Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISBN 1083-0464
Issue 1 July 2004, pp. 10 – 32
Submitted: September 10, 2003. Published: July 1, 2004

ORLICZ-SOBOLEV SPACES WITH ZERO BOUNDARY VALUES ON METRIC SPACES

NOUREDDINE AÏSSAOUI

ABSTRACT. In this paper we study two approaches for the definition of the first order Orlicz-Sobolev spaces with zero boundary values on arbitrary metric spaces. The first generalization, denoted by $M_{\Phi}^{1,0}(E)$, where E is a subset of the metric space X, is defined by the mean of the notion of the trace and is a Banach space when the N-function satisfies the Δ_2 condition. We give also some properties of these spaces. The second, following another definition of Orlicz-Sobolev spaces on metric spaces, leads us to three definitions that coincide for a large class of metric spaces and N-functions. These spaces are Banach spaces for any N-function.

A.M.S. (MOS) Subject Classification Codes.46E35, 31B15, 28A80.
Key Words and Phrases. Orlicz spaces, Orlicz-Sobolev spaces, modulus of a family of paths, capacities.

1. Introduction

This paper treats definitions and study of the first order Orlicz-Sobolev spaces with zero boundary values on metric spaces. Since we have introduce two definitions of Orlicz-Sobolev spaces on metric spaces, we are leading to examine two approaches.

The first approach follows the one given in the paper [7] relative to Sobolev spaces. This generalization, denoted by $M_{\Phi}^{1,0}(E)$, where E is a subset of the metric space X, is defined as Orlicz-Sobolev functions on X, whose trace on $X \setminus E$ vanishes.

Ecole Normale Supérieure, B.P 5206 Ben Souda, Fès, Maroc
n.aissaoui@caramail.com

This is a Banach space when the N-function satisfies the Δ_2 condition. For the definition of the trace of Orlicz-Sobolev functions we need the notion of Φ-capacity on metric spaces developed in [2]. We show that sets of Φ-capacity zero are removable in the Orlicz-Sobolev spaces with zero boundary values. We give some results closely related to questions of approximation of Orlicz-Sobolev functions with zero boundary values by compactly supported functions. The approximation is not valid on general sets. As in Sobolev case, we study the approximation on open sets. Hence we give sufficient conditions, based on Hardy type inequalities, for an Orlicz-Sobolev function to be approximated by Lipschitz functions vanishing outside an open set.

The second approach follows the one given in the paper [13] relative to Sobolev spaces; see also [12]. We need the rudiments developed in [3]. Hence we consider the set of Lipschitz functions on X vanishing on $X \setminus E$, and close that set under an appropriate norm. Another definition is to consider the space of Orlicz-Sobolev functions on X vanishing Φ-q.e. in $X \setminus E$. A third space is obtained by considering the closure of the set of compactly supported Lipschitz functions with support in E. These spaces are Banach for any N-function and are, in general, different. For a large class of metric spaces and a broad family of N-functions, we show that these spaces coincide.

2. Preliminaries

An $\mathcal{N}$*-function* is a continuous convex and even function Φ defined on $\mathbb{R}$, verifying $\Phi(t) > 0$ for $t > 0$, $\lim_{t\to 0} t^{-1}\Phi(t) = 0$ and $\lim_{t\to\infty} t^{-1}\Phi(t) = +\infty$.

We have the representation $\Phi(t) = \int\limits_0^{|t|} \varphi(x)d\mathfrak{L}(x)$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, with $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$, $\lim_{t\to 0^+} \varphi(t) = 0$ and $\lim_{t\to\infty} \varphi(t) = +\infty$. Here $\mathfrak{L}$ stands for the Lebesgue measure. We put in the sequel, as usually, $dx = d\mathfrak{L}(x)$.

The $\mathcal{N}$-function Φ^* *conjugate* to Φ is defined by $\Phi^*(t) = \int\limits_0^{|t|} \varphi^*(x)dx$, where φ^* is given by $\varphi^*(s) = \sup\{t : \varphi(t) \le s\}$.

Let (X, Γ, μ) be a measure space and Φ an $\mathcal{N}$-function. The *Orlicz class* $\mathcal{L}_{\Phi,\mu}(X)$ is defined by

$$\mathcal{L}_{\Phi,\mu}(X) = \left\{f : X \to \mathbb{R} \text{ measurable} : \int_X \Phi(f(x))d\mu(x) < \infty\right\}.$$

We define the *Orlicz space* $\mathbf{L}_{\Phi,\mu}(X)$ by

$$\mathbf{L}_{\Phi,\mu}(X) = \left\{ f : X \to \mathbb{R} \text{ measurable} : \int_X \Phi(\alpha f(x)) d\mu(x) < \infty \text{ for some } \alpha > 0 \right\}.$$

The Orlicz space $\mathbf{L}_{\Phi,\mu}(X)$ is a Banach space with the following norm, called the *Luxemburg norm,*

$$|||f|||_{\Phi,\mu,X} = \inf \left\{ r > 0 : \int_X \Phi\left(\frac{f(x)}{r}\right) d\mu(x) \le 1 \right\}.$$

If there is no confusion, we set $|||f|||_\Phi = |||f|||_{\Phi,\mu,X}$.

The Hölder inequality extends to Orlicz spaces as follows: if $f \in \mathbf{L}_{\Phi,\mu}(X)$ and $g \in \mathbf{L}_{\Phi^*,\mu}(X)$, then $fg \in \mathbf{L}^1$ and

$$\int_X |fg| \, d\mu \le 2|||f|||_{\Phi,\mu,X}.\ |||g|||_{\Phi^*,\mu,X}.$$

Let Φ be an $\mathcal{N}$-function. We say that Φ *verifies the* Δ_2 *condition* if there is a constant $C > 0$ such that $\Phi(2t) \le C\Phi(t)$ for all $t \ge 0$.

The Δ_2 condition for Φ can be formulated in the following equivalent way: for every $C > 0$ there exists $C' > 0$ such that $\Phi(Ct) \le C'\Phi(t)$ for all $t \ge 0$.

We have always $\mathcal{L}_{\Phi,\mu}(X) \subset \mathbf{L}_{\Phi,\mu}(X)$. The equality $\mathcal{L}_{\Phi,\mu}(X) = \mathbf{L}_{\Phi,\mu}(X)$ occurs if Φ verifies the Δ_2 condition.

We know that $\mathbf{L}_{\Phi,\mu}(X)$ is reflexive if Φ and Φ^* verify the Δ_2 condition.

Note that if Φ verifies the Δ_2 condition, then $\int \Phi(f_i(x))d\mu \to 0$ as $i \to \infty$ if and only if $|||f_i|||_{\Phi,\mu,X} \to 0$ as $i \to \infty$.

Recall that an $\mathcal{N}$-function Φ satisfies the Δ' condition if there is a positive constant C such that for all $x, y \ge 0$, $\Phi(xy) \le C\Phi(x)\Phi(y)$. See [9] and [12]. If an $\mathcal{N}$-function Φ satisfies the Δ' condition, then it satisfies also the Δ_2 condition.

Let Ω be an open set in $\mathbb{R}^N$, $\mathbf{C}^\infty(\Omega)$ be the space of functions which, together with all their partial derivatives of any order, are continuous on Ω, and $\mathbf{C}_0^\infty(\mathbb{R}^N) = \mathbf{C}_0^\infty$ stands for all functions in $\mathbf{C}^\infty(\mathbb{R}^N)$ which have compact support in $\mathbb{R}^N$. The space $\mathbf{C}^k(\Omega)$ stands for the space of functions having all derivatives of order $\le k$ continuous on Ω, and $\mathbf{C}(\Omega)$ is the space of continuous functions on Ω.

The (weak) partial derivative of f of order $|\beta|$ is denoted by

$$D^\beta f = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}.\partial x_2^{\beta_2}...\partial x_N^{\beta_N}} f.$$

Let Φ be an $\mathcal{N}$-function and $m \in \mathbb{N}$. We say that a function $f : \mathbb{R}^N \to \mathbb{R}$ has a distributional (weak partial) derivative of order m, denoted $D^\beta f$, $|\beta| = m$, if

$$\int f D^\beta \theta dx = (-1)^{|\beta|} \int (D^\beta f)\theta dx, \ \forall \theta \in \mathbf{C}_0^\infty.$$

Let Ω be an open set in $\mathbb{R}^N$ and denote $\mathbf{L}_{\Phi,\mathfrak{L}}(\Omega)$ by $\mathbf{L}_\Phi(\Omega)$. The *Orlicz-Sobolev space* $W^m\mathbf{L}_\Phi(\Omega)$ is the space of real functions f, such that f and its distributional derivatives up to the order m, are in $\mathbf{L}_\Phi(\Omega)$.

The space $W^m\mathbf{L}_\Phi(\Omega)$ is a Banach space equipped with the norm

$$|||f|||_{m,\Phi,\Omega} = \sum_{0\leq|\beta|\leq m} |||D^\beta f|||_\Phi,\ f \in W^m\mathbf{L}_\Phi(\Omega),$$

where $|||D^\beta f|||_\Phi = |||D^\beta f|||_{\Phi,\mathfrak{L},\Omega}$.

Recall that if Φ verifies the Δ_2 condition, then $\mathbf{C}^\infty(\Omega) \cap W^m\mathbf{L}_\Phi(\Omega)$ is dense in $W^m\mathbf{L}_\Phi(\Omega)$, and $\mathbf{C}_0^\infty(\mathbb{R}^N)$ is dense in $W^m\mathbf{L}_\Phi(\mathbb{R}^N)$.

