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ON THE APPROXIMATE SOLUTION OF SOME FREDHOLM INTEGRAL EQUATIONS BY NEWTON'S METHOD

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Abstract. The aim of this paper is to apply Newton's method to solve a kind of nonlinear integral equations of Fredholm type. The study follows two directions: firstly we give a theoretical result on existence and uniqueness of solution. Secondly we illustrate with an example the technique for constructing the functional sequence that approaches the solution.

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1. Introduction

In this paper we give an existence and uniqueness of solution result for a nonlinear integral equation of Fredholm type:

$$(1) \qquad \phi(x) = f(x) + \lambda \int_a^b K(x,t)\phi(t)^p \, dt, \quad x \in [a,b], \quad p \geq 2,$$

where λ is a real number, the kernel $K(x,t)$ is a continuous function in $[a,b] \times [a,b]$ and $f(x)$ is a given continuous function defined in $[a,b]$.

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There exist various results about Fredholm integral equations of second kind

$$\phi(x) = f(x) + \lambda \int_a^b K(x, t, \phi(t))\, dt, \quad x \in [a, b]$$

when the kernel $K(x, t, \phi(t))$ is linear in ϕ or it is of Lipschitz type in the third component. These two points have been considered, for instance, in [7] or [3] respectively. However the above equation (1) does not satisfy either of these two conditions.

In [3] we can also find a particular case of (1), for $f(x) = 0$ and $K(x, t)$ a degenerate kernel. In this paper we study the general case. The technique will consist in writing equation (1) in the form:

$$F(\phi) = 0, \tag{2}$$

where $F : \Omega \subseteq X \to Y$ is a nonlinear operator defined by

$$F(\phi)(x) = \phi(x) - f(x) - \lambda \int_a^b K(x, t)\phi(t)^p\, dt, \qquad p \geq 2,$$

and $X = Y = C([a, b])$ is the space of continuous functions on the interval $[a, b]$, equipped with the max-norm

$$\|\phi\| = \max_{x \in [0,1]} |\phi(x)|, \quad \phi \in X.$$

In addition, $\Omega = X$ if $p \in \mathbb{N}, \quad p \geq 2$, and when it will be necessary, $\Omega = C_+([a, b]) = \{\phi \in C([a, b]);\, \phi(t) > 0,\, t \in [a, b]\}$ for $p \in \mathbb{R}$, with $p > 2$.

The aim of this paper is to apply Newton's method to equation (2) in order to obtain a result on the existence and unicity of solution for such equation. This idea has been considered previously in different situations [1], [2], [4], [6].

At it is well known, Newton's iteration is defined by

$$\phi_{n+1} = \phi_n - \Gamma_n F(\phi_n), \quad n \geq 0, \tag{3}$$

where Γ_n is the inverse of the linear operator F'_{ϕ_n}. Notice that for each $\phi \in \Omega$, the first derivative F'_ϕ is a linear operator defined from X to Y by the following formula:

$$F'_\phi[\psi](x) = \psi(x) - \lambda p \int_a^b K(x, t)\phi(t)^{p-1}\psi(t)\, dt, \quad x \in [a, b], \quad \psi \in X. \tag{4}$$

In the second section we establish two main theorems, one about the existence of solution for (2) and other about the unicity of solution for the same equation. In the third section we illustrate these theoretical results with an example. For this particular case, we construct some iterates of Newton's sequence.

2. The main result

Let us denote $N = \max_{x\in[a,b]} \int_a^b |K(x,t)|\, dt$. Let ϕ_0 be a function in Ω such that $\Gamma_0 = [F'_{\phi_0}]^{-1}$ exists and $\|\Gamma_0 F(\phi_0)\| \leq \eta$. We consider the following auxiliary scalar function

(5) $f(t) = 2(\eta - t) + M(\|\phi_0\| + t)^{p-2}\left[(p-1)\eta t - 2(\eta - t)(\|\phi_0\| + t)\right]$,

where, $M = |\lambda| pN$. Let us note that if $p \in \mathbb{N}$, with $p \geq 2$, $f(t)$ is a polynomial of degree $p - 2$. Firstly, we establish the following two technical lemmas:

Lemma 2.1. *Let us assume that the equation $f(t) = 0$ has at least a positive real solution and let us denote by R the smaller one. Then we have the following relations:*

i) $\eta < R$.
ii) $a = M(\|\phi_0\| + R)^{p-1} < 1$.
iii) *If we denote* $b = \dfrac{(p-1)\eta}{2(\|\phi_0\| + R)}$ *and* $h(t) = \dfrac{1}{1-t}$*, then,* $abh(a) < 1$.
iv) $R = \dfrac{\eta}{1 - abh(a)}$.

