx}{(1-\overline{x}\varphi(z))^\alpha}=0$ for all $h(x)\in\overline{H_0^1}$. Hence, $\int g_x\left(e^{it}\right)h(x)dx=u(e^{it})\in\overline{H_0^1}$. □

The following lemmas are needed to prove the converse of Theorem 1.

Lemma 2. *Suppose $g_x\left(e^{it}\right)$ is a nonnegative L^1continuous function of x and let $\{\mu_n\}$ be a sequence of nonnegative Borel measures that are weak* convergent to μ. Define $w_n(t)=\int_{\mathbf{T}}g_x\left(e^{it}\right)d\mu_n(x)$ and $w(t)=\int_{\mathbf{T}}g_x\left(e^{it}\right)d\mu(x)$, then $\|w_n-w\|_{L^1}\longrightarrow 0$.*

Proof. Let

$$\begin{aligned} g_x(z) &= \int \operatorname{Re}\frac{(1+e^{-it}z)}{(1-e^{-it}z)}g_x\left(e^{it}\right)d(t)\ , \\ w_n(z) &= \int g_x(z)\,d\mu_n(x)\ \text{ and} \\ w(z) &= \int g_x(z)\,d\mu_n(x) \end{aligned}$$

where$|z| < 1$. Notice that all functions are positive and harmonic in $\mathbf{D}$ and that the radial limits of $w_n(z)$ and $w(z)$ are $w_n(t)$ and $w(t)$ respectively. Then, for $|z| \leq \rho < 1$,

$$|g_x(z) - g_y(z)| \leq \frac{1}{1-\rho} \left\| g_x\left(e^{it}\right) - g_y\left(e^{it}\right) \right\|_{L^1}$$

Then the continuity condition implies that $g_x(z)$ is uniformly continuous in x for all $|z| \leq \rho$. Hence, weak star convergence, implies that $w_n(z) \to w(z)$ uniformly on $|z| \leq \rho$ and consequently the convergence is locally uniformly on $\mathbf{D}$. In addition, we have $\|w_n(\rho e^{it})\|_{L^1} \to \|w(\rho e^{it})\|_{L^1}$. Hence we conclude that

$$\left\| w_n(\rho e^{it}) - w(\rho e^{it}) \right\|_{L^1} \longrightarrow 0 \text{ as } n \to \infty.$$

Now using Fatou's Lemma we conclude that

$$\left\| w_n(e^{it}) - w(e^{it}) \right\|_{L^1} \longrightarrow 0.$$

□

Lemma 3. *Let $g_x\left(e^{it}\right)$ be a nonnegative L^1 continuous function of x such that $\|g_x\|_{L^1} \leq a < \infty$ and $g_x(e^{it})$ defines a bounded operator on $\overline{H_0^1}$. If $f(z) = \displaystyle\int \frac{1}{(1-\overline{x}z)^\alpha} d\mu(x)$, let L be the operator given by*

$$L\left[f(z)\right] = \iint \frac{g_x\left(e^{it}\right)}{(1-e^{-it}z)^\alpha} dt d\mu(x)$$

then L is compact operator on $F_\alpha, \alpha \geq 1$.

Proof. First note that the condition that $g_x(e^{it})$ defines a bounded operator on $\overline{H_0^1}$ implies that the L operator is a well defined function on $\mathbf{F}_\alpha$. Let $\{f_n(z)\}$ be a bounded sequence in F_α and let $\{\mu_n\}$ be the corresponding norm bounded sequence of measures in $\mathbf{M}$. Since every norm bounded sequence of measures in $\mathbf{M}$ has a weak star convergent subsequence, let $\{\mu_n\}$ be such subsequence that is convergent to $\mu \in \mathbf{M}$. We want to show that $\{L(f_n)\}$ has a convergent subsequence in F_α.