For more details on the theory of Orlicz spaces, see [1, 8, 9, 10, 11].

In this paper, the letter C will denote various constants which may differ from one formula to the next one even within a single string of estimates.

3. Orlicz-Sobolev space with zero boundary values $M_\Phi^{1,0}(E)$

3.1. The Orlicz-Sobolev space $M_\Phi^1(X)$.

We begin by recalling the definition of the space $M_\Phi^1(X)$.

Let $u : X \to [-\infty, +\infty]$ be a μ-measurable function defined on X. We denote by $D(u)$ the set of all μ-measurable functions $g : X \to [0, +\infty]$ such that

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y)) \tag{3.1}$$

for every $x, y \in X \setminus F$, $x \neq y$, with $\mu(F) = 0$. The set F is called the exceptional set for g.

Note that the right hand side of (3.1) is always defined for $x \neq y$. For the points $x, y \in X$, $x \neq y$ such that the left hand side of (3.1) is undefined we may assume that the left hand side is $+\infty$.

Let Φ be an $\mathcal{N}$-function. The Dirichlet-Orlicz space $\mathbf{L}_\Phi^1(X)$ is the space of all μ-measurable functions u such that $D(u) \cap \mathbf{L}_\Phi(X) \neq \emptyset$. This space is equipped with the seminorm

$$|||u|||_{\mathbf{L}_\Phi^1(X)} = \inf\left\{|||g|||_\Phi : g \in D(u) \cap \mathbf{L}_\Phi(X)\right\}. \tag{3.2}$$

The Orlicz-Sobolev space $M_\Phi^1(X)$ is defined by $M_\Phi^1(X) = \mathbf{L}_\Phi(X) \cap \mathbf{L}_\Phi^1(X)$ equipped with the norm

$$|||u|||_{M_\Phi^1(X)} = |||u|||_\Phi + |||u|||_{\mathbf{L}_\Phi^1(X)}. \tag{3.3}$$

We define a *capacity* as an increasing positive set function C given on a σ-additive class of sets Γ, which contains compact sets and such that $C(\emptyset) = 0$ and $C(\bigcup\limits_{i\geq 1} X_i) \leq \sum\limits_{i\geq 1} C(X_i)$ for $X_i \in \Gamma$, $i = 1, 2, \dots$.

C is called outer capacity if for every $X \in \Gamma$,

$$C(X) = \inf \{C(O) : O \text{ open}, X \subset O\}.$$

Let C be a capacity. If a statement holds except on a set E where $C(E) = 0$, then we say that the statement holds C-quasieverywhere (abbreviated C-q.e.). A function $u : X \to [-\infty, \infty]$ is C-quasicontinuous in X if for every $\varepsilon > 0$ there is a set E such that $C(E) < \varepsilon$ and the restriction of u to $X \setminus E$ is continuous. When C is an outer capacity, we may assume that E is open.

Recall the following definition in [2]

Definition 1. *Let Φ be an $\mathcal{N}$-function. For a set $E \subset X$, define $C_\Phi(E)$ by*

$$C_\Phi(E) = \inf\{|||u|||_{M^1_\Phi(X)} : u \in B(E)\},$$

where $B(E) = \{u \in M^1_\Phi(X) : u \geq 1$ on a neighborhood of $E\}$.

If $B(E) = \emptyset$, we set $C_{\Phi,\mu}(E) = \infty$.

Functions belonging to $B(E)$ are called admissible functions for E.

In the definition of $C_\Phi(E)$, we can restrict ourselves to those admissible functions u such that $0 \leq u \leq 1$. On the other hand, C_Φ is an outer capacity.

Let Φ be an $\mathcal{N}$-function satisfying the Δ_2 condition, then by [2 Theorem 3.10] the set

$$Lip^1_\Phi(X) = \{u \in M^1_\Phi(X) : u \text{ is Lipschitz in } X\}$$

is a dense subspace of $M^1_\Phi(X)$. Recall the following result in [2, Theorem 4.10]

Theorem 1. *Let Φ be an $\mathcal{N}$-function satisfying the Δ_2 condition and $u \in M^1_\Phi(X)$. Then there is a function $v \in M^1_\Phi(X)$ such that $u = v$ μ-a.e. and v is C_Φ-quasicontinuous in X.*

The function v is called a C_Φ-quasicontinuous representative of u.

Recall also the following theorem, see [6]

Theorem 2. *Let C be an outer capacity on X and μ be a nonnegative, monotone set function on X such that the following compatibility condition is satisfied: If G is open and $\mu(E) = 0$, then*

$$C(G) = C(G \setminus E).$$

Let f and g be C-quasicontinuous on X such that

$$\mu(\{x : f(x) \neq g(x)\}) = 0.$$

Then $f = g$ C-quasi everywhere on X.

It is easily verified that the capacity C_Φ satisfies the compatibility condition. Thus from Theorem 2, we get the following corollary.

Corollary 1. *Let Φ be an $\mathcal{N}$-function. If u and v are C_Φ-quasicontinuous on an open set O and if $u = v$ μ-a.e. in O, then $u = v$ C_Φ-q.e. in O.*

Corollary 1 make it possible to define the trace of an Orlicz-Sobolev function to an arbitrary set.

Definition 2. *Let Φ be an $\mathcal{N}$-function, $u \in M^1_\Phi(X)$ and E be such that $C_\Phi(E) > 0$. The trace of u to E is the restriction to E of any C_Φ-quasicontinuous representative of u.*

Remark 1. *Let Φ be an $\mathcal{N}$-function. If u and v are C_Φ-quasicontinuous and $u \leq v$ μ-a.e. in an open set O, then $\max(u - v, 0) = 0$ μ-a.e. in O and $\max(u - v, 0)$ is C_Φ-quasicontinuous. Hence by Corollary 1, $\max(u - v, 0) = 0$ C_Φ-q.e. in O, and consequently $u \leq v$ C_Φ-q.e. in O.*

Now we give a characterization of the capacity C_Φ in terms of quasicontinuous functions. We begin by a definition

Definition 3. *Let Φ be an $\mathcal{N}$-function. For a set $E \subset X$, define $D_\Phi(E)$ by*

$$D_\Phi(E) = \inf\{|||u|||_{M^1_\Phi(X)} : u \in \mathcal{B}(E)\},$$

where

$$\mathcal{B}(E) = \{u \in M^1_\Phi(X) : u \text{ is } C_\Phi\text{-quasicontinuous and } u \geq 1\ C_\Phi\text{-q.e. in } E\}.$$

If $\mathcal{B}(E) = \emptyset$, we set $D_\Phi(E) = \infty$.

Theorem 3. *Let Φ be an $\mathcal{N}$-function and E a subset in X. Then*

$$C_\Phi(E) = D_\Phi(E).$$

Proof. Let $u \in M^1_\Phi(X)$ be such that $u \geq 1$ on an open neighborhood O of E. Then, by Remark 1, the C_Φ-quasicontinuous representative v of u satisfies $v \geq 1$ C_Φ-q.e. on O, and hence $v \geq 1$ C_Φ-q.e. on E. Thus $D_\Phi(E) \leq C_\Phi(E)$.

For the reverse inequality, let $v \in \mathcal{B}(E)$. By truncation we may assume that $0 \leq v \leq 1$. Let ε be such that $0 < \varepsilon < 1$ and choose an open set V such that $C_\Phi(V) < \varepsilon$ with $v = 1$ on $E \setminus V$ and $v\big|_{X\setminus V}$ is continuous. We can find, by topology, an open set $U \subset X$ such that $\{x \in X : v(x) > 1 - \varepsilon\} \setminus V = U \setminus V$. We have $E \setminus V \subset U \setminus V$. We choose $u \in B(V)$ such that $|||u|||_{M^1_\Phi(X)} < \varepsilon$ and that $0 \leq u \leq 1$. We define $w = \frac{v}{1-\varepsilon} + u$. Then $w \geq 1$ μ-a.e. in $(U \setminus V) \cup V = U \cup V$, which is an open neighbourhood of E. Hence $w \in B(E)$. This implies that

$$\begin{aligned} C_\Phi(E) &\leq |||w|||_{M^1_\Phi(X)} \leq \frac{1}{1-\varepsilon} |||v|||_{M^1_\Phi(X)} + |||u|||_{M^1_\Phi(X)} \\ &\leq \frac{1}{1-\varepsilon} |||v|||_{M^1_\Phi(X)} + \varepsilon. \end{aligned}$$

We get the desired inequality since ε and v are arbitrary. The proof is complete. ∎

We give a sharpening of [2, Theorem 4.8].

