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AN INTERACTING PARTICLES PROCESS FOR BURGERS EQUATION ON THE CIRCLE

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ABSTRACT. We adapt the results of Oelschläger (1985) to prove a weak law of large numbers for an interacting particles process which, in the limit, produces a solution to Burgers equation with periodic boundary conditions. We anticipate results of this nature to be useful in the development of Monte Carlo schemes for nonlinear partial differential equations.

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1. INTRODUCTION

Several propagation of chaos results have been proved for the Burgers equation (Calderoni and Pulvirenti 1983, Osada and Kotani 1985, Oelschläger 1985, Gutkin and Kac 1986, and Sznitman 1986) all using slightly different methods. Perhaps the best result for the Cauchy free-boundary problem is Sznitman's (1986) result which describes the particle interaction in terms of the average 'co-occupation time' of the randomly diffusing particles. For various reasons, we follow Oelschläger and prove a Law of Large Numbers type result for the measure valued process (MVP) where the interaction is given in terms of a kernel density estimate with bandwidth a function of the number N of interacting diffusions.

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The heuristics are as follows: The (nonlinear) partial differential equation

$$u_t = \frac{u_{xx}}{2} - \left(u(x,t) \int b(x-y)u(y,t)\,dy\right)_x \tag{1}$$

is the Kolmogorov forward equation for the diffusion $X = (X_t)$ which is the solution to the stochastic differential equation

$$\begin{aligned} dX_t &= dW_t + \left\{\int b(X_t - y)u(y,t)\,dy\right\}dt & (2)\\ &= dW_t + E(b(X_t - \bar{X}_t))dt & (3) \end{aligned}$$

where $u(x,t)\,dx$ is the density of X_t, W_t is standard Brownian motion (a Wiener process), $\bar{X}$ is an independent copy of X, and E is the expectation operator. Note the change in notation: for a stochastic process X, X_t denotes its location at time t not a (partial) derivative with respect to t.

The law of large numbers suggests that

$$E(b(X_t - \bar{X}_t)) = \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} b(X_t - X_t^j)$$

where the X^j are independent copies of X and this empirical approximation suggests looking at the system of N stochastic differential equations given by

$$dX_t^{i,N} = dW_t^{i,N} + \frac{1}{N}\sum_{j=1}^{N} b(X_t^{i,N} - X_t^{j,N})dt, \quad i = 1, \ldots, N$$

where the $W^{i,N}$ are independent Brownian motions. Now if b is bounded and Lipschitz and the N particles are started independently with distribution μ_0, then the system of N stochastic differential equations will have a unique solution (Karatzas and Shreve 1991) and the measure valued process

$$\mu_t^N = \frac{1}{N}\sum_{j=1}^{N} \delta_{X_t^{j,N}}$$

where δ_x is the point-mass at x will converge to a solution μ of (1) in the sense that for every bounded continuous function f on the real-line and every $t > 0$,

$$\int f(x)\mu_t^N(dx) = \frac{1}{N}\sum_{j=1}^{N} f(X_t^{j,N}) \to \int f(x)\mu_t(dx),$$

where μ_t has a density u so $\mu_t(dx) = u(x,t)dx$ and u solves (1).

By formal analogy, if we take $2b(x-y) = \delta_0(x-y)$, where δ_0 is the point-mass at zero, then

$$u_t = \frac{u_{xx}}{2} - \left(u(x,t)\int \frac{\delta_0(x-y)}{2}u(y,t)\,dy\right)_x \tag{4}$$

$$= \frac{u_{xx}}{2} - \left(\frac{u^2}{2}\right)_x \tag{5}$$

$$= \frac{u_{xx}}{2} - uu_x \tag{6}$$

which is the Burgers equation with viscosity parameter $\varepsilon = 1/2$. Unfortunately, δ_0 is neither bounded nor Lipschitz and a lot of work goes into dealing with this problem. This is covered in greater detail later in the paper.

Our interest in these models lies partially in their potential use as numerical methods for nonlinear partial differential equations. This idea has been the subject of a good deal of recent research, see Talay and Tubaro (1996). As noted there, and elsewhere, the Burgers equation is an excellent test for new numerical methods precisely because it does have an exact solution. In the next two sections, we prove the underlying Law of Large Numbers for the Burgers equation with periodic boundary conditions. Such boundary conditions seem natural for numerical work.