First, let us assume that $d\mu_n(x) >> 0$ for all n, and let $w_n(t) = \int g_x\left(e^{it}\right) d\mu_n(x)$ and $w(t) = \int g_x\left(e^{it}\right) d\mu(x)$,then we know from the previous lemma that $w_n(t), w(t) \in L^1$ for all n, and $w_n(t) \to w(t)$ in L^1. Now since $g_x\left(e^{it}\right)$ is a nonnegative continuous function in x and

$\{\mu_n\}$ is weak star convergent to μ, then

$$L(f_n(z)) = \iint \frac{g_x\left(e^{it}\right) d(t)}{(1-e^{-it}z)^{\alpha}} d\mu_n\left(x\right) = \int \frac{w_n(t)}{(1-e^{-it}z)^{\alpha}} dt$$
$$L(f(z)) = \iint \frac{g_x\left(e^{it}\right) d(t)}{(1-e^{-it}z)^{\alpha}} d\mu\left(x\right) = \int \frac{w(t)}{(1-e^{-it}z)^{\alpha}} dt$$

Furthermore because $w_n(t)$ is nonnegative then

$$\|L(f_n)\|_{F_\alpha} = \|w_n\|_{L^1}$$
$$\|L(f)\|_{F_\alpha} = \|w\|_{L^1}$$

Now since$\|w_n - w\|_{L^1} \to 0$ then $\|L(f_n) - L(f)\|_{F_\alpha} \to 0$ which shows that $\{L(f_n)\}$ has convergent subsequence in F_α and thus L is a compact operator for the case where μ is a positive measure.
In the case where μ is complex measure we write $d\mu_n\left(x\right) = (d\mu_n^1\left(x\right) - d\mu_n^2\left(x\right)) + i(d\mu_n^3\left(x\right) - d\mu_n^4\left(x\right))$,
where each $d\mu_n^j\left(x\right) >> 0$ and define $w_n^j(t) = \int g_x\left(e^{it}\right) d\mu_n^j\left(x\right)$ then $w_n(t) = \int g_x\left(e^{it}\right) d\mu_n\left(x\right) = \left(w_n^1(t) - w_n^2(t)\right) + i\left(w_n^3(t) - w_n^4(t)\right)$.
Using an argument similar to the one above we get that $w_n^j(t)$, $w^j(t) \in L^1$, and $\|w_n^j - w^j\|_{L^1} \longrightarrow 0$. Consequently, $\|w_n - w\|_{L^1} \longrightarrow 0$, where $w(t) = \left(w^1(t) - w^2(t)\right) + i\left(w^3(t) - w^4(t)\right) = \int g_x\left(e^{it}\right) d\mu\left(x\right)$.
Hence, $\|L(f_n) - L(f)\|_{F_\alpha} \leq \|w_n - w\|_{L^1} \longrightarrow 0$.
Finally, we conclude that the operator is compact. □

The following is the converse of Theorem 1.

Theorem 2. *For a holomorphic self-map φ of the unit disc $\mathbf{D}$, if*

$$\frac{1}{(1-\overline{x}\varphi(z))^{\alpha}} = \int \frac{g_x\left(e^{it}\right)}{(1-e^{-it}z)^{\alpha}} dt$$

where $g_x \in L^1$, nonnegative, $\|g_x\|_{L^1} \leq a < \infty$ for all $x \in \mathbf{T}$ and g_x is an L^1continuous function of x , then C_φ is compact on F_α.

Proof. We want to show that C_φ is compact on F_α. Let $f(z) \in F_\alpha$ then there exists a measure μ in $\mathbf{M}$ such that for every z in D

$$f(z) = \int \frac{1}{(1-\overline{x}z)^{\alpha}} d\mu(x)$$

Using the assumption of the theorem we get that

$$(f \circ \varphi)(z) = \int \frac{1}{(1-\overline{x}\varphi(z))^{\alpha}} d\mu_n\left(x\right) = \iint \frac{g_x\left(e^{it}\right)}{(1-e^{-it}z)^{\alpha}} dt d\mu_n\left(x\right)$$

which by the previous lemma was shown to be compact on F_α. □

Now we give some examples:

Corollary 2. *Let $\varphi \in H(\mathbf{D})$, with $\|\varphi\|_\infty < 1$. Then C_φ is compact on F_α, $\alpha \geq 1$.*