Theorem 4. *Let Φ be an $\mathcal{N}$-function and $(u_i)_i$ be a sequence of C_Φ-quasicontinuous functions in $M^1_\Phi(X)$ such that $(u_i)_i$ converges in $M^1_\Phi(X)$ to a C_Φ-quasicontinuous function u. Then there is a subsequence of $(u_i)_i$ which converges to u C_Φ-q.e. in X.*

Proof. There is a subsequence of $(u_i)_i$, which we denote again by $(u_i)_i$, such that

$$\sum_{i=1}^{\infty} 2^i \, |||u_i - u|||_{M^1_\Phi(X)} < \infty. \tag{3.4}$$

We set $E_i = \{x \in X : |u_i(x) - u(x)| > 2^{-i}\}$ for $i = 1, 2, ...$, and $F_j = \bigcup_{i=j}^{\infty} E_i$. Then $2^i |u_i - u| \in \mathcal{B}(E_i)$ and by Theorem 3 we obtain $C_\Phi(E_i) \leq 2^i \, |||u_i - u|||_{M^1_\Phi(X)}$. By subadditivity we get

$$C_\Phi(F_j) \leq \sum_{i=j}^{\infty} C_\Phi(E_i) \leq \sum_{i=j}^{\infty} 2^i \, |||u_i - u|||_{M^1_\Phi(X)} \, .$$

Hence

$$C_\Phi(\bigcap_{j=1}^{\infty} F_j) \leq \lim_{j \to \infty} C_\Phi(F_j) = 0.$$

Thus $u_i \to u$ pointwise in $X \setminus \bigcap_{j=1}^{\infty} F_j$ and the proof is complete. ∎

3.2. **The Orlicz-Sobolev space with zero boundary values $M^{1,0}_\Phi(E)$.**

Definition 4. *Let Φ be an $\mathcal{N}$-function and E a subspace of X. We say that u belongs to the Orlicz-Sobolev space with zero boundary values, and denote $u \in M^{1,0}_\Phi(E)$, if there is a C_Φ-quasicontinuous function $\widetilde{u} \in M^1_\Phi(X)$ such that $\widetilde{u} = u$ μ-a.e. in E and $\widetilde{u} = 0$ C_Φ-q.e. in $X \setminus E$.*

In other words, u belongs to $M^{1,0}_\Phi(E)$ if there is $\widetilde{u} \in M^1_\Phi(X)$ as above such that the trace of $\widetilde{u}$ vanishes C_Φ-q.e. in $X \setminus E$.

The space $M^{1,0}_\Phi(E)$ is equipped with the norm

$$|||u|||_{M^{1,0}_\Phi(E)} = |||\widetilde{u}|||_{M^1_\Phi(X)}\,.$$

Recall that $C_\Phi(E) = 0$ implies that $\mu(E) = 0$ for every $E \subset X$; see [2]. It follows that the norm does not depend on the choice of the quasicontinuous representative.

Theorem 5. *Let Φ be an $\mathcal{N}$-function satisfying the Δ_2 condition and E a subspace of X. Then $M^{1,0}_\Phi(E)$ is a Banach space.*

Proof. Let $(u_i)_i$ be a Cauchy sequence in $M^{1,0}_\Phi(E)$. Then for every u_i, there is a C_Φ-quasicontinuous function $\widetilde{u_i} \in M^1_\Phi(X)$ such that $\widetilde{u_i} = u_i$ μ-a.e. in E and $\widetilde{u_i} = 0$ C_Φ-q.e. in $X \setminus E$. By [2, Theorem 3.6] $M^1_\Phi(X)$ is complete. Hence there is $u \in M^1_\Phi(X)$ such that $\widetilde{u_i} \to u$ in $M^1_\Phi(X)$ as $i \to \infty$. Let $\widetilde{u}$ be a C_Φ-quasicontinuous representative of u given by Theorem 1. By Theorem 4 there is a subsequence $(\widetilde{u_i})_i$ such that $\widetilde{u_i} \to \widetilde{u}$ C_Φ-q.e. in X as $i \to \infty$. This implies that $\widetilde{u} = 0$ C_Φ-q.e. in $X \setminus E$ and hence $u \in M^{1,0}_\Phi(E)$. The proof is complete. ∎

Moreover the space $M^{1,0}_\Phi(E)$ has the following lattice properties. The proof is easily verified.

Lemma 1. *Let Φ be an $\mathcal{N}$-function and let E be a subset in X. If $u, v \in M^{1,0}_\Phi(E)$, then the following claims are true.*

1) *If* $\alpha \geq 0$, *then* $\min(u,\alpha) \in M^{1,0}_\Phi(E)$ *and* $|||\min(u,\alpha)|||_{M^{1,0}_\Phi(E)} \leq |||u|||_{M^{1,0}_\Phi(E)}$.
2) *If* $\alpha \leq 0$, *then* $\max(u,\alpha) \in M^{1,0}_\Phi(E)$ *and* $|||\max(u,\alpha)|||_{M^{1,0}_\Phi(E)} \leq |||u|||_{M^{1,0}_\Phi(E)}$.
3) $|u| \in M^{1,0}_\Phi(E)$ *and* $||||u||||_{M^{1,0}_\Phi(E)} \leq |||u|||_{M^{1,0}_\Phi(E)}$.
4) $\min(u,v) \in M^{1,0}_\Phi(E)$ *and* $\max(u,v) \in M^{1,0}_\Phi(E)$.

Theorem 6. *Let Φ be an $\mathcal{N}$-function satisfying the Δ_2 condition and E a μ-measurable subset in X. If $u \in M^{1,0}_\Phi(E)$ and $v \in M^1_\Phi(X)$ are such that $|v| \leq u$ μ-a.e. in E, then $v \in M^{1,0}_\Phi(E)$.*

Proof. Let w be the zero extension of v to $X \setminus E$ and let $\widetilde{u} \in M^1_\Phi(X)$ be a C_Φ-quasicontinuous function such that $\widetilde{u} = u$ μ-a.e. in E and that $\widetilde{u} = 0$ C_Φ-q.e. in $X \setminus E$. Let $g_1 \in D(\widetilde{u}) \cap \mathbf{L}_\Phi(X)$ and $g_2 \in D(v) \cap \mathbf{L}_\Phi(X)$. Define the function g_3 by

$$g_3(x) = \begin{cases} \max(g_1(x), g_2(x)), & x \in E \\ g_1(x), & x \in X \setminus E. \end{cases}$$

Then it is easy to verify that $g_3 \in D(w) \cap \mathbf{L}_\Phi(X)$. Hence $w \in M^1_\Phi(X)$. Let $\widetilde{w} \in M^1_\Phi(X)$ be a C_Φ-quasicontinuous function such that $\widetilde{w} = w$ μ-a.e. in X given by Theorem 1. Then $|\widetilde{w}| \leq \widetilde{u}$ μ-a.e. in X. By Remark 1 we get $|\widetilde{w}| \leq \widetilde{u}$ C_Φ-q.e. in X and consequently $\widetilde{w} = 0$ C_Φ-q.e. in $X \setminus E$. This shows that $v \in M^{1,0}_\Phi(E)$. The proof is complete. ∎

The following lemma is easy to verify.

Lemma 2. *Let Φ be an $\mathcal{N}$-function and let E be a subset in X. If $u \in M^{1,0}_\Phi(E)$ and $v \in M^1_\Phi(X)$ are bounded functions, then $uv \in M^{1,0}_\Phi(E)$.*

We show in the next theorem that the sets of capacity zero are removable in the Orlicz-Sobolev spaces with zero boundary values.

Theorem 7. *Let Φ be an $\mathcal{N}$-function and let E be a subset in X. Let $F \subset E$ be such that $C_\Phi(F) = 0$. Then $M^{1,0}_\Phi(E) = M^{1,0}_\Phi(E \setminus F)$.*

Proof. It is evident that $M^{1,0}_\Phi(E \setminus F) \subset M^{1,0}_\Phi(E)$. For the reverse inclusion, let $u \in M^{1,0}_\Phi(E)$, then there is a C_Φ-quasicontinuous function $\widetilde{u} \in M^1_\Phi(X)$ such that $\widetilde{u} = u$ μ-a.e. in E and that $\widetilde{u} = 0$ C_Φ-q.e. in $X \setminus E$. Since $C_\Phi(F) = 0$, we get that $\widetilde{u} = 0$ C_Φ-q.e. in $X \setminus (E \setminus F)$. This implies that $u_{|E\setminus F} \in M^{1,0}_\Phi(E \setminus F)$. Moreover we have $\left|\left|\left|u_{|E\setminus F}\right|\right|\right|_{M^{1,0}_\Phi(E\setminus F)} = |||u|||_{M^{1,0}_\Phi(E)}$. The proof is complete. ∎

As in the Sobolev case, we have the following remark.

Remark 2. 1) *If $C_\Phi(\partial F) = 0$, then $M^{1,0}_\Phi(int\ E) = M^{1,0}_\Phi(\overline{E})$.*

2) *We have the equivalence: $M^{1,0}_\Phi(X \setminus F) = M^{1,0}_\Phi(X) = M^1_\Phi(X)$ if and only if $C_\Phi(F) = 0$.*

The converse of Theorem 7 is not true in general. In fact it suffices to take $\Phi(t) = \frac{1}{p}t^p$ $(p > 1)$ and consider the example in [7].

Nevertheless the converse of Theorem 7 holds for open sets.