Proof: First, notice that *iv)* follows from the relation $f(R) = 0$. So, as $R > 0$, we deduce that $abh(a) < 1$, and *iii)* holds. Moreover, $1 > 1 - abh(a) > 0$, then $1 < \dfrac{1}{1 - abh(a)}$, so $\eta < R$, and *i)* also holds.

To prove *ii)*, we consider the relation $f(R) = 0$ that can be written in the form:

$$2(\eta - R)\left[1 - M(\|\phi_0\| + R)^{p-1}\right] = -M\eta(p-1)R(\|\phi_0\| + R)^{p-2} < 0.$$

As $\eta - R < 0$, $1 - M(\|\phi_0\| + R)^{p-1} = 1 - a > 0$, and therefore $a < 1$.

Let us denote $B(\phi_0, R) = \{\phi \in X;\ \|\phi - \phi_0\| < R\}$ and $\overline{B(\phi_0, R)} = \{\phi \in X;\ \|\phi - \phi_0\| \leq R\}$.

Lemma 2.2. *If $B(\phi_0, R) \subseteq \Omega$, the following conditions hold*

i) *For all $\phi \in B(\phi_0, R)$ there exists $[F'_\phi]^{-1}$ and $\|[F'_\phi]^{-1}\| \leq h(a)$.*

ii) *If* $\phi_n, \phi_{n-1} \in B(\phi_0, R)$*, then*

$$\|F(\phi_n)\| \leq \frac{(p-1)a}{2(\|\phi_0\|+R)}\|\phi_n - \phi_{n-1}\|^2.$$

Proof: To prove *i)* we apply the Banach lemma on invertible operators [5]. Taking into account

$$(I - F'_\phi)\psi(x) = \lambda p \int_a^b K(x,t)\phi(t)^{p-1}\psi(t)\,dt,$$

then

$$\|I - F'_\phi\| \leq |\lambda| pN \|\phi\|^{p-1} \leq M(\|\phi_0\|+R)^{p-1} = a < 1,$$

therefore, there exists $[F'_\phi]^{-1}$ and $\|[F'_\phi]^{-1}\| \leq \dfrac{1}{1-a} = h(a)$.

To prove *ii)*, using Taylor's formula, we have

$$F(\phi_n)(x) = \int_0^1 [F'_{\phi_{n-1}+s(\phi_n-\phi_{n-1})} - F'_{\phi_{n-1}}](\phi_n - \phi_{n-1})(x)\,ds$$

$$= -\lambda p \int_0^1 \int_a^b K(x,t)\left[\rho_n(s,t)^{p-1} - \phi_{n-1}(t)^{p-1}\right](\phi_n(t) - \phi_{n-1}(t))\,dt\,ds,$$

$$-\lambda p \int_0^1 \int_a^b K(x,t)\left[\sum_{j=0}^{p-2} \rho_n(s,t)^{p-2-j}\phi_{n-1}(t)^j\right][\phi_n(t) - \phi_{n-1}(t)]^2 s\,dt\,ds,$$

where $\rho_n(s,t) = \phi_{n-1}(t) + s(\phi_n - \phi_{n-1})$ and we have considered the equality

$$x^{p-1} - y^{p-1} = \left(\sum_{j=0}^{p-2} x^{p-2-j}y^j\right)(x-y), \quad x, y \in \mathbb{R}.$$

As $\phi_{n-1}, \phi_n \in B(\phi_0, R)$, for each $s \in [0,1]$, $\rho_n(s,\cdot) \in B(\phi_0, R)$, then $\|\rho_n(s,\cdot)\| \leq \|\phi_0\| + R$. Consequently

$$\|F(\phi_n)\| \leq \frac{|\lambda| pN}{2}\left(\sum_{j=0}^{p-2} (\|\phi_0\|+R))^{p-2-j}\|\phi_{n-1}\|^j\right)\|\phi_n - \phi_{n-1}\|^2$$

$$\leq |\lambda|\frac{p(p-1)N}{2}[\|\phi_0\|+R]^{p-2}\|\phi_n - \phi_{n-1}\|^2 = \frac{(p-1)a}{2(\|\phi_0\|+R)}\|\phi_n - \phi_{n-1}\|^2,$$

and the proof is complete.