Proof. $(C_\varphi \circ K_x^\alpha)(z) = \dfrac{1}{(1-\overline{x}\varphi(z))^\alpha} \in H^\infty \cap F_\alpha \subset F_{\alpha a}$ and is subordinate to $\dfrac{1}{(1-z)^\alpha}$, hence

$$(C_\varphi \circ K_x^\alpha)(z) = \int K_x^\alpha(z) g_x\left(e^{it}\right) dt$$

with $g_x\left(e^{it}\right) \geq 0$ and since $1 = (C_\varphi \circ K_x^\alpha)(0) = \displaystyle\int g_x\left(e^{it}\right) dt$ we get that $\|g_x\left(e^{it}\right)\|_1 = 1$. □

Remark 1. *In fact one can show that C_φ, as in the above corollary, is compact from F_α, $\alpha \geq 1$ into F_1. In other words a contraction.*

Corollary 3. *If C_φ is compact on F_α, $\alpha \geq 1$ and $\lim\limits_{r\to 1}\left|\varphi(re^{i\theta})\right| = 1$ then $\left|\dfrac{1}{\varphi'(e^{i\theta})}\right| = 0$.*

Proof. If C_φ is compact then

$$(C_\varphi \circ K_x^\alpha)(z) = \int K_x^\alpha(z) g_x\left(e^{it}\right) dt$$

Hence, if $z = e^{i\theta}$ and $\varphi(e^{i\theta}) = x$ then

$$\lim_{r\to 1}\frac{(e^{i\theta} - re^{i\theta})^\alpha}{(1-\overline{x}\varphi(re^{i\theta}))^\alpha} = 0.$$

□

Corollary 4. *If $C_\varphi \in K(F_\alpha, F_\alpha)$ for $\alpha \geq 1$, then C_φ is contraction.*

4. Miscellaneous Results

We first start by giving another characterization of compactness on F_α.

Lemma 4. *Let $\varphi \in C(F_\alpha, F_\alpha), \alpha > 0$ then $\varphi \in K(F_\alpha, F_\alpha)$ if and only if for any bounded sequence (f_n) in F_α with $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$ as $n \to \infty$, $\|C_\varphi(f_n)\|_{F_\alpha} \to 0$ as $n \to \infty$.*

Proof. Suppose $C_\varphi \in K(F_\alpha, F_\alpha)$ and let (f_n) be a bounded sequence (f_n) in F_α with $\lim\limits_{n\to\infty} f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$. If the conclusion is false then there exists an $\epsilon > 0$ and a subsequence $n_1 < n_2 < n_3 < \cdots$ such that

$$\left\|C_\varphi(f_{n_j})\right\|_{F_\alpha} \geq \epsilon, \text{ for all } j = 1, 2, 3, \ldots$$

Since (f_n) is bounded and C_φ is compact, one can find a another subsequence $n_{j1} < n_{j2} < n_{j3} < \cdots$ and f in F_α such that

$$\lim_{k\to\infty} \left\| C_\varphi(f_{n_{j_k}}) - f \right\|_{F_\alpha} = 0$$

Since point functional evaluation are continuous in F_α then for any $z \in \mathbf{D}$ there exist $A > 0$ such that

$$\left| \left(C_\varphi(f_{n_{j_k}}) - f \right)(z) \right| \leq A \left\| C_\varphi(f_{n_{j_k}}) - f \right\|_{F_\alpha} \to 0 \text{ as } k \to \infty$$

Hence

$$\lim_{k\to\infty} \left[C_\varphi(f_{n_{j_k}}) - f \right] \to 0$$

uniformly on compact subsets of $\mathbf{D}$. Moreover since $f_{n_{j_k}} \to 0$ uniformly on compact subsets of $\mathbf{D}$, then $f = 0$ i.e. $C_\varphi(f_{n_{j_k}}) \to 0$ on compact subsets of F_α. Hence