Theorem 8. *Let Φ be an $\mathcal{N}$-function and suppose that μ is finite in bounded sets and that O is an open set. Then $M^{1,0}_\Phi(O) = M^{1,0}_\Phi(O \setminus F)$ if and only if $C_\Phi(F \cap O) = 0$.*

Proof. We must show only the necessity. We can assume that $F \subset O$. Let $x_0 \in O$ and for $i \in \mathbb{N}^*$, pose $O_i = B(x_0, i) \cap \{x \in O : \operatorname{dist}(x, X \setminus O) > 1/i\}$. We define for $i \in \mathbb{N}^*$, $u_i : X \to \mathbb{R}$ by $u_i(x) = \max(0, 1 - \operatorname{dist}(x, F \cap O_i))$. Then $u_i \in M^1_\Phi(X)$, u_i is continuous, $u_i = 1$ in $F \cap O_i$ and $0 \leq u_i \leq 1$. For $i \in \mathbb{N}^*$, define $v_i : O_i \to \mathbb{R}$ by $v_i(x) = \operatorname{dist}(x, X \setminus O_i)$. Then $v_i \in M^{1,0}_\Phi(O_i) \subset M^{1,0}_\Phi(O)$. By Lemma 2 we have, for every $i \in \mathbb{N}^*$, $u_i v_i \in M^{1,0}_\Phi(O) = M^{1,0}_\Phi(O \setminus F)$. If w is a C_Φ-quasicontinuous function such that $w = u_i v_i$ μ-a.e. in $O \setminus F$, then $w = u_i v_i$ μ-a.e. in O since

$\mu(F) = 0$. By Corollary 1 we get $w = u_i v_i$ C_Φ-q.e. in O. In particular $w = u_i v_i > 0$ C_Φ-q.e. in $F \cap O_i$. Since $u_i v_i \in M_\Phi^{1,0}(O \setminus F)$ we may define $w = 0$ C_Φ-q.e. in $X \setminus (O \setminus F)$. Hence $w = 0$ C_Φ-q.e. in $F \cap O_i$. This is possible only if $C_\Phi(F \cap O_i) = 0$ for every $i \in \mathbb{N}^*$. Hence $C_\Phi(F) \leq \sum_{i=1}^{\infty} C_\Phi(F \cap O_i) = 0$. The proof is complete. ∎

3.3. **Some relations between $H_\Phi^{1,0}(E)$ and $M_\Phi^{1,0}(E)$.** We would describe the Orlicz-Sobolev space with zero boundary values on $E \subset X$ as the completion of the set $Lip_\Phi^{1,0}(E)$ defined by

$$Lip_\Phi^{1,0}(E) = \{u \in M_\Phi^1(X) : u \text{ is Lipschitz in } X \text{ and } u = 0 \text{ in } X \setminus E\}$$

in the norm defined by (3.3). Since $M_\Phi^1(X)$ is complete, this completion is the closure of $Lip_\Phi^{1,0}(E)$ in $M_\Phi^1(X)$. We denote this completion by $H_\Phi^{1,0}(E)$.

Let Φ be an $\mathcal{N}$-function satisfying the Δ_2 condition and E a subspace of X. By [2, Theorem 3.10] we have $H_\Phi^{1,0}(X) = M_\Phi^{1,0}(X)$. Since $Lip_\Phi^{1,0}(E) \subset M_\Phi^{1,0}(E)$ and $M_\Phi^{1,0}(E)$ is complete, then $H_\Phi^{1,0}(E) \subset M_\Phi^{1,0}(E)$. When $\Phi(t) = \frac{1}{p}t^p$ $(p > 1)$, simple examples show that the equality is not true in general; see [7]. Hence for the study of the equality, we restrict ourselves to open sets as in the Sobolev case. We begin by a sufficient condition.

Theorem 9. *Let Φ be an $\mathcal{N}$-function satisfying the Δ_2 condition, O an open subspace of X and suppose that $u \in M_\Phi^1(O)$. Let v be the function defined on O by $v(x) = \dfrac{u(x)}{dist(x, X \setminus O)}$. If $v \in \mathbf{L}_\Phi(O)$, then $u \in H_\Phi^{1,0}(O)$.*

Proof. Let $g \in D(u) \cap \mathbf{L}_\Phi(O)$ and define the function $\overline{g}$ by

$$\begin{aligned} \overline{g}(x) &= \max(g(x), v(x)) \text{ if } x \in O \\ \overline{g}(x) &= 0 \text{ if } x \in X \setminus O. \end{aligned}$$

Then $\overline{g} \in \mathbf{L}_\Phi(X)$. Define the function $\overline{u}$ as the zero extension of u to $X \setminus O$. For μ-a.e. $x, y \in O$ or $x, y \in X \setminus O$, we have

$$|\overline{u}(x) - \overline{u}(y)| \leq d(x,y)(\overline{g}(x) + \overline{g}(y)).$$

For μ-a.e. $x \in O$ and $y \in X \setminus O$, we get

$$|\overline{u}(x) - \overline{u}(y)| = |u(x)| \leq d(x,y)\frac{|u(x)|}{\mathrm{dist}(x, X \setminus O)} \leq d(x,y)(\overline{g}(x) + \overline{g}(y)).$$

Thus $\overline{g} \in D(\overline{u}) \cap \mathbf{L}_\Phi(X)$ which implies that $\overline{u} \in M_\Phi^1(O)$. Hence

$$|\overline{u}(x) - \overline{u}(y)| \leq d(x,y)(\overline{g}(x) + \overline{g}(y)) \tag{3.5}$$

for every $x, y \in X \setminus F$ with $\mu(F) = 0$.

For $i \in \mathbb{N}^*$, set

$$F_i = \{x \in O \setminus F : |\overline{u}(x)| \leq i,\ \overline{g}(x) \leq i\} \cup X \setminus O. \tag{3.6}$$

From (3.5) we see that $\overline{u}_{|F_i}$ is $2i$-Lipschitz and by the McShane extension

$$\overline{u}_i(x) = \inf\{\overline{u}(y) + 2id(x, y) : y \in F_i\}$$

we extend it to a $2i$-Lipschitz function on X. We truncate $\overline{u}_i$ at the level i and set $u_i(x) = \min(\max(\overline{u}_i(x), -i), i)$. Then u_i is such that u_i is $2i$-Lipschitz function in X, $|u_i| \leq i$ in X and $u_i = \overline{u}$ in F_i and, in particular, $u_i = 0$ in $X \setminus O$. We show that $u_i \in M^1_\Phi(X)$. Define the function g_i by

$$\begin{aligned} g_i(x) &= \overline{g}(x), \text{ if } x \in F_i, \\ g_i(x) &= 2i, \text{ if } \ x \in X \setminus F_i. \end{aligned}$$

We begin by showing that

$$|u_i(x) - u_i(y)| \leq d(x, y)(g_i(x) + g_i(y)), \tag{3.7}$$

for $x, y \in X \setminus F$. If $x, y \in F_i$, then (3.7) is evident. For $y \in X \setminus F_i$, we have

$$\begin{aligned} |u_i(x) - u_i(y)| &\leq 2id(x, y) \leq d(x, y)(g_i(x) + g_i(y)), \text{ if } x \in X \setminus F_i, \\ |u_i(x) - u_i(y)| &\leq 2id(x, y) \leq d(x, y)(\overline{g}(x) + 2i), \text{ if } x \in X \setminus F_i. \end{aligned}$$

This implies that (3.7) is true and thus $g_i \in D(u_i)$. Now we have

$$\begin{aligned} |||g_i|||_\Phi &\leq |||g_i|||_{\Phi,F_i} + 2i|||1|||_{\Phi,X\setminus F_i} \\ &\leq |||\overline{g}|||_{\Phi,F_i} + \frac{2i}{\Phi^{-1}(\frac{1}{\mu(X\setminus F_i)})} < \infty, \end{aligned}$$

and

$$\begin{aligned} |||u_i|||_\Phi &\leq |||\overline{u}|||_{\Phi,F_i} + 2i|||1|||_{\Phi,X\setminus F_i} \\ &\leq |||\overline{u}|||_{\Phi,F_i} + \frac{2i}{\Phi^{-1}(\frac{1}{\mu(X\setminus F_i)})} < \infty. \end{aligned}$$

Hence $u_i \in M^1_\Phi(X)$. It follows that $u_i \in Lip^{1,0}_\Phi(O)$.

It remains to prove that $u_i \to \overline{u}$ in $M^1_\Phi(X)$. By (3.6) we have

$$\mu(X \setminus F_i) \leq \mu(\{x \in X : |\overline{u}(x)| > i\}) + \mu(\{x \in X : \ \overline{g}(x) > i\}).$$

Since $\overline{u} \in \mathbf{L}_\Phi(X)$ and Φ satisfies the Δ_2 condition, we get

$$\int_{\{x\in X:|\overline{u}(x)|>i\}} \Phi(\overline{u}(x))d\mu(x) \geq \Phi(i)\mu\{x \in X : |\overline{u}(x)| > i\},$$

which implies that $\Phi(i)\mu\{x \in X : |\overline{u}(x)| > i\} \to 0$ as $i \to \infty$.

By the same argument we deduce that $\Phi(i)\mu\{x \in X : \overline{g}(x) > i\} \to 0$ as $i \to \infty$.

