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ON THE CONVERGENCE OF NEWTON'S METHOD FOR ANALYTIC OPERATORS

Ioannis K. Argyros

Abstract. We approximate a solution of an operator equation using Newton's method. The new local and semilocal convergence theorems are based on the analyticity of the operator involved.

A.M.S. (MOS) Subject Classification Codes. 65B05, 65N30, 47H17, 49M15.

Key Words and Phrases. Newton's method, Banach space, analytic operator, majorizing sequence, Banach lemma on invertible operators.

1. Introduction

In this study we are concerned with the problem of approximating a solution x^* of equation

$$F(x) = 0, \tag{1}$$

where F is an analytic operator defined on an open convex subset of a Banach space X with values in a Banach space Y.

Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad (x_0 \in D) \quad (n \geq 0) \tag{2}$$

has been used extensively to generate a sequence approximating x^*. Until 1986 the domain condition in the Kantorovich theorem was used to show convergence of Newton's method to x^*. In 1986 S. Smale [6] and then others [5], [7] replaced the domain condition with the analytic property of F. Here moving in this direction we provide new semilocal and local convergence theorems. It turns out that in the semilocal case under our weaker conditions we provide a more precise information on the location of the solution x^* as well as on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ $(n \geq 0)$ than before [7]. In the local case we can provide a larger radius of convergence than [7]. This observation is very important in computational mathematics and finds applications in projection methods and in the selection of efficient mesh independence strategies [1], [2], [8].

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2. Semilocal Analysis of Newton's Method

We need the following result on majorizing sequences for Newton's method.

Theorem 1. *Let η, γ_i $(i \geq 2)$ be non-negative parameters. Suppose there exists $\delta \in [0,1]$ such that*

$$h_0 = 2\int_0^1 (1-\theta)\sum_{j=0}^{\infty}(j+1)(j+2)\gamma_{j+2}(\theta\eta)^j\,d\theta + \delta\sum_{j=1}^{\infty}(j+1)\eta^j \leq \delta. \tag{3}$$

Then, iteration $\{t_n\}$ $(n \geq 0)$ given by

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+2} = t_{n+1} + \frac{\int_0^1 (1-\theta)\sum\limits_{j=0}^{\infty}(j+1)(j+2)\gamma_{j+2}(t_n+\theta(t_{n+1}-t_n))^j(t_{n+1}-t_n)d\theta}{1-\sum\limits_{j=1}^{\infty}(j+1)\gamma_{j+1}t_{n+1}^j} \tag{4}$$

is non-decreasing, bounded above by

$$t^{**} = \frac{2\eta}{2-\delta}, \tag{5}$$

and converges to some t^* such that

$$0 \leq t^* \leq t^{**}. \tag{6}$$

Moreover, the following error bounds hold for all $n \geq 0$:

$$0 \leq t_{n+2} - t_{n+1} \leq \frac{\delta}{2}(t_{n+1}-t_n) \leq \left(\frac{\delta}{2}\right)^{n+1}\eta. \tag{7}$$

Proof. Set

$$h_n = 2\int_0^1 (1-\theta)\sum_{j=0}^{\infty}(j+1)(j+2)\gamma_{j+2}(t_n+\theta(t_{n+1}-t_n))^j(t_{n+1}-t_n)d\theta + \delta\sum_{j=1}^{\infty}(j+1)\gamma_{j+1}t_{n+1}^j \quad (n \geq 0). \tag{8}$$

We must show using induction on the integer $k \geq 0$

$$h_k \leq \delta, \quad t_{k+1}-t_k \geq 0, \quad 1-\sum_{j=1}^{\infty}(j+1)\gamma_{j+1}t_{k+1}^j > 0. \tag{9}$$

Note that $h_0 \le \delta$ (by (3)), $t_1 - t_0 \ge 0$, $1 - \sum_{j=1}^{\infty}(j+1)\gamma_{j+1}t_1^j > 0$ (by (3)). Hence (9) hold for $k = 0$. But then (4) gives

$$0 \le t_2 - t_1 \le \frac{\delta}{2}(t_1 - t_0). \tag{10}$$

Assume that (9) holds for all $k \le n+1$. It can easily be seen that

$$h_k \le h_0 \quad (k \ge 0), \tag{11}$$

which shows the first estimate in (9) for all $k \ge 0$. The second follows from (4).