2. The Setup and Goal.

We are interested in looking at the dynamics of the measure valued process

$$\mu_t^N = \sum_{j=1}^N \delta_{Y_t^{j,N}} \tag{7}$$

with δ_x the point-mass at x,

$$Y_t^{j,N} = \varphi(X_t^{j,N}) \tag{8}$$

where $\varphi(x) = x - [x]$ and $[x]$ is the largest integer less than or equal to x, with the $X_t^{j,N}$ satisfying the following system of stochastic differential equations

$$dX_t^{j,N} = dW_t^{j,N} + F\left(\frac{1}{N}\sum_{l=1}^N b^N(X_t^{j,N} - X_t^{l,N})\right)\,dt \tag{9}$$

where the $W_t^{j,N}$ are independent standard Brownian motion processes,

$$F(x) = \frac{x \wedge \|u_0\|}{2},$$

u_0 is a bounded measurable density function on $S = [0, 1)$, $\|\cdot\|$ is the supremum norm, $\|f\| = \sup_S |f(x)|$, and $b^N(x) > 0$ is an infinitely-differentiable one-periodic function on the real line $I\!R$ such that

$$\int_0^1 b^N(x)\,dx = 1 \tag{10}$$

for all $N = 1, 2, \ldots$, and for any continuous bounded one-periodic function f

$$\int_{-1/2}^{1/2} f(x)b^N(x)\,dx \to f(0) \tag{11}$$

as $N \to \infty$. We call a function f on $I\!R$ one-periodic if $f(x) = f(x+1)$ for every $x \in I\!R$.

For any x and y in S, let

$$\rho(x, y) = |x - y - 1| \wedge |x - y| \wedge |x - y + 1| \tag{12}$$

and note that (S, ρ) is a complete, separable, and compact metric space. Let $C_b(S)$ denote the space of all continuous bounded functions on (S, ρ). Note that if f is a continuous one-periodic function on $I\!R$ and g is the restriction of f to S, then $g \in C_b(S)$. Additionally, for any one-periodic function f on $I\!R$ we have $f(Y_t^{j,N}) = f(X_t^{j,N})$ and therefore

$$\begin{aligned}
\langle \mu_t^N, f\rangle &= \int_S f(x)\mu_t^N(dx) \\
&= \frac{1}{N}\sum_{j=1}^N f(Y_t^{j,N}) \\
&= \frac{1}{N}\sum_{j=1}^N f(X_t^{j,N})
\end{aligned}$$

for any one-periodic function f on $I\!R$.

To study the dynamics of the process μ_t^N as $N \to \infty$ we will need to study, for any f which is both one-periodic and twice-differentiable with bounded first and second derivatives, the dynamics of the processes $\langle \mu_t^N, f\rangle$. These dynamics are obtained from (7), (9), and Itô's formula (see Karatzas and Shreve 1991, p.153)

$$\begin{aligned}
\langle \mu_t^N, f\rangle = \langle \mu_0^N, f\rangle &+ \int_0^t \langle \mu_s^N, F(g_s^N(\cdot))f' + \frac{1}{2}f''\rangle\,ds \\
&+ \frac{1}{N}\sum_{j=1}^N \int_0^t f'(X_s^{j,N})\,dW_s^{j,N}
\end{aligned} \tag{13}$$

where the use the notation

$$\langle \mu, f\rangle = \int_S f(x)\mu(dx)$$

with μ a measure on S,

$$g_t^N(x) = \frac{1}{N}\sum_{l=1}^{N} b^N(x - X_t^{l,N}) \tag{14}$$

and the fact that because b^N is one-periodic, $b^N(Y_t^{j,N} - Y_t^{l,N}) = b^N(X_t^{j,N} - X_t^{l,N})$.

Given any metric space (M, m), let $\mathcal{M}_1(M)$ be the space of probability measures on M equipped with the usual weak topology:

$$\lim_{k\to\infty} \mu^k = \mu$$

if and only if

$$\lim_{k\to\infty} \int_M f(x)\mu^k(dx) = \int_M f(x)\mu(dx)$$

for every f in $C_b(M)$, where $C_b(M)$ is the space of all continuous bounded and real-valued functions f on M under the supremum norm $\|f\| = \sup_M |f(x)|$.