$$\lim_{k\to\infty} \left\| C_\varphi(f_{n_{j_k}}) \right\|_{F_\alpha} = 0$$

which contradicts our assumption. Thus we must have

$$\lim_{n\to\infty} \|C_\varphi(f_n)\|_{F_\alpha} = 0.$$

Conversely, let (f_n) be a bounded sequence in the closed unit ball of F_α. We want to show that $C_\varphi(f_n)$ has a norm convergent subsequence. The closed unit ball of F_α is compact subset of F_α in the topology of uniform convergence on compact subsets of $\mathbf{D}$. Therefore there is a subsequence (f_{n_k}) such that

$$f_{n_k} \to f$$

uniformly on compact subsets of D. Hence by hypothesis

$$\|C_\varphi(f_{n_k}) - C_\varphi(f)\|_{F_\alpha} \to 0 \text{ as } k \to \infty$$

which completes the proof. $\square$

Proposition 2. *If $C_\varphi \in C(F_\alpha, F_\alpha)$ then $C_\varphi \in K(F_\alpha, F_\beta)$ for all $\beta > \alpha > 0$.*

Proof. Let (f_n) be a bounded sequence in the closed unit ball of F_α. Then $(f_n \circ \varphi)$ is bounded in F_α and since the inclusion map $i : F_\alpha \to F_{\beta a}$ is compact, $(f_n \circ \varphi)$ has a convergent subsequence in F_β. $\square$

Proposition 3. *$C_\varphi(f) = (f \circ \varphi)$ is compact on F_α if and only if the operator $\varphi' C_\varphi(g) = \varphi'(g \circ \varphi)$ is compact on $F_{\alpha+1}$.*

Proof. Suppose that $C_\varphi(f) = (f \circ \varphi)$ is compact on F_α. It is known from [6] that $\varphi' C_\varphi(g) = \varphi'(g \circ \varphi)$ is bounded on $F_{\alpha+1}$. Let (g_n) be a bounded sequence in $F_{\alpha+1}$ with $g_n \to 0$ uniformly on compact subsets of $\mathbf{D}$ as $n \to \infty$. We want to show that $\lim_{n\to\infty} \|\varphi'(g_n \circ \varphi)\|_{F_{\alpha+1}} = 0$. Let (f_n) be the sequence defined by $f_n(z) = \int_0^z g_n(w)dw$. Then $f_n \in F_\alpha$ and $\|f_n\|_{F_\alpha} \leq \frac{2}{\alpha} \|g_n\|_{F_{\alpha+1}}$, thus (f_n) is a bounded sequence in F_α. Furthermore, using the Lebesgue dominated convergence theorem we get that $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$. Thus

$$\begin{aligned} \|\varphi'(g_n \circ \varphi)\|_{F_{\alpha+1}} &= \|\varphi'(f_n' \circ \varphi)\|_{F_{\alpha+1}} \\ &= \|(f_n \circ \varphi)'\|_{F_{\alpha+1}} \\ &\leq \alpha \|(f_n \circ \varphi)\|_{F_\alpha} \to 0 \text{ as } n \to \infty \end{aligned}$$

which shows that $\varphi' C_\varphi(g) = \varphi'(g \circ \varphi)$ is compact on $F_{\alpha+1}$.
Conversely, assume that $\varphi' C_\varphi(g) = \varphi'(g \circ \varphi)$ is compact on $F_{\alpha+1}$. Then in particular $\varphi' C_\varphi(f') = \varphi'(f' \circ \varphi) = (f \circ \varphi)'$ is a compact for every $f \in F_\alpha$. Now since $\|(f \circ \varphi)\|_{F_\alpha} \leq \frac{2}{\alpha} \|(f \circ \varphi)'\|_{F_{\alpha+1}}$. Let (f_n) be a bounded sequence in F_α with $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$ as $n \to \infty$. We want to show that $\lim_{n\to\infty} \|(f_n \circ \varphi)\|_{F_\alpha} = 0$. Since any bounded sequence of F_α is also a bounded sequence of $F_{\alpha+1}$, then $\|(f_n \circ \varphi)\|_{F_\alpha} \leq \frac{2}{\alpha} \|(f_n \circ \varphi)'\|_{F_{\alpha+1}} \to 0$ as $n \to \infty$ and the proof is complete. □

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