We must prove:

$$t_k \le t^{**} \quad (k \ge 0). \tag{12}$$

We get

$$t_0 = 0 \le t^{**}, \quad t_1 = \eta \le t^{**} \quad \text{and} \quad t_2 \le \eta + \frac{\delta}{2}\eta = \frac{2+\delta}{2}\eta \le t^{**}.$$

Assume (12) holds for all $k \le n+1$. It follows from (7) that

$$\begin{aligned} t_{k+2} &\le t_{k+1} + \frac{\delta}{2}(t_{k+1} - t_k) \\ &\le t_k + \frac{\delta}{2}(t_k - t_{k-1}) + \frac{\delta}{2}(t_{k+1} - t_k) \\ &\le \cdots \le t_1 + \frac{\delta}{2}(t_1 - t_0) + \cdots + \frac{\delta}{2}(t_{k-1} - t_k) + \frac{\delta}{2}(t_{k+1} - t_k) \\ &\le n + \frac{\delta}{2}\eta + \left(\frac{\delta}{2}\right)^2 \eta + \cdots + \left(\frac{\delta}{2}\right)^{k+1} \eta \\ &= \frac{1 - \left(\frac{\delta}{2}\right)^{k+2}}{1 - \frac{\delta}{2}}\eta \le \frac{2}{2-\delta}\eta = t^{**}. \end{aligned} \tag{13}$$

Hence, iteration $\{t_n\}$ $(n \ge 0)$ is bounded above by t^{**}. The third estimate in (9) holds by (13) and (3). Moreover it follows from (4) that $\{t_n\}$ $(n \ge 0)$ is also non-decreasing and as such it converges to some t^* satisfying (6).

That completes the proof of Theorem 1.

We show the semilocal convergence theorem for Newton's method and analytic operators.

Theorem 2. *Let $F\colon D \subseteq X \to Y$ be an analytic operator. Assume there exist a point $x_0 \in D$ and parameters $n \ge 0$, $\delta \in [0,1]$ such that*

$$F'(x_0)^{-1} \in L(Y,X) \tag{14}$$

$$\|F'(x_0)^{-1}F(x_0)\| \le \eta, \tag{15}$$

$$h_0 \le \delta, \tag{16}$$

where, h_0, t^* are given by (3), (6) respectively

$$\gamma_j = \left\| \frac{F'(x_0)^{-1}F^{(j)}(x_0)}{j!} \right\| \quad (j \geq 2) \tag{17}$$

and

$$\overline{U}(x_0, t^*) = \{x \in X \mid \|x - x_0\| \leq t^*\} \subseteq D. \tag{18}$$

Then, Newton's method $\{x_n\}$ $(n \geq 0)$ generated by (2) is well defined, remains in $\overline{U}(x_0, t^*)$ for all $n \geq 0$ and converges to a solution $x^* \in \overline{U}(x_0, t^*)$ of equation $F(x) = 0$.

Moreover the following error bounds hold for all $n \geq 0$

$$\|x_{n+2} - x_{n+1}\| \leq t_{n+2} - t_{n+1} \tag{19}$$

and

$$\|x_n - x^*\| \leq t^* - t_n \tag{20}$$

where, iteration $\{t_n\}$ $(n \geq 0)$ is given by (4).

Proof. Let us prove (19) and

$$\overline{U}(x_{n+1}, t^* - t_{n+1}) \leq \overline{U}(x_n, t^* - t_n) \tag{21}$$

hold for all $n \geq 0$. If $z \in \overline{U}(x_1, t^* - t_1)$,

$$\|z - x_0\| \leq \|z - x_1\| + \|x_1 - x_0\| \leq t^* - t_1 + t_1 = t^* - t_0$$

implies $z \in \overline{U}(x_0, t^* - t_0)$. Since also

$$\|x_1 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq \eta = t_1$$

(19) and (21) hold for $n = 0$. Given they hold for $n = 0, 1, \ldots, k+1$, then

$$\|x_{k+1} - x_0\| \leq \sum_{i=1}^{k+1} \|x_i - x_{i-1}\| \leq \sum_{i=1}^{k+1} (t_i - t_{i-1}) = t_{k+1} - t_0 = t_{k+1}$$

and

$$\|x_k + \theta(x_{k+1} - x_k) - x_0\| \leq t_k + \theta(t_{k+1} - t_k) < t^*, \quad \theta \in [0, 1].$$