On the space (S, ρ) the weak topology is generated by the bounded Lipschitz metric

$$\|\mu^1 - \mu^2\|_H = \sup_{f\in H} |\langle \mu^1, f\rangle - \langle \mu^2, f\rangle|$$

where

$$H = \{f \in C_b(S) \,:\, \|f\| \leq 1,\ |f(x) - f(y)| < \rho(x,y) \text{ for all } x, y \in S\}$$

(Pollard 1984, or Dudley 1966).

Fix a positive $T < \infty$ and take $C([0,T], \mathcal{M}_1(S))$ to be the space of all continuous functions $\mu = (\mu_t)$ from $[0,T]$ to $\mathcal{M}_1(S)$ with the metric

$$m(\mu^1, \mu^2) = \sup_{0\leq t\leq T} \|\mu_t^1 - \mu_t^2\|_H,$$

then the empirical processes μ_t^N with $0 \leq t \leq T$ are random elements of the space $C([0,T], \mathcal{M}_1(S))$. Indeed, take any sequence $(t_k) \subset [0,T]$ with

$t_k \to t$, then for any f in H we have

$$\begin{aligned}
|\langle \mu_t^N, f\rangle - \langle \mu_{t_k}^N, f\rangle| &= \left| \frac{1}{N}\sum_{j=1}^N f(Y_t^{j,N}) - f(Y_{t_k}^{j,N}) \right| \\
&\le \frac{1}{N}\sum_{j=1}^N |f(Y_t^{j,N}) - f(Y_{t_k}^{j,N})| \\
&\le \frac{1}{N}\sum_{j=1}^N \rho(Y_t^{j,N}, Y_{t_k}^{j,N}) \\
&\le \frac{1}{N}\sum_{j=1}^N |X_t^{j,N} - X_{t_k}^{j,N}| \\
&= \frac{1}{N}\sum_{j=1}^N \left| [W_t^{j,N} - W_{t_k}^{j,N}] + \int_{t_k}^t F(g_s^N(X_s^{j,N}))\, ds \right| \to 0
\end{aligned}$$

because the $W_t^{j,N}$ are continuous in t and $\|F\| < \infty$. This means that the distributions $\mathcal{L}(\mu^N)$ of the processes $\mu^N = (\mu_t^N)$ can be considered random elements of the space $\mathcal{M}_1(C([0,T], \mathcal{M}_1(S)))$.

Our goal is to prove the following Law of Large Numbers type result.

Theorem 1. *Under the conditions that*

(i): b^N *is one-periodic, positive and infinitely-differentiable with*

$$\int_0^1 b^N(x)\, dx = 1, \tag{15}$$

and

$$\int_{-1/2}^{1/2} f(x) b^N(x)\, dx \to f(0) \tag{16}$$

for every continuous, bounded, and one-periodic function f *on* $\mathbb{R}$,

(ii): $\|b^N\| \le AN^\alpha$ *for some* $0 < \alpha < 1/2$ *and some constant* $A < \infty$,

(iii): *there is a* β *with* $0 < \beta < (1 - 2\alpha)$ *such that*

$$\sum_\lambda |\tilde{b}^N(\lambda)|^2 (1 + |\lambda|^\beta) < \infty \tag{17}$$

where $\lambda = 2k\pi$, *with* $k \in \mathbf{Z}$, *and* $\tilde{b}^N(\lambda) = \int_0^1 e^{i\lambda x} b^N(x)\, dx$ *is the Fourier transform of* b^N,

(iv): u_0 *is a density function on* $[0,1)$ *with* $\|u_0\| < \infty$, *and*

(v): $\langle \mu_0^N, f\rangle = \frac{1}{N}\sum_{j=1}^N f(Y_0^{j,N}) = \frac{1}{N}\sum_{j=1}^N f(X_0^{j,N}) \to \int_0^1 f(x) u_0(x)\, dx$ *for every* $f \in C_b(S)$.

then there is a deterministic family of measures $\mu = (\mu_t)$ on $[0,1)$ such that

$$\mu^N \to \mu \tag{18}$$

in probability as $N \to \infty$, for every t in $[0,T]$, with $\mu^N = (\mu_t^N)$, μ_t is absolutely continuous with respect to Lebesgue measure on S with density function $g_t(x) = u(x,t)$ satisfying the Burgers equation

$$u_t + uu_x = \frac{1}{2}u_{xx} \tag{19}$$

with periodic boundary conditions.