Next, we give the following results on existence and uniqueness of solutions for the equation (2). Besides, we obtain that the sequence given by Newton's method has R-order two.

Theorem 2.3. *Let us assume that equation $f(t) = 0$, with f defined in (5) has at least a positive solution and let R be the smaller one. If $B(\phi_0, R) \subseteq \Omega$, then there exists at least a solution ϕ^* of (2) in $\overline{B(\phi_0, R)}$. In addition, the Newton's sequence (3) converges to ϕ^* with at least R-order two.*

Proof: Firstly, as $\|\phi_1 - \phi_0\| \leq \eta < R$, we have $\phi_1 \in B(\phi_0, R)$. Then, Γ_1 exists and $\|\Gamma_1\| \leq h(a)$. In addition,

$$\|F(\phi_1)\| \leq \frac{(p-1)a}{2(\|\phi_0\| + R)}\|\phi_1 - \phi_0\|^2 = ab\eta$$

and therefore

$$\|\phi_2 - \phi_1\| \leq abh(a)\eta.$$

Then, applying *iv)* from Lemma 2.1,

$$\|\phi_2 - \phi_0\| \leq \|\phi_2 - \phi_1\| + \|\phi_1 - \phi_0\| \leq (1 - (abh(a))^2)R < R,$$

and we have that $x_2 \in B(\phi_0, R)$. By induction is easy to prove that

$$\|\phi_n - \phi_{n-1}\| \leq (abh(a))^{2^{n-1}-1}\|\phi_1 - \phi_0\|. \tag{6}$$

In addition, taking into account Bernoulli's inequality, we also have:

$$\|\phi_n - \phi_0\| \leq \left(\sum_{j=0}^{n-1}(abh(a))^{2^j-1}\right)\|\phi_1 - \phi_0\| < \left(\sum_{j=0}^{\infty}(abh(a))^{2^j-1}\right)\eta$$

$$< \left(\sum_{j=0}^{\infty}(abh(a))^{j}\right)\eta = R$$

Consequently, $\phi_n \in B(\phi_0, R)$ for all $n \geq 0$.

Next, we prove that $\{\phi_n\}$ is a Cauchy sequence. From (6), Lemma 2.1 and Bernouilli's inequality, we deduce

$$\|\phi_{n+m} - \phi_n\| \leq \|\phi_{n+m} - \phi_{n+m-1}\| + \|\phi_{n+m-1} - \phi_{n+m-2}\| + \cdots + \|\phi_n - \phi_{n-1}\|$$

$$\leq \left[(abh(a))^{2^{n+m-1}-1} + (abh(a))^{2^{n+m-2}-1} + \cdots + (abh(a))^{2^n-1}\right]\|\phi_1 - \phi_0\|$$

$$\leq (abh(a))^{2^n-1}\left[(abh(a))^{2^n(2^{m-1}-1)} + (abh(a))^{2^n(2^{m-2}-1)} + \cdots + (abh(a))^{2^n} + 1\right]\eta$$

$$< (abh(a))^{2^n-1}\left[(abh(a))^{2^n(m-1)} + (abh(a))^{2^n(m-2)} + \cdots + (abh(a))^{2^n} + 1\right]\eta$$

$$= (abh(a))^{2^n-1}\frac{1 - (abh(a))^{2^n m}}{1 - (abh(a))^{2^n}}\eta.$$

But this last quantity goes to zero when $n \to \infty$. Let $\phi^* = \lim_{n\to\infty} \phi_n$, then, by letting $m \to \infty$, we have

$$\|\phi^*-\phi_n\| \leq (abh(a))^{2^n-1}\frac{\eta}{1-(abh(a))^{2^n}} = \frac{\eta}{(1-(abh(a))^{2^n})(abh(a))}(abh(a))^{2^n}$$

$$\leq \frac{\eta}{(1-(abh(a)))(abh(a))}(abh(a))^{2^n} = C\gamma^{2^n}$$

with $C > 0$ and $\gamma = abh(a) < 1$. This inequality guarantees that $\{\phi_n\}$ has at least R-order of convergence two [8].