Thus

$$\Phi(i)\mu(X \setminus F_i) \to 0 \text{ as } i \to \infty. \tag{3.8}$$

Using the convexity of Φ and the fact that Φ satisfies the Δ_2 condition, we get

$$\begin{aligned}\int_X \Phi(\overline{u} - u_i)d\mu &\leq \int_{X\setminus F_i} \Phi(|\overline{u}| + |u_i|)d\mu \\ &\leq \tfrac{C}{2}[\int_{X\setminus F_i} \Phi \circ |\overline{u}|\, d\mu + \Phi(i)\mu(X \setminus F_i)] \to 0 \text{ as } i \to \infty.\end{aligned}$$

On the other hand, for each $i \in \mathbb{N}^*$ we define the function h_i by

$$\begin{aligned} h_i(x) &= \overline{g}(x) + 3i, \text{ if } x \in X \setminus F_i, \\ h_i(x) &= 0, \text{ if } x \in F_i. \end{aligned}$$

We claim that $h_i \in D(\overline{u} - u_i) \cap \mathbf{L}_\Phi(X)$. In fact, the only nontrivial case is $x \in F_i$ and $y \in X \setminus F_i$; but then

$$\begin{aligned} |(\overline{u} - u_i)(x) - (\overline{u} - u_i)(y)| &\leq d(x, y)(\overline{g}(x) + \overline{g}(y) + 2i) \\ &\leq d(x, y)(\overline{g}(y) + 3i). \end{aligned}$$

By the convexity of Φ and by the Δ_2 condition we have

$$\begin{aligned}\int_X \Phi \circ h_i d\mu &\leq \int_{X\setminus F_i} \Phi \circ (\overline{g} + 3i)d\mu \\ &\leq C[\int_{X\setminus F_i} \Phi \circ \overline{g} d\mu + \Phi(i)\mu(X \setminus F_i)] \to 0 \text{ as } i \to \infty.\end{aligned}$$

This implies that $|||h_i|||_\Phi \to 0$ as $i \to \infty$ since Φ verifies the Δ_2 condition.

Now

$$|||\overline{u} - u_i|||_{\mathbf{L}^1_\Phi(X)} \leq |||h_i|||_\Phi \to 0 \text{ as } i \to \infty.$$

Thus $\overline{u} \in H^{1,0}_\Phi(O)$. The proof is complete. ∎

Definition 5. *A locally finite Borel measure μ is doubling if there is a positive constant C such that for every $x \in X$ and $r > 0$,*

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Definition 6. *A nonempty set $E \subset X$ is uniformly μ-thick if there are constants $C > 0$ and $0 < r_0 \leq 1$ such that*

$$\mu(B(x, r) \cap E) \geq C\mu(B(x, r)),$$

for every $x \in E$, and $0 < r < r_0$.

Now we give a Hardy type inequality in the context of Orlicz-Sobolev spaces.

Theorem 10. *Let Φ be an $\mathcal{N}$-function such that Φ^* satisfies the Δ_2 condition and suppose that μ is doubling. Let $O \subset X$ be an open set such that $X \setminus O$ is uniformly μ-thick. Then there is a constant $C > 0$ such that for every $u \in M_\Phi^{1,0}(O)$,*

$$|||v|||_{\Phi,O} \leq C|||u|||_{M_\Phi^{1,0}(O)},$$

where v is the function defined on O by $v(x) = \dfrac{u(x)}{dist(x, X \setminus O)}$. The constant C is independent of u.

Proof. Let $u \in M_\Phi^{1,0}(O)$ and $\widetilde{u} \in M_\Phi^1(O)$ be Φ-quasicontinuous such that $u = \widetilde{u}$ μ-a.e. in O and $\widetilde{u} = 0$ Φ-q.e. in $X \setminus O$. Let $g \in D(\widetilde{u}) \cap \mathbf{L}_\Phi(X)$ and set $O' = \{x \in O : \text{dist}(x, X \setminus O) < r_0\}$. For $x \in O'$, we choose $x_0 \in X \setminus O$ such that $r_x = \text{dist}(x, X \setminus O) = d(x, x_0)$. Recall that the Hardy-Littlewood maximal function of a locally μ-integrable function f is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)d\mu(y).$$

Using the uniform μ-thickness and the doubling condition, we get

$$\begin{aligned}
\frac{1}{\mu(B(x_0,r_x) \setminus O)} \int_{B(x_0,r_x)\setminus O} g(y)d\mu(y) &\leq \frac{C}{\mu(B(x_0,r_x))} \int_{B(x_0,r_x)} g(y)d\mu(y) \\
&\leq \frac{C}{\mu(B(x,2r_x))} \int_{B(x,2r_x)} g(y)d\mu(y) \\
&\leq C\mathcal{M}g(x).
\end{aligned}$$

On the other hand, for μ-a.e. $x \in O'$ there is $y \in B(x_0, r_x) \setminus O$ such that

$$\begin{aligned}
|u(x)| &\leq d(x,y)(g(x) + \frac{1}{\mu(B(x_0,r_x) \setminus O)} \int_{B(x_0,r_x)\setminus O} g(y)d\mu(y)) \\
&\leq Cr_x(g(x) + \mathcal{M}g(x)) \\
&\leq C\text{dist}(x, X \setminus O)\mathcal{M}g(x).
\end{aligned}$$

By [5], $\mathcal{M}$ is a bounded operator from $\mathbf{L}_\Phi(X)$ to itself since Φ^* satisfies the Δ_2 condition. Hence

$$|||v|||_{\Phi,O'} \leq C|||\mathcal{M}g|||_\Phi \leq C|||g|||_\Phi.$$

On $O \setminus O'$ we have

$$|||v|||_{\Phi,O\setminus O'} \leq r_0^{-1}|||u|||_{\Phi,O}.$$

Thus

$$|||v|||_{\Phi,O} \leq C(|||\widetilde{u}|||_{\Phi} + |||g|||_{\Phi}).$$

By taking the infimum over all $g \in D(\widetilde{u}) \cap \mathbf{L}_{\Phi}(X)$, we get the desired result. ∎

By Theorem 9 and Theorem 10 we obtain the following corollaries

Corollary 2. *Let Φ be an $\mathcal{N}$-function such that Φ and Φ^* satisfy the Δ_2 condition and suppose that μ is doubling. Let $O \subset X$ be an open set such that $X \setminus O$ is uniformly μ-thick. Then $M_{\Phi}^{1,0}(O) = H_{\Phi}^{1,0}(O)$.*

Corollary 3. *Let Φ be an $\mathcal{N}$-function such that Φ and Φ^* satisfy the Δ_2 condition and suppose that μ is doubling. Let $O \subset X$ be an open set such that $X \setminus O$ is uniformly μ-thick and let $(u_i)_i \subset M_{\Phi}^{1,0}(O)$ be a bounded sequence in $M_{\Phi}^{1,0}(O)$. If $u_i \to u$ μ-a.e., then $u \in M_{\Phi}^{1,0}(O)$.*

In the hypotheses of Corollary 3 we get $M_{\Phi}^{1,0}(O) = H_{\Phi}^{1,0}(O)$. Hence the following property $(\mathcal{P})$ is satisfied for sets E whose complement is μ-thick:
$(\mathcal{P})$ Let $(u_i)_i$ be a bounded sequence in $H_{\Phi}^{1,0}(E)$. If $u_i \to u$ μ-a.e., then $u \in H_{\Phi}^{1,0}(E)$.

Remark 3. *If $M_{\Phi}^{1}(X)$ is reflexive, then by Mazur's lemma closed convex sets are weakly closed. Hence every open subset O of X satisfies property $(\mathcal{P})$. But in general we do not know whether the space $M_{\Phi}^{1}(X)$ is reflexive or not.*

Recall that a space X is proper if bounded closed sets in X are compact.

Theorem 11. *Let Φ be an $\mathcal{N}$-function satisfying the Δ_2 condition and suppose that X is proper. Let O be an open set in X satisfying property $(\mathcal{P})$. Then $M_{\Phi}^{1,0}(O) = H_{\Phi}^{1,0}(O)$.*

Proof. It suffices to prove that $M_{\Phi}^{1,0}(O) \subset H_{\Phi}^{1,0}(O)$. Let $u \in M_{\Phi}^{1,0}(O)$ be a Φ-quasicontinuous function from $M_{\Phi}^{1}(X)$ such that $u = 0$ Φ-q.e. on $X \setminus O$. By using the property $(\mathcal{P})$, we deduce, by truncating and considering the positive and the negative parts separately, that we can assume that u is bounded and non-negative. If $x_0 \in O$ is a fixed point, define the sequence $(\eta_i)_i$ by

$$\eta_i(x) = \begin{cases} 1 & \text{if } d(x_0, x) \leq i - 1, \\ i - d(x_0, x) & \text{if } i - 1 < d(x_0, x) < i \\ 0 & \text{if } d(x_0, x) \geq i. \end{cases}$$

If we define the sequence $(v_i)_i$ by $v_i = u\eta_i$, then since $v_i \to u$ μ-a.e. in X and $|||v_i|||_{M^1_\Phi(X)} \leq 2|||u|||_{M^1_\Phi(X)}$, by the property $(\mathcal{P})$ it clearly suffices to show that $v_i \in H^{1,0}_\Phi(O)$. Remark that

$$\begin{aligned} |v_i(x) - v_i(y)| &\leq |u(x) - u(y)| + |\eta_i(x) - \eta_i(y)| \\ &\leq d(x,y)(g(x) + g(y) + u(x). \end{aligned}$$

Hence $v_i \in M^1_\Phi(X)$.