The proof has three parts. First, we establish the fact that the sequence of probability laws $\mathcal{L}(\mu^N)$ is relatively compact in $\mathcal{M}_1(C([0,T],\mathcal{M}_1(S)))$ and therefore every subsequence of (μ^{N_k}) of (μ^N) has a further subsequence that converges in law to some μ in $C([0,T],\mathcal{M}_1(S))$. Second, we prove that any such limit process μ must satisfy a certain integral equation, and finally, that this integral equation has a unique solution. We follow rather closely the arguments of Oelschläger (1985) and apply his result (Theorem 5.1, p.31) in the final step of the argument.

3. THE LAW OF LARGE NUMBERS.

Relative Compactness. The first step in the proof of Theorem 1 is to show that the sequence of probability laws $\mathcal{L}(\mu^N)$, $N = 1, 2, \ldots$, is relatively compact in $\mathcal{M} = \mathcal{M}_1(C([0,T],\mathcal{M}_1(S)))$. Since S is a compact metric space $\mathcal{M}_1(S)$ is as well (Stroock 1983, p.122) and therefore for any $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset \mathcal{M}_1(S)$ such that

$$\inf_N P\left(\mu_t^N \in K_\varepsilon, \forall t \in [0,T]\right) \geq 1 - \varepsilon; \tag{20}$$

in particular, we may take $K_\varepsilon = \mathcal{M}_1(S)$ regardless of $\varepsilon \geq 0$. Furthermore, for $0 \leq s \leq t \leq T$ and some constant $C > 0$ we have

$$\begin{aligned}
\|\mu_t^N - \mu_s^N\|_H^4 &= \sup_{f\in H}(\langle \mu_t^N, f\rangle - \langle \mu_s^N, f\rangle)^4 \\
&= \sup_{f\in H}\left(\frac{1}{N}\sum_{j=1}^N f(Y_t^{j,N}) - f(Y_s^{j,N})\right)^4 \\
&\leq \left(\frac{1}{N}\sum_{j=1}^N \rho(Y_t^{j,N}, Y_s^{j,N})\right)^4 \\
&\leq \left(\frac{1}{N}\sum_{j=1}^N |X_t^{j,N} - X_s^{j,N}|\right)^4 \\
&\leq \frac{1}{N}\sum_{j=1}^N |X_t^{j,N} - X_s^{j,N}|^4 \\
&= \frac{1}{N}\sum_{j=1}^N \left|(W_t^{j,N} - W_s^{j,N}) + \int_s^t F\left(g_u^N(X_u^{j,N})\right)\, du\right|^4 \\
&\leq C\left(\frac{1}{N}\sum_{j=1}^N \left|W_t^{j,N} - W_s^{j,N}\right|^4 + \frac{1}{N}\sum_{j=1}^N \left|\int_s^t F\left(g_u^N(X_u^{j,N})\right)\, du\right|^4\right)
\end{aligned}$$

and therefore

$$E\|\mu_t^N - \mu_s^N\|_H^4 \leq C(3(t-s)^2 + \|u_0\|^4(t-s)^4) < 3C\|u_0\|^4(t-s)^2 \tag{21}$$

for $t - s$ small. Together equations (20) and (21) imply that the sequence of probability laws $\mathcal{L}(\mu^N)$ is relatively compact (Gikhman and Skorokhod 1974, VI, 4) as desired.

Almost Sure Convergence. Now the relative compactness of the sequence of laws $\mathcal{L}(\mu^N)$ in $\mathcal{M}$ implies that there is an increasing subsequence $(N_k) \subset (N)$ such that $\mathcal{L}(\mu^{N_k})$ converges in $\mathcal{M}$ to some limit $\mathcal{L}(\mu)$ which is the distribution of some measure valued process $\mu = (\mu_t)$. For ease of notation, we assume at this point that $(N_k) = (N)$. The Skorokhod representation theorem implies now that after choosing the proper probability space, we may define μ^N and μ so that

$$\lim_{N\to\infty}\sup_{t\leq T}\|\mu_t^N - \mu_t\|_H = 0 \tag{22}$$

P-almost surely. This leaves us with the task of describing the possible limit processes, μ.