Finally, for $n = 0$, we obtain

$$\|\phi^* - \phi_0\| < \frac{\eta}{1-abh(a)} = R$$

then, $\phi^* \in B(\phi_0, R)$. Moreover, as

$$\|F(\phi_n)\| \leq \frac{1}{2}M(p-1)(\|\phi_0\| + R)^{p-2}\|\phi_n - \phi_{n-1}\|^2,$$

when $n \to \infty$ we obtain $F(\phi^*) = 0$, and ϕ^* is a solution of $F(x) = 0$.

Now we give a uniqueness result:

Theorem 2.4. *Let $\|\Gamma_0\| \leq \beta$, then the solution of (2) is unique in $B(\phi_0, \overline{R}) \bigcap \Omega$, with $\overline{R}$ is the bigger positive solution of the equation*

$$\frac{M\beta(p-1)}{2}(2\|\phi_0\| + R + x)^{p-2}(R + x) = 1. \tag{7}$$

Proof: To show the uniqueness, we suppose that $\gamma^* \in B(\phi_0, \overline{R}) \bigcap \Omega$ is another solution of (2). Then

$$0 = \Gamma_0 F(\gamma^*) - \Gamma_0 F(\phi^*) = \int_0^1 \Gamma_0 F'_{\phi^*+s(\gamma^*-\phi^*)}\, ds(\gamma^* - \phi^*).$$

We are going to prove that A^{-1} exists, where A is a linear operator defined by

$$A = \int_0^1 \Gamma_0 F'_{\phi^*+s(\gamma^*-\phi^*)}\, ds,$$

then $\gamma^* = \phi^*$. For this, notice that for each $\psi \in X$ and $x \in [a, b]$, we have

$$(A - I)(\psi)(x) = \int_0^1 \Gamma_0[F'_{\phi^*+s(\gamma^*-\phi^*)} - F'_{\phi_0}]\psi(x)\, ds,$$

$$= -\lambda p \int_0^1 \Gamma_0 \int_a^b K(x,t) \left[\rho^*(s,t)^{p-1} - \phi_0(t)^{p-1}\right] \psi(t)\, dt\, ds$$

$$= -\lambda p \int_0^1 \Gamma_0 \int_a^b K(x,t) \left[\sum_{j=0}^{p-2} \rho^*(s,t)^{p-2-j} \phi_0(t)^j \right] (\rho^*(s,t) - \phi_0(t)) \psi(t)\, dt\, ds,$$

where $\rho^*(s,t) = \phi^*(t) + s(\gamma^*(t) - \phi^*(t))$.

Taking into account that

$$|\rho^*(s,t) - \phi_0(t)| \leq \|\phi^* - \phi_0 + s(\gamma^* - \phi^*)\| \leq (1-s)\|\phi^* - \phi_0\| + s\|\gamma^* - \phi_0\| < (1-s)R + s\overline{R},$$

we obtain

$$\|(A-I)\psi\| \leq |\lambda| pN \|\Gamma_0\| \left[\int_0^1 \left(\sum_{j=0}^{p-2} \|\rho^*(s,\cdot)\|^{p-2-j} \|\phi_0\|^j \right) ((1-s)R + s\overline{R})\, ds \right] \|\psi\|.$$

Therefore, as

$$\|\rho^*(s,\cdot)\| \leq (1-s)\|\phi^*\| + s\|\gamma^*\| \leq (1-s)(\|\phi_0\| + R) + s(\|\phi_0\| + \overline{R}) \leq 2\|\phi_0\| + R + \overline{R},$$

we have, from (7),

$$\|A-I\| \leq \frac{\|\Gamma_0\| M}{2} (R + \overline{R}) \left[\sum_{j=0}^{p-2} \left(\frac{\|\phi_0\|}{2\|\phi_0\| + R + \overline{R}} \right)^j \right] (2\|\phi_0\| + R + \overline{R})^{p-2}$$

$$< \frac{M\beta}{2} (R + \overline{R})(p-1)(2\|\phi_0\| + R + \overline{R})^{p-2} = 1.$$

So, the operator $\displaystyle\int_0^1 F'(\phi^* + t(\gamma^* - \phi^*))\, dt$ has an inverse and consequently, $\gamma^* = \phi^*$. Then, the proof is complete.