Now fix i and set $v = v_i$. Since v vanishes outside a bounded set, we can find a bounded open subset $U \subset O$ such that $v = 0$ Φ-q.e. in $X \setminus U$. We choose a sequence $(w_j) \subset M^1_\Phi(X)$ of quasicontinuous functions such that $0 \leq w_j \leq 1$, $w_j = 1$ on an open set O_j, with $|||w_j|||_{M^1_\Phi(X)} \to 0$, and so that the restrictions $v_{|X\setminus O_j}$ are continuous and $v = 0$ in $X \setminus (U \cup O_j)$. The sequence $(s_j)_j$, defined by $s_j = (1 - w_j)\max(v - \frac{1}{j}, 0)$, is bounded in $M^1_\Phi(X)$, and passing if necessary to a subsequence, $s_j \to v$ μ-a.e. Since $v_{|X\setminus O_j}$ is continuous, we get

$$\overline{\{x \in X : s_j(x) \neq 0\}} \subset \left\{x \in X : v(x) \geq \frac{1}{j}\right\} \setminus O_j \subset U.$$

This means that $\overline{\{x \in X : s_j(x) \neq 0\}}$ is a compact subset of O, whence by Theorem 9, $s_j \in H^{1,0}_\Phi(O)$. The property $(\mathcal{P})$ implies $v \in H^{1,0}_\Phi(O)$ and the proof is complete. ∎

Corollary 4. *Let Φ be an $\mathcal{N}$-function satisfying the Δ_2 condition and suppose that X is proper. Let O be an open set in X and suppose that $M^1_\Phi(X)$ is reflexive. Then $M^{1,0}_\Phi(O) = H^{1,0}_\Phi(O)$.*

Proof. By Remark 3, O satisfies property $(\mathcal{P})$, and Theorem 11 gives the result. ∎

4. Orlicz-Sobolev space with zero boundary values $N^{1,0}_\Phi(E)$

4.1. **The Orlicz-Sobolev space $N^1_\Phi(X)$.** We recall the definition of the space $N^1_\Phi(X)$.

Let (X, d, μ) be a metric, Borel measure space, such that μ is positive and finite on balls in X.

If I is an interval in $\mathbb{R}$, a path in X is a continuous map $\gamma : I \to X$. By abuse of language, the image $\gamma(I) =: |\gamma|$ is also called a path. If $I = [a, b]$ is a closed interval, then the length of a path $\gamma : I \to X$ is

$$l(\gamma) = \text{length}(\gamma) = \sup \sum_{i=1}^{n} |\gamma(t_{i+1}) - \gamma(t_i)|,$$

where the supremum is taken over all finite sequences $a = t_1 \leq t_2 \leq \ldots \leq t_n \leq t_{n+1} = b$. If I is not closed, we set $l(\gamma) = \sup l(\gamma\,|_J)$, where the supremum is taken over all closed sub-intervals J of I. A path is said to be rectifiable if its length is a finite number. A path $\gamma : I \to X$ is locally rectifiable if its restriction to each closed sub-interval of I is rectifiable.

For any rectifiable path γ, there are its associated length function $s_\gamma : I \to [0, l(\gamma)]$ and a unique 1-Lipschitz continuous map $\gamma_s : [0, l(\gamma)] \to X$ such that $\gamma = \gamma_s \circ s_\gamma$. The path γ_s is the arc length parametrization of γ.

Let γ be a rectifiable path in X. The line integral over γ of each non-negative Borel function $\rho : X \to [0, \infty]$ is $\int_\gamma \rho ds = \int_0^{l(\gamma)} \rho \circ \gamma_s(t) dt$.

If the path γ is only locally rectifiable, we set $\int_\gamma \rho ds = \sup \int_{\gamma'} \rho ds$, where the supremum is taken over all rectifiable sub-paths γ' of γ. See [5] for more details.

Denote by Γ_{rect} the collection of all non-constant compact (that is, I is compact) rectifiable paths in X.

Definition 7. *Let Φ be an $\mathcal{N}$-function and Γ be a collection of paths in X. The Φ-modulus of the family Γ, denoted $Mod_\Phi(\Gamma)$, is defined as*

$$\inf_{\rho \in \mathcal{F}(\Gamma)} |||\rho|||_\Phi,$$

where $\mathcal{F}(\Gamma)$ is the set of all non-negative Borel functions ρ such that $\int_\gamma \rho ds \geq 1$ for all rectifiable paths γ in Γ. Such functions ρ used to define the Φ-modulus of Γ are said to be admissible for the family Γ.

From the above definition the Φ-modulus of the family of all non-rectifiable paths is 0.

A property relevant to paths in X is said to hold for Φ-*almost all paths* if the family of rectifiable compact paths on which that property does not hold has Φ-modulus zero.

Definition 8. *Let u be a real-valued function on a metric space X. A non-negative Borel-measurable function ρ is said to be an upper gradient of u if for all compact rectifiable paths γ the following inequality holds*

$$|u(x) - u(y)| \leq \int_\gamma \rho ds, \tag{4.1}$$

where x and y are the end points of the path.

Definition 9. *Let Φ be an $\mathcal{N}$-function and let u be an arbitrary real-valued function on X. Let ρ be a non-negative Borel function on X. If there exists a family $\Gamma \subset \Gamma_{rect}$ such that $Mod_\Phi(\Gamma) = 0$ and the inequality (4.1) is true for all paths γ in $\Gamma_{rect} \setminus \Gamma$, then ρ is said to* *be a Φ-weak upper gradient of u. If inequality (4.1) holds true for Φ-modulus almost all paths in a set $B \subset X$, then ρ is said to be a Φ-weak upper gradient of u on B.*

Definition 10. *Let Φ be an $\mathcal{N}$-function and let the set $\widetilde{N^1_\Phi}(X,d,\mu)$ be the collection of all real-valued function u on X such that $u \in \mathbf{L}_\Phi$ and u have a Φ-weak upper gradient in $\mathbf{L}_\Phi$. If $u \in \widetilde{N^1_\Phi}$, we set*

$$|||u|||_{\widetilde{N^1_\Phi}} = |||u|||_\Phi + \inf_\rho |||\rho|||_\Phi, \tag{4.2}$$

where the infimum is taken over all Φ-weak upper gradient, ρ, of u such that $\rho \in \mathbf{L}_\Phi$.

Definition 11. *Let Φ be an $\mathcal{N}$-function. The Orlicz-Sobolev space corresponding to Φ, denoted $N^1_\Phi(X)$, is defined to be the space $\widetilde{N^1_\Phi}(X,d,\mu)/\backsim$, with norm $|||u|||_{N^1_\Phi} := |||u|||_{\widetilde{N^1_\Phi}}$.*

For more details and developments, see [3].

4.2. The Orlicz-Sobolev space with zero boundary values $N^{1,0}_\Phi(E)$.

Definition 12. *Let Φ be an $\mathcal{N}$-function. For a set $E \subset X$ define $Cap_\Phi(E)$ by*

$$Cap_\Phi(E) = \inf\left\{|||u|||_{N^1_\Phi} : u \in \mathcal{D}(E)\right\},$$

where $\mathcal{D}(E) = \{u \in N^1_\Phi : u\,|_E \geq 1\}$.

If $\mathcal{D}(E) = \emptyset$, we set $Cap_\Phi(E) = \infty$. Functions belonging to $\mathcal{D}(E)$ are called admissible functions for E.

Definition 13. *Let Φ be an $\mathcal{N}$-function and E a subset of X. We define $\widetilde{N^{1,0}_\Phi}(E)$ as the set of all functions $u : E \to [-\infty,\infty]$ for which there exists a function $\widetilde{u} \in \widetilde{N^1_\Phi}(E)$ such that $\widetilde{u} = u$ μ-a.e. in E and $\widetilde{u} = 0$ Cap_Φ-q.e. in $X \setminus E$; which means $Cap_\Phi(\{x \in X \setminus E : \widetilde{u}(x) \neq 0\}) = 0$.*

Let $u, v \in \widetilde{N^{1,0}_\Phi}(E)$. We say that $u \sim v$ if $u = v$ μ-a.e. in E. The relation $\sim$ is an equivalence relation and we set $N^{1,0}_\Phi(E) = \widetilde{N^{1,0}_\Phi}(E)/\backsim$. We equip this space with the norm $|||u|||_{N^{1,0}_\Phi(E)} := |||u|||_{N^1_\Phi(X)}$.

It is easy to see that for every set $A \subset X$, $\mu(A) \leq Cap_\Phi(A)$. On the other hand, by [3, Corollary 2] if $\widetilde{u}$ and $\widetilde{u}'$ both correspond to u in the above definition, then $|||\widetilde{u} - \widetilde{u}'|||_{N^1_\Phi(X)} = 0$. This means that the norm on $N^{1,0}_\Phi(E)$ is well defined.

Definition 14. *Let Φ be an $\mathcal{N}$-function and E a subset of X. We set* $Lip^{1,0}_{\Phi,N}(E) = \left\{u \in N^1_\Phi(X) : u \text{ is Lipschitz in } X \text{ and } u = 0 \text{ in } X \setminus E\right\}$, *and*

$$Lip^{1,0}_{\Phi,C}(E) = \left\{u \in Lip^{1,0}_{\Phi,N}(E) : u \text{ has compact support}\right\}.$$

We let $H^{1,0}_{\Phi,N}(E)$ be the closure of $Lip^{1,0}_{\Phi,N}(E)$ in the norm of $N^1_\Phi(X)$, and $H^{1,0}_{\Phi,C}(E)$ be the closure of $Lip^{1,0}_{\Phi,C}(E)$ in the norm of $N^1_\Phi(X)$.

By definition $H^{1,0}_{\Phi,N}(E)$ and $H^{1,0}_{\Phi,C}(E)$ are Banach spaces. We prove that $N^{1,0}_\Phi(E)$ is also a Banach space.