An Integral Equation. We know from Ito's formula that for any $f \in C_b^2(S)$, μ^N satisfies

$$\begin{aligned}\langle \mu_t^N, f\rangle - \langle \mu_0^N, f\rangle &- \int_0^t \langle \mu_s^N, F(g_s^N(\cdot))f' + \frac{1}{2}f''\rangle \, ds \\ &= \frac{1}{N}\sum_{j=1}^N \int_0^t f'(X_s^{j,N})\, dW_s^{j,N} \qquad (23)\end{aligned}$$

where the right hand side is a martingale. Because $f \in C_b^2(S)$, the weak convergence of μ^N to μ gives us that

$$\langle \mu_t^N, f\rangle \to \langle \mu_t, f\rangle \qquad (24)$$

as $N \to \infty$ for all $0 \le t \le T$ and we have

$$\langle \mu_0^N, f\rangle \to \langle \mu_0, f\rangle \qquad (25)$$

as $N \to \infty$ by assumption. Furthermore, Doob's inequality (Stroock 1983, p.355) implies

$$\begin{aligned}E\left[\sup_{t\le T}\left(\frac{1}{N}\sum_{j=1}^N \int_0^t f'(X_s^{j,N})\, dW_s^{j,N}\right)^2\right] &\le 4E\left[\left(\frac{1}{N}\sum_{j=1}^N \int_0^T f'(X_s^{j,N})\, dW_s^{j,N}\right)^2\right] \\ &\le \frac{4}{N}T\|f'\|^2\end{aligned}$$

and therefore the right hand side of (23) vanishes as $N \to \infty$. Clearly now, the integral term third in equation (23) must converge as well and the goal at present is to find out to what.

First, because $f \in C_b^2(S)$, the weak convergence of μ^N to μ gives us that

$$\frac{1}{2}\int_0^t \langle \mu_s^N, f''\rangle \, ds \to \frac{1}{2}\int_0^t \langle \mu_s, f''\rangle \, ds \qquad (26)$$

as $N \to \infty$. Now only the $\int_0^t \langle \mu_s^N, F(g_s^N(\cdot))f'\rangle \, ds$-term remains and this is indeed the most troublesome because of the interaction between the μ_s^N and g_s^N terms. To study this term we will need to work out the convergence properties of the 'density' g_s^N. We start by working on some L^2 bounds.

The Convergence of the Density g_s^N. Note that

$$\langle g_s^N(\cdot), e^{i\lambda\cdot}\rangle = \langle \mu_s^N, e^{i\lambda\cdot}\rangle \, \tilde{b}^N(\lambda),$$

where $\tilde{b}^N$ is the Fourier transform of the interaction kernel b^N.

Ito's formula implies that for any $\lambda \in (2k\pi)$ with $k \in \mathbf{Z}$

$$\begin{aligned}
|\langle \mu_t^N, e^{i\lambda\cdot}\rangle|^2 e^{\lambda^2(t-\tau)} \; - \; & \int_0^t (((\langle \mu_s^N, e^{-i\lambda\cdot}\rangle\langle \mu_s^N, F(g_s^N(\cdot))(i\lambda)e^{i\lambda\cdot} - \frac{\lambda^2}{2}e^{i\lambda\cdot}\rangle \\
& + \langle \mu_s^N, e^{i\lambda\cdot}\rangle\langle \mu_s^N, F(g_s^N(\cdot))(-i\lambda)e^{-i\lambda\cdot} - \frac{\lambda^2}{2}e^{-i\lambda\cdot}\rangle)e^{\lambda^2(s-\tau)} \\
& + |\langle \mu_s^N, e^{i\lambda\cdot}\rangle|^2\lambda^2 e^{\lambda^2(s-\tau)} + \frac{1}{N}\lambda^2 e^{\lambda^2(s-\tau)})\, ds \\
= \; |\langle \mu_t^N, e^{i\lambda\cdot}\rangle|^2 e^{\lambda^2(t-\tau)} - & \int_0^t (((\langle \mu_s^N, e^{-i\lambda\cdot}\rangle\langle \mu_s^N, F(g_s^N(\cdot))(i\lambda)e^{i\lambda\cdot}\rangle \\
& + \langle \mu_s^N, e^{i\lambda\cdot}\rangle\langle \mu_s^N, F(g_s^N(\cdot))(-i\lambda)e^{-i\lambda\cdot}\rangle)e^{\lambda^2(s-\tau)} \\
& + \frac{\lambda^2}{N}e^{\lambda^2(s-\tau)})\, ds
\end{aligned} \tag{27}$$

is a martingale.