3. An example

To illustrate the above theoretical results, we consider the following example

$$\phi(x) = \sin(\pi x) + \frac{1}{5} \int_0^1 \cos(\pi x) \sin(\pi t) \phi(t)^3\, dt, \quad x \in [0,1]. \tag{8}$$

Let $X = C[0,1]$ be the space of continuous functions defined on the interval $[0,1]$, with the max-norm and let $F : X \to X$ be the operator given by

(9)

$$F(\phi)(x) = \phi(x) - \sin(\pi x) - \frac{1}{5} \int_0^1 \cos(\pi x) \sin(\pi t) \phi((t)^3\, dt, \quad x \in [0,1].$$

By differentiating (9) we have:

$$F'_\phi[u](x) = u(x) - \frac{3}{5} \cos(\pi x) \int_0^1 \sin(\pi t) \phi(t)^2 u(t)\, dt. \tag{10}$$

With the notation of section 2,

$$\lambda=\frac{1}{5},\quad N=\max_{x\in[0,1]}\int_0^1|\sin(\pi t)|\,dt=1\quad\text{and}\quad M=|\lambda|pN=\frac{3}{5}.$$

We take as starting-point $\phi_0(x)=\sin(\pi x)$, then we obtain from (10)

$$F'_{\phi_0}[u](x)=u(x)-\frac{3}{5}\cos(\pi x)\int_0^1\sin^3(\pi t)u(t)\,dt$$

If $F'_\phi[u](x)=\omega(x)$, then $[F'_\phi]^{-1}[\omega](x)=u(x)$ and $u(x)=\omega(x)+\frac{3}{5}\cos(\pi x)J_u$, where

$$J_u=\int_0^1\sin(\pi t)\phi(t)^2u(t)\,dt.$$

Therefore the inverse of F'_{ϕ_0} is given by

$$[F'_{\phi_0}]^{-1}[\omega](x)=\omega(x)+\frac{3}{5}\,\frac{\int_0^1\sin^3(\pi t)w(t)\,dt}{1-\frac{3}{5}\int_0^1\cos(\pi t)\sin^3(\pi t)\,dt}\cos(\pi x).$$

Then

$$\|\Gamma_0\|\le\|I+\frac{4}{5\pi}\cos(\pi x)\|\le 1.25468\cdots=\beta,$$

and $\|F(\phi_0)\|\le\frac{3}{40}=0.075$. Consequently $\|\Gamma_0F(\phi_0)\|\le 0.094098\cdots=\eta$.

The equation $f(t)=0$, with f given by (5) is now

$$1.2t^3+2.4t^2-0.912918t+0.0752789=0.$$

This equation has two positive solutions. The smaller one is $R=0.129115\ldots$. Then, by Theorem 2.3, we know there exists a solution of (8) in $\overline{B(\phi_0,R)}$. To obtain the uniqueness domain we consider the equation (7) whose positive solution is the uniqueness ratio. In this case, the solution is unique in $B(\phi_0,0.396793\ldots)$.

Finally, we are going to deal with the computational aspects to solve (8) applying Newton's method (3). To calculate the iterations $\phi_{n+1}(x)=\phi_n(x)-[F'_{\phi_n}]^{-1}[F(\phi_n)](x)$ with the function $\phi_0(x)$ as starting-point, we proceed in the following way:

(1) First we compute the integrals

$$A_n=\int_0^1\sin(\pi t)\phi_n(t)^3\,dt;\quad B_n=\int_0^1\sin(\pi t)^2\phi_n(t)^2\,dt;$$

$$C_n=\int_0^1\cos(\pi t)\sin(\pi t)\phi_n(t)^2\,dt.$$

(2) Next we define

$$\phi_{n+1}(x) = \sin(\pi x) + \frac{1}{5}\frac{-2A_n + 3B_n}{1 - \frac{3}{5}C_n}\cos(\pi x).$$

So we obtain the following approximations

$$\begin{aligned}
\phi_0(x) &= \sin \pi x, \\
\phi_1(x) &= \sin \pi x + 0.075 \cos \pi x, \\
\phi_2(x) &= \sin \pi x + 0.07542667509481667 \cos \pi x, \\
\phi_3(x) &= \sin \pi x + 0.07542668890493719 \cos \pi x, \\
\phi_4(x) &= \sin \pi x + 0.07542668890493714 \cos \pi x, \\
\phi_5(x) &= \sin \pi x + 0.07542668890493713 \cos \pi x,
\end{aligned}$$

As we can see, in this case Newton's method converges to the solution

$$\phi^*(x) = \sin \pi x + \frac{20 - \sqrt{391}}{3}\cos \pi x.$$

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