Theorem 12. *Let Φ be an $\mathcal{N}$-function and E a subset of X. Then $N^{1,0}_\Phi(E)$ is a Banach space.*

Proof. Let $(u_i)_i$ be a Cauchy sequence in $N^{1,0}_\Phi(E)$. Then there is a corresponding Cauchy sequence $(\widetilde{u}_i)_i$ in $N^1_\Phi(X)$, where $\widetilde{u}_i$ is the function corresponding to u_i as in the definition of $N^{1,0}_\Phi(E)$. Since $N^1_\Phi(X)$ is a Banach space, see [3, Theorem 1], there is a function $\widetilde{u} \in N^1_\Phi(X)$, and a subsequence, also denoted $(\widetilde{u}_i)_i$ for simplicity, so that as in the proof of [3, Theorem 1], $\widetilde{u}_i \to \widetilde{u}$ pointwise outside a set T with $Cap_\Phi(T) = 0$, and also in the norm of $N^1_\Phi(X)$. For every i, set $A_i = \{x \in X \setminus E : \widetilde{u}_i(x) \neq 0\}$. Then $Cap_\Phi(\cup_i A_i) = 0$. Moreover, on $(X \setminus E) \setminus (\cup_i A_i \cup T)$, we have $\widetilde{u}(x) = \lim\limits_{i\to\infty} \widetilde{u}_i(x) = 0$.

Since $Cap_\Phi(\cup_i A_i \cup T) = 0$, the function $u = \widetilde{u}_{|E}$ is in $N^{1,0}_\Phi(E)$. On the other hand we have

$$|||u - u_i|||_{N^{1,0}_\Phi(E)} = |||\widetilde{u} - \widetilde{u}_i|||_{N^1_\Phi(X)} \to 0 \text{ as } i \to \infty.$$

Thus $N^{1,0}_\Phi(E)$ is a Banach space and the proof is complete. ∎

Proposition 1. *Let Φ be an $\mathcal{N}$-function and E a subset of X. Then the space $H^{1,0}_{\Phi,N}(E)$ embeds isometrically into $N^{1,0}_\Phi(E)$, and the space $H^{1,0}_{\Phi,C}(E)$ embeds isometrically into $H^{1,0}_{\Phi,N}(E)$.*

Proof. Let $u \in H^{1,0}_{\Phi,N}(E)$. Then there is a sequence $(u_i)_i \subset N^1_\Phi(X)$ of Lipschitz functions such that $u_i \to u$ in $N^1_\Phi(X)$ and for each integer i, $u_{i|X\setminus E} = 0$. Considering if necessary a subsequence of $(u_i)_i$, we proceed as in the proof of [3, Theorem 1], we can consider the function $\widetilde{u}$ defined outside a set S with $Cap_\Phi(S) = 0$, by $\widetilde{u} = \frac{1}{2}(\limsup\limits_i u_i + \liminf\limits_i u_i)$. Then $\widetilde{u} \in N^1_\Phi(X)$ and $u_{|E} = \widetilde{u}_{|E}$ μ-a.e and $\widetilde{u}_{|(X\setminus E)\setminus S} = 0$. Hence $u_{|E} \in N^{1,0}_\Phi(E)$, with the two norms equal. Since $H^{1,0}_{\Phi,C}(E) \subset Lip^{1,0}_{\Phi,N}(E)$, it is easy to see that $H^{1,0}_{\Phi,C}(E)$ embeds isometrically into $H^{1,0}_{\Phi,N}(E)$. The proof is complete. ∎

When $\Phi(t) = \frac{1}{p}t^p$, there are examples of spaces X and $E \subset X$ for which $N^{1,0}_{\Phi}(E)$, $H^{1,0}_{\Phi,N}(E)$ and $H^{1,0}_{\Phi,C}(E)$ are different. See [13]. We give, in the sequel, sufficient conditions under which these three spaces agree. We begin by a definition and some lemmas.

Definition 15. *Let Φ be an $\mathcal{N}$-function. The space X is said to support a $(1,\Phi)$-Poincaré inequality if there is a constant $C > 0$ such that for all balls $B \subset X$, and all pairs of functions u and ρ, whenever ρ is an upper gradient of u on B and u is integrable on B, the following inequality holds*

$$\frac{1}{\mu(B)}\int_B |u - u_E| \le C diam(B)\, |||g|||_{\mathbf{L}_\Phi(B)}\, \Phi^{-1}(\frac{1}{\mu(B)}).$$

Lemma 3. *Let Φ be an $\mathcal{N}$-function and Y a metric measure space with a Borel measure μ that is finite on bounded sets. Let $u \in N^1_\Phi(Y)$ be non-negative and define the sequence $(u_i)_i$ by $u_i = \min(u, i)$, $i \in \mathbb{N}$. Then $(u_i)_i$ converges to u in the norm of $N^1_\Phi(Y)$.*

Proof. Set $E_i = \{x \in Y : u(x) > i\}$. If $\mu(E_i) = 0$, then $u_i = u$ μ-a.e. and since $u_i \in N^1_\Phi(Y)$, by [3, Corollary 2] the $N^1_\Phi(Y)$ norm of $u - u_i$ is zero for sufficiently large i. Now, suppose that $\mu(E_i) > 0$. Since μ is finite on bounded sets, it is an outer measure. Hence there is an open set O_i such that $E_i \subset O_i$ and $\mu(O_i) \le \mu(E_i) + 2^{-i}$.

We have

$$\frac{1}{i}\, |||u|||_{\mathbf{L}_\Phi(E_i)} \ge |||1|||_{\mathbf{L}_\Phi(E_i)} = \frac{1}{\Phi^{-1}(\frac{1}{\mu(E_i)})}.$$

Since Φ^{-1} is continuous, increasing and verifies $\Phi(x) \to \infty$ as $x \to \infty$, we get

$$\frac{1}{\Phi^{-1}(\frac{1}{\mu(O_i)-2^{-i}})} \le \frac{1}{i}\, |||u|||_{\mathbf{L}_\Phi} \to 0 \text{ as } i \to \infty,$$

and

$$\mu(O_i) \to 0 \text{ as } i \to \infty.$$

Note that $u = u_i$ on $Y \setminus O_i$. Thus $u - u_i$ has $2g\chi_{O_i}$ as a weak upper gradient whenever g is an upper gradient of u and hence of u_i as well; see [3, Lemma 9]. Thus $u_i \to u$ in $N^1_\Phi(Y)$. The proof is complete. ∎

Remark 4. *By [3, Corollary 7], and in conditions of this corollary, if $u \in N^1_\Phi(X)$, then for each positive integer i, there is a $w_i \in N^1_\Phi(X)$ such that $0 \le w_i \le 1$, $|||w_i|||_{N^1_\Phi(X)} \le 2^{-i}$, and $w_{i|F_i} = 1$, with F_i an open subset of X such that u is continuous on $X \setminus F_i$.*

We define, as in the proof of Theorem 11, for $i \in \mathbb{N}^*$, the function t_i by

$$t_i = (1 - w_i)\max(u - \tfrac{1}{i}, 0).$$

Lemma 4. *Let Φ be an $\mathcal{N}$-function satisfying the Δ' condition. Let X be a proper doubling space supporting a $(1,\Phi)$-Poincaré inequality, and let $u \in N^1_\Phi(X)$ be such that $0 \le u \le M$, where M is a constant. Suppose that the set $A = \{x \in X : u(x) \neq 0\}$ is a bounded subset of X. Then $t_i \to u$ in $N^1_\Phi(X)$.*

Proof. Set $E_i = \left\{x \in X : u(x) < \frac{1}{i}\right\}$. By [3, Corollary 7] and by the choice of F_i, there is an open set U_i such that $E_i \setminus F_i = U_i \setminus F_i$. Pose $V_i = U_i \cup F_i$ and remark that $w_{i|F_i} = 1$ and $u_{|E_i} < \frac{1}{i}$. Then $\{x \in X : t_i(x) \neq 0\} \subset A \backslash V_i \subset A$. If we set $v_i = u - t_i$, then $0 \le v_i \le M$ since $0 \le t_i \le u$. We can easily verify that $t_i = (1 - w_i)(u - 1/i)$ on $A \setminus V_i$, and $t_i = 0$ on V_i. Therefore

$$v_i = w_i u + (1 - w_i)\big/ i \text{ on } A \setminus V_i, \tag{4.3}$$

and

$$v_i = u \text{ on } V_i. \tag{4.4}$$

Let $x, y \in X$. Then

$$\begin{aligned} |w_i(x)u(x) - w_i(y)u(y)| &\le |w_i(x)u(x) - w_i(x)u(y)| + |w_i(x)u(y) - w_i(y)u(y)| \\ &\le w_i(x)\,|u(x) - u(y)| + M\,|w_i(x) - w_i(y)|\,. \end{aligned}$$

Let ρ_i be an upper gradient of w_i such that $|||\rho_i|||_{\mathbf{L}_\Phi} \le 2^{-i+1}$ and let ρ be an upper gradient of u belonging to $\mathbf{L}_\Phi$. If γ is a path connecting two points $x, y \in X$, then

$$|w_i(x)u(x) - w_i(y)u(y)| \le w_i(x)\int_\gamma \rho ds + M\int_\gamma \rho_i ds.$$