Now take $\tau = t + h$ and

$$k_h^N(\lambda, t) = |\langle \mu_t^N, e^{i\lambda\cdot}\rangle|^2 |\tilde{b}^N(\lambda)|^2 e^{-\lambda^2 h}$$

then the martingale property above gives

$$\begin{aligned}
E[k_h^N(\lambda, t)] \;=\; & E[k_{t+h}^N(\lambda, 0)] + \int_0^t E[\langle \mu_s^N, e^{-i\lambda\cdot}\rangle\langle \mu_s^N, F(g_s^N(\cdot))(i\lambda)e^{i\lambda\cdot}\rangle \\
& + \langle \mu_s^N, e^{i\lambda\cdot}\rangle\langle \mu_s^N, F(g_s^N(\cdot))(-i\lambda)e^{-i\lambda\cdot}\rangle \\
& + \frac{\lambda^2}{N}]e^{-\lambda^2(t+h-s)}|\tilde{b}^N(\lambda)|^2\, ds \\
\leq \; & E[k_{t+h}^N(\lambda, 0)] \\
& + \int_0^t (E[2|\langle \mu_s^N, e^{i\lambda\cdot}\rangle||\langle \mu_s^N, F(g_s^N(\cdot))e^{i\lambda\cdot}\rangle \\
& \cdot|\lambda|\, e^{-\lambda^2(t+h-s)}|\tilde{b}^N(\lambda)|^2] \\
& + \frac{\lambda^2}{N}e^{-\lambda^2(t+h-s)}|\tilde{b}^N(\lambda)|^2)\, ds \\
\leq \; & E[k_{t+h}^N(\lambda, 0)] \\
& + \int_0^t (2\|u_0\| E[|\langle \mu_s^N, e^{i\lambda\cdot}\rangle|^2|\lambda|e^{-\lambda^2(t+h-s)}|\tilde{b}^N(\lambda)|^2] \\
& + \frac{\lambda^2}{N}e^{-\lambda^2(t+h-s)}|\tilde{b}^N(\lambda)|^2)\, ds.
\end{aligned} \tag{28}$$

Summing over $\lambda \in (\lambda_k)$ gives

$$\begin{aligned}
\sum_\lambda E[k_h^N(\lambda,t)] &\leq \sum_\lambda E[k_{t+h}^N(\lambda,0)] \\
&\quad +2\|u_0\| \sum_\lambda \int_0^t E\left[|\langle \mu_s^N, e^{i\lambda\cdot}\rangle|^2 |\lambda| e^{-\lambda^2(t+h-s)} |\tilde{b}^N(\lambda)|^2\right] ds \\
&\quad +\sum_\lambda \int_0^t \left(\frac{\lambda^2}{N} e^{-\lambda^2(t+h-s)} |\tilde{b}^N(\lambda)|^2\right) ds \\
&= A_{I} + A_{II} + A_{III}.
\end{aligned}$$

Now, of course,

$$\sum_\lambda k_{t+h}^N(\lambda,0) \leq \sum_\lambda e^{-\lambda^2(t+h)} \leq (t+h)^{-1/2}$$

and therefore

$$A_{I} = \sum_\lambda E[k_{t+h}^N(\lambda,0)] \leq (t+h)^{-1/2}.$$

For A_{III}, using hypothesis (ii) from Theorem 1, we have

$$\begin{aligned}
A_{III} &= \frac{1}{N}\sum_\lambda |\tilde{b}^N(\lambda)|^2 \int_0^t \lambda^2 e^{-\lambda^2(t+h-s)}\, ds = \frac{1}{N}\sum_\lambda |\tilde{b}^N(\lambda)|^2 e^{-\lambda^2 h} \\
&\leq \frac{1}{N}\sum_\lambda |\tilde{b}^N(\lambda)|^2 = \frac{1}{N}\int_0^1 (b^N(x))^2\, dx \\
&\leq \frac{2N^{2\alpha}}{N} C \leq 2C
\end{aligned}$$

