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PROVING MATRIX EQUATIONS

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Abstract. Students taking an undergraduate Linear Algebra course may face problems like this one (ref[1]):
Given $A_\lambda = (\lambda - A)^{-1}$ and $A_\mu = (\mu - A)^{-1}$
then prove

$$(\lambda - \mu)A_\lambda A_\mu = A_\mu - A_\lambda \tag{1}$$

where λ and μ are scalars and A_λ, A_μ and A are invertible $n \times n$ matrices.

The purpose of the note is to present a general method for determining the truth of symbolic matrix equations where 0 or more such equations are given as true. The idea behind the method is to write the equation to be proved in terms of independent variables only, removing all the dependent variables, effectively reducing the problem to the case of 0 equations given as true. It should then be a simple matter to determine the truth of the equation to be proved, as it must be true for all values of any variable in the equation.

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The Method.

- Determine dependent and independent variables in the given equations. In the example A_λ and A_μ are dependent variables and A, λ, and μ are independent.

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- Rewrite the given equations, if necessary, to express dependent variables in terms of only independent variables, for any dependent variables which appear in the equation to prove. In the example the dependent variables A_λ and A_μ are already expressed in terms of the independent variables λ, μ, and A. So the given equations need not be rewritten.
- Substitute independent variables for dependent variables in the equation to prove. Then we will have an equation that is totally expressed in independent variables, i.e. we have transformed the problem to the case of 0 equations given. In the example equation (1) is now
  $(\lambda - \mu)(\lambda - A)^{-1}(\mu - A)^{-1} = (\mu - A)^{-1} - (\lambda - A)^{-1}$
  It must prove true for any λ, μ, and A.
- Multiply and distribute as necessary to express the equation to prove in normal form (i.e. no parentheses) as follows: if an outer term has an exponent > 0 then multiply and distribute the primitives. If the exponent is < 0 then multiply the equation by the positive exponent of the same term to remove the negative exponent. For example $A(B + C)^2(A - C)^{-2}$ in an equation would be reduced to normal form by first distributing the $(B + C)^2$ to $(B^2 + BC + CB + C^2)(A - C)^{-2}$. Then multiply the equation from the right by $(A - C)^2$ to remove the negative exponent. Continue to multiply and distribute terms as necessary to reduce the level (i.e. number of parentheses) of the equation until the equation is in normal form.
- Cancel terms until the resulting equation is $0 = 0$. If the resulting equation differs from $0 = 0$ then the equation to prove is not true.

The Example. Equation (1) in the example problem would be reduced as follows:

- $(\lambda - \mu)(\lambda - A)^{-1}(\mu - A)^{-1} = (\mu - A)^{-1} - (\lambda - A)^{-1}$

- Multiply from right by $(\mu - A)(\lambda - A)$ to achieve
  $(\lambda - \mu) = ((\mu - A)^{-1} - (\lambda - A)^{-1})(\mu - A)(\lambda - A)$

- Distribute from right to achieve
  $(\lambda - \mu) = (\lambda - A) - (\lambda - A)^{-1}(\mu - A)(\lambda - A)$

- Multiply from left by $(\lambda - A)$ to achieve
  $(\lambda - A)(\lambda - \mu) = (\lambda - A)(\lambda - A) - (\mu - A)(\lambda - A)$

- At this point the nested inverses have been removed and the terms can simply be distributed to achieve
$\lambda^2 - \lambda\mu - A\lambda + A\mu = \lambda^2 - \lambda A - A\lambda + A^2 - (\mu\lambda - \mu A - A\lambda + A^2)$
- which reduces to normal form:
$\lambda^2 - \lambda\mu - A\lambda + A\mu = \lambda^2 - \lambda A - A\lambda + A^2 - \mu\lambda + \mu A + A\lambda - A^2$
- Finally cancel terms until the equation reduces to $0 = 0$.

It seems curious that textbooks for the introductory course in linear algebra do not include this simple but handy method.

References. 1. Michael O'Nan, *Linear Algebra* Volume **2A** (1971), Problem no. 39, page 57.