Hence, if $z \in |\gamma|$, then

$$\begin{aligned} |w_i(x)u(x) - w_i(y)u(y)| &\le |w_i(x)u(x) - w_i(z)u(z)| + |w_i(z)u(z) - w_i(y)u(y)| \\ &\le w_i(z)\int_{\gamma_{xz}} \rho ds + M\int_{\gamma_{xz}} \rho_i ds + w_i(z)\int_{\gamma_{zy}} \rho ds + M\int_{\gamma_{zy}} \rho_i ds \\ &\le w_i(z)\int_\gamma \rho ds + M\int_\gamma \rho_i ds, \end{aligned}$$

where γ_{xz} and γ_{zy} are such that the concatenation of these two segments gives the original path γ back again. Therefore

$$|w_i(x)u(x) - w_i(y)u(y)| \le \int_\gamma \left(\inf_{z\in|\gamma|} w_i(z)\rho + M\rho_i\right) ds.$$

Thus

$$|w_i(x)u(x) - w_i(y)u(y)| \leq \int_\gamma (w_i(z)\rho + M\rho_i)ds.$$

This means that $w_i\rho + M\rho_i$ is an upper gradient of w_iu. Since $|||w_i|||_{\mathbf{L}_\Phi} \leq 2^{-i}$, we get that $w_i \to 0$ μ-a.e. On the other hand $w_i\rho \leq \rho$ on X implies that $w_i\rho \in \mathbf{L}_\Phi$ and hence $\Phi \circ (w_i\rho) \in \mathbf{L}^1$ because Φ verifies the Δ_2 condition. Since Φ is continuous, $\Phi \circ (w_i\rho) \to 0$ μ-a.e. The Lebesgue dominated convergence theorem gives $\int_X \Phi \circ (w_i\rho)dx \to 0$ as $i \to \infty$. Thus $|||w_i\rho|||_{\mathbf{L}_\Phi} \to 0$ as $i \to \infty$ since Φ verifies the Δ_2 condition.

Let B be a bounded open set such that $A \subset BT$. Then $O_i = (A \cup F_i) \cap B$ is a bounded open subset of A and $O_i \subset A$. Therefore since $O_i \cap V_i \subset (E_i \cap A) \cup F_i$, we get

$$\begin{aligned} \mu(O_i \cap V_i) &\leq \mu(E_i \cap A) + \mu(F_i) \\ &\leq \mu\left(\left\{x \in X : 0 < u(x) < \frac{1}{i}\right\}\right) + Cap_\Phi(F_i). \end{aligned}$$

Hence $\mu(O_i \cap V_i) \to 0$ as $i \to \infty$, since bounded sets have finite measure and therefore $\mu\left(\left\{x \in X : 0 < u(x) < \frac{1}{i}\right\}\right) \to \mu(\emptyset) = 0$ as $i \to \infty$. Thus $|||\rho|||_{\mathbf{L}_\Phi(O_i \cap V_i)} \to 0$ as $i \to \infty$.

By [3, Lemma 8] and equations (4.3) and (4.4), we get

$$g_i := \left(w_i\rho + M\rho_i + \frac{1}{i}\rho_i\right)\chi_{O_i} + \rho\chi_{O_i \cap V_i}$$

is a weak upper gradient of v_i and since

$$|||g_i|||_{\mathbf{L}_\Phi} \leq |||w_i\rho|||_{\mathbf{L}_\Phi} + (M + \tfrac{1}{i})\,|||\rho_i|||_{\mathbf{L}_\Phi} + |||\rho|||_{\mathbf{L}_\Phi(O_i \cap V_i)},$$

we infer that $|||g_i|||_{\mathbf{L}_\Phi} \to 0$ as $i \to \infty$.

On the other hand, we have

$$\begin{aligned} |||v_i|||_{\mathbf{L}_\Phi} &= |||u - t_i|||_{\mathbf{L}_\Phi} \leq |||w_iu|||_{\mathbf{L}_\Phi(A \setminus V_i)} + \frac{1}{i}\,|||1 - w_i|||_{\mathbf{L}_\Phi(A \setminus V_i)} + |||u|||_{\mathbf{L}_\Phi(O_i \cap V_i)} \\ &\leq M|||w_i|||_{N^1_\Phi(X)} + \frac{1}{i}\frac{1}{\Phi^{-1}(\frac{1}{\mu(A)})} + |||u|||_{\mathbf{L}_\Phi(O_i \cap V_i)}. \end{aligned}$$

Since $|||w_i|||_{N^1_\Phi(X)} \to 0$ and $|||u|||_{\mathbf{L}_\Phi(O_i \cap V_i)} \to 0$ as $i \to \infty$, we conclude that $|||v_i|||_{\mathbf{L}_\Phi} \to 0$ as $i \to \infty$, and hence $t_i \to u$ in $N^1_\Phi(X)$. The proof is complete. ∎

Theorem 13. *Let Φ be an $\mathcal{N}$-function satisfying the Δ' condition. Let X be a proper doubling space supporting a $(1, \Phi)$-Poincaré inequality and E an open subset of X. Then $N^{1,0}_\Phi(E) = H^{1,0}_{\Phi,N}(E) = H^{1,0}_{\Phi,C}(E)$.*

Proof. By Proposition 1 we know that $H^{1,0}_{\Phi,C}(E) \subset H^{1,0}_{\Phi,N}(E) \subset N^{1,0}_{\Phi}(E)$. It suffices to prove that $N^{1,0}_{\Phi}(E) \subset H^{1,0}_{\Phi,C}(E)$. Let $u \in N^{1,0}_{\Phi}(E)$, and identify u with its extension $\widetilde{u}$. By the lattice properties of $N^1_\Phi(X)$ it is easy to see that u^+ and u^- are both in $N^{1,0}_\Phi(E)$ and hence it suffices to show that u^+ and u^- are in $H^{1,0}_{\Phi,C}(E)$. Thus we can assume that $u \geq 0$. On the other hand, since $N^{1,0}_\Phi(E)$ is a Banach space that is isometrically embedded in $N^1_\Phi(X)$, if $(u_n)_n$ is a sequence in $N^{1,0}_\Phi(E)$ that is Cauchy in $N^1_\Phi(X)$, then its limit, u, lies in $N^{1,0}_\Phi(E)$. Hence by Lemma 3, it also suffices to consider u such that $0 \leq u \leq M$, for some constant M. By [3, Lemma 17], it suffices to consider u such that $A = \{x \in X : u(x) \neq 0\}$ is a bounded set. By Lemma 4, it suffices to show that for each positive integer i, the function $\varphi_i = (1 - w_i)\max(u - \frac{1}{i}, 0)$ is in $H^{1,0}_{\Phi,C}(E)$.

On the other hand, if O_i and F_i are open subsets of X and $Cap_\Phi(F_i) \leq 2^{-i}$, as in the proof of Lemma 4, we have $A \cup F_i = O_i \cup F_i$. Since u has bounded support, we can choose O_i as bounded sets contained in E. We have $w_i\,|_{F_i} = 1$ and hence $\varphi_i\,|_{F_i} = 0$. Set $E_i = \left\{x \in X : u(x) < \frac{1}{i}\right\}$. Then, as in the proof of Lemma 4, there is an open set $U_i \subset E$ such that $E_i \setminus F_i = U_i \setminus F_i$ and $\varphi_i\,|_{F_i \cup U_i} = 0$. Thus

$\overline{\{x : \varphi_i(x) \neq 0\}} \subset \{x \in E : u(x) \geq 1/i\} \setminus F_i = O_i \setminus (E_i \cup F_i) \subset O_i \subset E$.

The support of φ_i is compact because X is proper, and hence $\delta =$dist(supp φ_i, $X \setminus E) > 0$. By [3, Theorem 5], φ_i is approximated by Lipschitz functions in $N^1_\Phi(X)$. Let g_i be an upper gradient of φ_i. By [3, Lemma 9] we can assume that $g_i\,\big|_{X \setminus O_i} = 0$. As in [3], define the operator $\mathcal{M}'$ by $\mathcal{M}'(f)(x) = \sup\limits_B \dfrac{1}{\mu(B)}\Phi(|||f|||_{\mathbf{L}_\Phi(B)})$, where the supremum is taken over all balls $B \subset X$ such that $x \in B$. Then if $x \in X \setminus E$, we get

$$\mathcal{M}'(g_i)(x) = \sup_{x \in B,\ \mathrm{rad}B > \delta/2} \frac{1}{\mu(B)}\Phi(|||g_i|||_{\mathbf{L}_\Phi(B)}) \leq \frac{C'}{(\delta/2)^s}\Phi(|||g_i|||_{\mathbf{L}_\Phi}) < \infty,$$

where $s = \frac{LogC}{Log2}$, C being the doubling constant, and C' is a constant depending only on C and A. We know from [3, Proposition 4] that if $f \in \mathbf{L}_\Phi$, then $\lim\limits_{\lambda \to \infty} \lambda\mu\left\{x \in X : \mathcal{M}'(f)(x) > \lambda\right\} = 0$. Hence in the proof of [3, Theorem 5], choosing $\lambda > \frac{C'}{(\delta/2)^s}\Phi(|||g_i|||_{\mathbf{L}_\Phi})$ ensures that the corresponding Lipschitz approximations agree with the functions φ_i on $X \setminus E$. Thus these Lipschitz approximations are in $H^{1,0}_{\Phi,N}(E)$, and therefore so is φ_i. Moreover, these Lipschitz approximations have compact support in E, and hence $\varphi_i \in H^{1,0}_{\Phi,C}(E)$. The proof is complete. ∎

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