Suppose $\|x - x_0\| \leq t < t^*$

$$\begin{aligned} \|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| &\leq \sum_{j=1}^{\infty} \frac{\|F'(x_0)^{-1}F^{(j+1)}(x_0)\|}{j!} \|x - x_0\|^j \\ &\leq \sum_{j=1}^{\infty} (j+1)\gamma_{j+1} t^j < 1 \quad \text{(by (3))}. \end{aligned} \tag{22}$$

It follows by the Banach Lemma on invertible operators [4] $F'(x)^{-1}$ exists and

$$\|F'(x)^{-1}F'(x_0)\| \leq \frac{1}{1-\sum\limits_{j=1}^{\infty}(j+1)\gamma_{j+1}t^j}. \tag{23}$$

Moreover from (2), (17) and (19) we get

$$\begin{aligned}
&\|F'(x_0)^{-1}F(x_{k+1})\| \\
&\quad = \|F'(x_0)^{-1}\{F(x_{k+1})-F(x_k)-F'(x_k)(x_{k+1}-x_k)\}\| \\
&\quad = \left\|\int_0^1 (1-\theta)F'(x_0)^{-1}F''[x_k+\theta(x_{k+1}-x_k)](x_{k+1}-x_k)^2 d\theta\right\| \\
&\quad = \left\|\int_0^1 (1-\theta)\sum_{j=0}^{\infty}\frac{F'(x_0)^{-1}F^{(j+2)}(x_0)}{j!}\right. \\
&\qquad \left.\cdot\ [x_k-x_0+\theta(x_{k+1}-x_k)]^j(x_{k+1}-x_k)^2 d\theta\right\| \\
&\quad \leq \int_0^1 (1-\theta)\sum_{j=0}^{\infty}(j+1)(j+2)\gamma_{j+2}(t_k+\theta(t_{k+1}-t_k))^j(t_{k+1}-t_k)^2 d\theta.
\end{aligned} \tag{24}$$

Combining (23) and (24), we get

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &= \|[F'(x_{n+1})^{-1}F'(x_0)][F'(x_0)^{-1}F(x_{n+1})]\| \\
&\leq \|F'(x_{n+1})^{-1}F'(x_0)\|\,\|F'(x_0)^{-1}F(x_{n+1})\| \\
&\leq t_{n+2}-t_{n+1}.
\end{aligned} \tag{25}$$

This completes the induction. Theorem 1 implies that $\{t_n\}$ $(n \geq 0)$ is a Cauchy sequence. From (19) and (21) $\{x_n\}$ $(n \geq 0)$ becomes a Cauchy sequence too, and as such it converges to some $x^* \in \overline{U}(x_0, t^*)$ (since $U(x_0, t^*)$ is a closed set) so that (20) holds.

That completes the proof of Theorem 2.

3. Local Convergence Analysis of Newton's Method

We finally provide the local convergence theorem for Newton's method and analytic operators.

Theorem 3. *Let $F\colon D \subseteq X \to Y$ be an analytic operator. Assume there exists a simple zero $x^* \in D$, a parameter $\overline{\delta} \in [0,1]$, and a minimum non-negative parameter r^* satisfying*

$$\overline{h}_0 \leq \overline{\delta}, \tag{26}$$

and

$$\overline{U}(x^*, r^*) \subseteq D, \tag{27}$$

where,

$$\overline{h}_0 = \overline{h}_0(r) = \int_0^1 \sum_{j=0}^{\infty}(j+1)(j+2)\overline{\gamma}_{j+2}[(1-\theta)r]^{j+1}d\theta + \overline{\delta}\sum_{j=1}^{\infty}(j+1)\overline{\gamma}_{j+1}r, \tag{28}$$

$$\overline{\gamma}_j = \left\| \frac{F'(x^*)^{-1}F^{(j)}(x^*)}{j!} \right\| \quad (j \geq 2). \tag{29}$$