for some constant $C > 0$. Now

$$\begin{aligned}
&2\|u_0\| \int_0^t E[|\langle \mu_s^N, e^{i\lambda\cdot}\rangle|^2 |\tilde{b}^N(\lambda)|^2 |\lambda| e^{-\lambda^2(t+h-s)}]\, ds \\
&= 2\|u_0\| \int_0^t E[|\langle \mu_s^N, e^{i\lambda\cdot}\rangle|^2 |\tilde{b}^N(\lambda)|^2 e^{-\lambda^2(t+h-s)/2} |\lambda| e^{-\lambda^2(t+h-s)/2}]\, ds \\
&\leq 2\|u_0\| C \int_0^t E[|\langle \mu_s^N, e^{i\lambda\cdot}\rangle|^2 |\tilde{b}^N(\lambda)|^2 e^{-\lambda^2(t+h-s)/2}]\, ds \\
&= 2\|u_0\| C \int_0^t E[k_{(t+h-s)/2}^N(\lambda,s)]\, ds \\
&\leq 2\|u_0\| C \int_0^t e^{-\lambda^2(t+h-s)/2}\, ds \\
&\leq \frac{4\|u_0\| C}{\lambda^2}
\end{aligned}$$

for some other constant $C > 0$ and therefore

$$A_{II} \leq 4\|u_0\| C \sum_{\lambda \neq 0} \lambda^{-2} \leq 4\|u_0\| D$$

for some constant $D < \infty$. Hence

$$\begin{aligned}\sum_\lambda E[k_h^N(\lambda,t)] &= \sum_\lambda E[|\langle \mu_t^N, e^{i\lambda\cdot}\rangle|^2 |\tilde{b}^N(\lambda)|^2] e^{-\lambda^2 h} \\ &= A_{I} + A_{II} + A_{III} \\ &\le (t+h)^{-1/2} + C(\|u_0\|+1)\end{aligned}$$

uniformly in $h > 0$ for some constant $C < \infty$. Letting h go to zero gives

$$\begin{aligned}\sum_\lambda E|\tilde{g}_t^N(\lambda)|^2 &= \sum_\lambda E[k_0^N(\lambda,t)] \\ &= \lim_{h\to 0}\sum_\lambda E[k_h^N(\lambda,t)] \le t^{-1/2} + C(\|u_0\|+1).\end{aligned}$$

From the martingale property (27) we have

$$\begin{aligned}E[k_0^N(\lambda,t)] \le E[k_{t/2}^N(\lambda,t/2)] &+ 2\|u_0\|\int_{t/2}^t E[|\langle \mu_s^N, e^{i\lambda\cdot}\rangle|^2|\tilde{b}^N(\lambda)|^2]|\lambda|e^{-\lambda^2(t-s)}\,ds \\ &+ \frac{\lambda^2}{N}\int_{t/2}^t e^{-\lambda^2(t-s)}|\tilde{b}^N(\lambda)|^2\,ds\end{aligned}$$

and for $\beta \in (0, 1-2\alpha)$ we have

$$\begin{aligned}(1+|\lambda|^\beta)E[k_0^N(\lambda,t)] &\le (1+|\lambda|^\beta)E[k_{t/2}^N(\lambda,t/2)] \\ &\quad +2\|u_0\|\int_{t/2}^t E[|\langle \mu_s^N, e^{i\lambda\cdot}\rangle|^2|\tilde{b}^N(\lambda)|^2] \\ &\qquad \cdot |\lambda|(1+|\lambda|^\beta)e^{-\lambda^2(t-s)}\,ds \\ &\quad +(1+|\lambda|^\beta)\frac{\lambda^2}{N}\int_{t/2}^t e^{-\lambda^2(t-s)}|\tilde{b}^N(\lambda)|^2\,ds \\ &\le (1+|\lambda|^\beta)e^{-\lambda^2 t/2} \\ &\quad +2\|u_0\|C\int_{t/2}^t E[|\langle \mu_s^N, e^{i\lambda\cdot}\rangle|^2|\tilde{b}^N(\lambda)|^2]e^{-\lambda^2(t-s)/2}\,ds \\ &\quad +(1+|\lambda|^\beta)\frac{\lambda^2}{N}\int_{t/2}^t e^{-\lambda^2(t-s)}|\tilde{b}^N(\lambda)|^2\,ds\end{aligned}$$

for some constant $C < \infty$ and we know that

$$\sum_\lambda (1+|\lambda|^\beta)e^{-\lambda^2 t/2} < \infty,$$

$$2\|u_0\|C\int_{t/2}^t \sum_\lambda E[k_{(t-s)/2}^N(\lambda,s)]\,ds \le 2\|u_0\|C\int_{t/2}^t \sum_\lambda e^{-\lambda^2(t-s)/2}\,ds < \infty,$$

and, from hypothesis (iii) of Theorem 1,

$$\frac{1}{N}\sum_\lambda (1+|\lambda|^\beta)|\tilde{b}^N(\lambda)|^2 < \infty$$

and therefore

$$\sum_\lambda (1+|\lambda|^\beta)E|\tilde{g}_t^N(\lambda)|^2 = \sum_\lambda (1+|\lambda|^\beta)E[k_0^N(\lambda,t)] < \infty. \tag{29}$$