Then, Newton's method $\{x_n\}$ $(n \geq 0)$ *generated by (2) is well defined, remains in* $U(x^*, r^*)$ *for all* $n \geq 0$ *and converges to* x^* *provided that* $x_0 \in U(x^*, r^*)$. *Moreover the following error bounds hold for all* $n \geq 0$

$$\|x_{n+1} - x^*\| \leq \frac{\int_0^1 \sum_{j=0}^{\infty}(j+1)(j+2)\overline{\gamma}_{j+2}(1-\theta)^{j+1}\|x_n - x^*\|^{j+2}d\theta}{1 - \sum_{j=1}^{\infty}(j+1)\overline{\gamma}_{j+1}\|x_n - x^*\|^j}. \tag{30}$$

Proof. Let $x \in U(x^*, r^*)$. By the analyticity of F we get

$$\begin{aligned} &\|F'(x^*)^{-1}[F'(x) - F'(x^*)]\| \\ &= \left\| \sum_{j=1}^{\infty} \frac{F'(x^*)^{-1}F^{(j+1)}(x^*)}{j!}(x - x^*)^j \right\| \\ &\leq \sum_{j=1}^{\infty}(j+1)\overline{\gamma}_{j+1}\|x - x^*\|^j < 1 \quad \text{(by (26))}. \end{aligned} \tag{31}$$

It follows by the Banach Lemma on invertible operators

$$\|F'(x)^{-1}F'(x^*)\| \leq \frac{1}{1 - \sum_{j=1}^{\infty}(j+1)\overline{\gamma}_{j+1}\|x - x^*\|^j}. \tag{32}$$

Using (2) we obtain in turn the approximation

$$\begin{aligned} x_{n+1} - x^* &= x_n - x^* - F'(x_n)^{-1}F(x_n) \\ &= [F'(x_n)^{-1}F'(x^*)]F'(x^*)^{-1} \\ &\quad \cdot \left\{ \int_0^1 [F'(x_n + \theta(x^* - x_n)) - F'(x_n)](x^* - x_n)d\theta \right\} \\ &= [F'(x_n)^{-1}F'(x^*)]F'(x^*)^{-1} \int_0^1 (1-\theta)F''[x_n + \theta(x^* - x_n)] \\ &\quad \cdot (x^* - x_n)^2 d\theta. \end{aligned} \tag{33}$$

Moreover, by (29) we get

(34)
$$\begin{aligned}&\left\|F'(x^*)^{-1}\int_0^1(1-\theta)F''[x_n+\theta(x^*-x_n)](x^*-x_n)^2d\theta\right\|\\&\quad=\left\|\int_0^1(1-\theta)\frac{F'(x^*)^{-1}F^{(j+2)}(x^*)}{j!}[x_n-x^*+\theta(x^*-x_n)]^j(x^*-x_n)^2d\theta\right\|\\&\quad\le\int_0^1\sum_{j=0}^{\infty}(j+1)(j+2)\overline{\gamma}_{j+2}(1-\theta)^{j+1}\|x_n-x^*\|^{j+2}d\theta.\end{aligned}$$

Estimate (30) follows from (2), (32), (33), and (34). Furthermore, by the choice of r^*

$$\alpha\equiv\frac{\int_0^1\sum_{j=0}^{\infty}(j+1)(j+2)\overline{\gamma}_{j+2}[(1-\theta)\|x_0-x^*\|]^{j+1}d\theta}{1-\sum_{j=1}^{\infty}(j+1)\overline{\gamma}_{j+1}\|x_0-x^*\|}\in[0,1)\tag{35}$$

and

$$\|x_{n+1}-x^*\|\le\alpha\|x_n-x^*\|\le\alpha^{n+1}\|x_0-x^*\|,\tag{36}$$

which shows $x_n\in U(x^*,r^*)$ and $\lim_{n\to\infty}x_n=x^*$.

That completes the proof of Theorem 3.

The results in [5] and [6] used the assumption that the sequence $\{\gamma_j\}$ $(j\ge 2)$ is bounded by

$$\gamma=\sup_{j\ge 2}\gamma_j^{\frac{1}{j-1}}.\tag{37}$$

This assumption is not true for many interesting types of operators [7]. In [7] they showed a semilocal result where $\{\gamma_j\}$ $(j\ge 2)$ is not considered to be bounded by γ, whereas in the local case they used the bound

$$\gamma^*=\sup_{j\ge 2}\left\|\frac{F'(x^*)^{-1}F^{(j)}(x^*)}{j!}\right\|^{\frac{1}{j-1}}.\tag{38}$$

The convergence radius is given by

$$r_1^*=\frac{1}{2\gamma^*}(3-2\sqrt{2}).\tag{39}$$

Note that in our local result (Theorem 3) we do not use $\{\overline{\gamma}_j\}$ $(j\ge 2)$ is bounded by γ^*.

Define functions φ, ψ by

$$\varphi(t)=\eta-\psi(t),\tag{40}$$

$$\psi(t)=t-\sum_{j=2}^{\infty}\gamma_jt^j\tag{41}$$

and iteration $\{v_n\}$ $(n \geq 0)$ by

$$v_{n+1} = v_n - \frac{\varphi(v_n)}{\varphi'(v_n)} \quad (n \geq 0). \tag{42}$$

If $\varphi(t)$ has a unique zero v^* in $[0, +\infty)$, i.e., if

$$\max_{t>0} \psi(t) \quad \text{exists.} \tag{43}$$

they showed convergence of (2) to x^* and

$$\|x_{n+1} - x_n\| \leq v_{n+1} - v_n, \tag{44}$$

$$\|x_{n+1} - x_n\| \leq v^* - v_n. \tag{45}$$

It can easily be seen that (3) implies (43) but not vice versa. Moreover using induction on n we can easily show that for all $n \geq 0$

$$t_n \leq v_n \tag{46}$$

$$t_{n+1} - t_n \leq v_{n+1} - v_n \tag{47}$$

$$t^* - t_n \leq v^* - v_n \tag{48}$$

and

$$t^* \leq v^*, \tag{49}$$

which justify the claims made at the introduction.

We complete this study with an example to show that our convergence radius r^* can be larger than r_1^*.

Example 1. Let $X = Y = \mathbf{R}$, $D = U(0,1)$, and define function F on D by

$$F(x) = e^x - 1. \tag{50}$$

By (38) we obtain $\gamma^* = .5$. Hence (39) gives

$$r_1^* = 3 - 2\sqrt{2} = .171572875. \tag{41}$$

Using (29) we obtain $\overline{\gamma}_j = \frac{1}{j!}$ $(j \geq 2)$. Moreover for $\delta = 1$ we obtain from (28)

$$\begin{aligned} h_0(r) &= \int_0^1 \sum_{j=0}^{\infty} \frac{(j+1)(j+2)}{(j+2)!}(1-\theta)^{j+1} r^{j+1} d\theta \\ &\quad + \sum_{j=1}^{\infty} (j+1) \frac{1}{(j+1)!} r^j \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} r^{j+1} \int_0^1 (1-\theta)^{j+1} d\theta + \sum_{j=1}^{\infty} \frac{r^j}{j!} \\ &= \sum_{j=0}^{\infty} \frac{r^{j+1}}{(j+1)!} \frac{j+1}{j+2} + \sum_{j=1}^{\infty} \frac{r^j}{j!} \\ &\leq \sum_{j=0}^{\infty} \frac{r^{j+1}}{(j+1)!} + \sum_{j=1}^{\infty} \frac{r^j}{j!} \\ &= 2(e^r - 1). \end{aligned} \tag{52}$$

Solving the inequality

$$2(e^r - 1) \leq 1, \tag{53}$$

we obtain

$$r^* = .405465108. \tag{54}$$

That is,

$$r_1^* < r^*. \tag{55}$$

Hence, our local results provide a wider choice for x_0 than the corresponding ones in [7]. This observation is very important in computational mathematics and finds applications in projection methods and in step-length selection in predictor–corrector continuation procedures [1], [2], [8].

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FLUTTER PANEL EQUATION IN NON CYLINDRICAL DOMAIN

H. R. Clark[1], J. L. Ferrel, M. R. Clark[2]

Abstract. The motion for small-amplitude of a thin elastic panel with both linear-weak and linear-strong damping are investigated in one-dimensional domain with moving boundary. Existence and uniqueness of global solutions are shown. Asymptotic behavior of energies are also established.