Finally, from (29) it is easy to work out the convergence properties of g^N. Indeed,

$$\begin{aligned}
\lim_{N,M\to\infty} \quad & E\left[\int_0^T \int_0^1 |g_t^N(x) - g_t^N(x)|^2\,dx\,dt\right] \\
&= \lim_{N,M\to\infty} E\left[\int_0^T \sum_\lambda |\tilde{g}_t^N(\lambda) - \tilde{g}_t^M(\lambda)|^2\,dt\right] \\
&\leq \lim_{N,M\to\infty} E\left[\int_0^T \sum_{|\lambda|\leq K} |\tilde{g}_t^N(\lambda) - \tilde{g}_t^M(\lambda)|^2\,dt\right] \\
&\qquad + \lim_{N,M\to\infty} 2E\left[\int_0^T \sum_{|\lambda|>K} (|\tilde{g}_t^N(\lambda)|^2 + |\tilde{g}_t^M(\lambda)|^2)\,dt\right] \\
&\leq \lim_{N,M\to\infty} 4E\left[\int_0^T \sum_{|\lambda|\leq K} |\langle \mu_t^N - \mu_t^M, e^{i\lambda\cdot}\rangle|^2\,dt\right] \\
&\qquad +4(1+K^\beta)^{-1}\sup_N E\left[\int_0^T \sum_\lambda |\tilde{g}_t^N(\lambda)|^2(1+|\lambda|^\beta)\,dt\right] \\
&\leq C(1+K^\beta)^{-1}T
\end{aligned}$$

for some constant $C < \infty$ and the right hand side of this last inequality can be made smaller than any given $\varepsilon > 0$ by the choice of K. So, by the completeness of L^2, we have proved the existence of a positive random function $g_t(x)$ such that

$$\lim_{N\to\infty} E\left[\int_0^T \int_0^1 |g_t^N(x) - g_t(x)|^2\,dx\,dt\right] = 0. \tag{30}$$

Of course, this means that for any $f \in C_b(S)$ we have

$$\begin{aligned}
\int_0^1 f(x)g_t(x)\,dx &= \lim_{N\to\infty}\int_0^1 f(x)g_t^N(x)\,dx \;=\; \lim_{N\to\infty}\langle \mu_t^N * b^N, f\rangle \\
&= \lim_{N\to\infty}\langle \mu_t^N, f * b^N\rangle \;=\; \langle \mu_t, f\rangle \;=\; \int_0^1 f(x)\,\mu_t(dx)
\end{aligned}$$

and therefore μ_t is absolutely continuous with respect to Lebesgue measure on S with derivative g_t.

Conclusion. Finally, combining (23-26), and (30), implies

$$\langle \mu_t, f \rangle - \langle \mu_0, f \rangle = \int_0^t \langle \mu_s, F(g_s(\cdot)) f' + \frac{1}{2} f'' \rangle \, ds \tag{31}$$

and from Proposition 3.5 of Oelschläger (1985) we know that the integral equation (31) has a unique solution μ_t absolutely continuous with respect to Lebesgue measure on S with density g_t. We note also that the solution $g_t(x) = u(x,t)$ of the Burgers equation

$$u_t + u u_x = \frac{1}{2} u_{xx}$$

with periodic boundary conditions

$$u(x,t) = u(x+1,t),$$

for all real x, and all $t > 0$, and initial condition u_0, satisfies the integral equation

$$\langle g_t(\cdot), f \rangle - \langle u_0(\cdot), f \rangle = \int_0^t \langle g_s(\cdot), \frac{1}{2} g_s(\cdot) f' + \frac{1}{2} f'' \rangle \, ds$$

and from the Hopf-Cole solution (II.67) we see that

$$\|g_t\| \leq \|u_0\|$$

and therefore $g_t(x)$ satisfies (31) as well. The uniqueness result for solutions to the periodic boundary problem for the Burgers equation then completes the proof of Theorem 1. ∎

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