A.M.S. (MOS) Subject Classification Codes. 35B40, 35G30, 35K30.

Key Words and Phrases. Flutter panel equation, non cylindrical domain, existence of solutions, uniqueness, asymptotic behavior.

1. Introduction

The one-dimensional equation of motion of a thin panel fixed at both ends and undergoing cylindrical bending can be written as

(1.1)
$$\begin{aligned} &u_{tt}(x,t) - \Big(\zeta_0 + \zeta_1 \int_\Omega |u_x(t)|^2 dx + \sigma \int_\Omega u_x(t) u_{xt}(t) dx\Big) u_{xx}(x,t) + \\ &\sqrt{\rho}\delta u_t(x,t) + \rho u_x(x,t) + u_{xxxx}(x,t) + \nu u_{xxxxt}(x,t) = 0 \ \text{ in } \ Q, \end{aligned}$$

where u is the plate transverse displacement, x is the spatial coordinate in the direction of the fluid flow, t is the time, the viscoelastic structural damping terms are denotes by ν and σ, $\sqrt{\rho}\delta$ represents an aerodynamic damping, ζ_1 is the nonlinear stiffness of membrane, ρ is the aerodynamic pressure, ζ_0 is an in-plane tensile load, and (x,t) belongs to $Q = \Omega \times [0,\infty[$ with $\Omega = (0,1)$. All quantities are physically non-dimensionalized and ν, σ, δ, ζ_1 are fixed positive and ζ_0, ρ are fixed non-negative.

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Equation (1.1) arise in a wind tunnel experiment for a panel at supersonic speeds, for a derivation of this model see, for instance, Dowell [9].

Equation (1.1) include in its mathematical formulation some special situations in elasticity. Omitting the dissipative terms and the aerodynamic pressure ρ we have the beam equation

(1.2)
$$u_{tt}(x,t)-\Big(\zeta_0+\zeta_1\int_\Omega |u_x(t)|^2dx\Big)u_{xx}(x,t)+u_{xxxx}(x,t)=0 \text{ in } Q.$$

The beam equation (1.2) has been studied by several authors, among them, Ball [2], Biler [3], Brito [4], Medeiros [15] and Pereira [18].

By omission of the term u_{xxxx} in (1.2) we have the well known Kirchhoff equation

$$u_{tt}(x,t)-\Big(\zeta_0+\zeta_1\int_\Omega |u_x(t)|^2dx\Big)u_{xx}(x,t)=0 \text{ in } Q. \tag{1.3}$$

Equation (1.3) has also been extensively studied by several authors in both $\{1,2,\cdots,n\}$-dimensional cases and general mathematical model over a Hilbert space H. Local solutions and global solutions has been shown to exist in several physical-mathematical contexts. Among them, Arosio-Spagnolo [1], Clark [5, 6], Dickey [7], Dix [8], Kirchhoff [13], Ferrel-Medeiros [10], Matos-Pereira [14], Pohozaev [19].

The historical evolution of the equation (1.3) with an excellent bibliography was presented by Medeiros at al [16], and a new model associated with Kirchhoff equation (1.3) was also deduced, which we denominate Medeiros-Kirchhoff equation, namely

$$u_{tt}(x,t)-\Big(f(t)+g(t)\int_{\Omega_t} |u_x(t)|^2dx\Big)u_{xx}(x,t)=0 \text{ in } Q. \tag{1.4}$$

where the coefficients $f(t)=\dfrac{\tau_0}{m}+\dfrac{k\big(\gamma(t)-\gamma_0\big)}{m\gamma_0}$ and $g(t)=\dfrac{k}{2m\gamma(t)}$ are defined in §1 of Medeiros at al [16], Ω_t and Q_t are defined by

$$\Omega_t=\{x\in\mathbb{R}:\ \alpha(t)<x<\beta(t) \quad\text{and}\quad t\in[0,\infty[\,\},$$
$$Q_t=\big\{(x,t)\in\mathbb{R}^2:\ x\in\Omega_t \quad\text{and}\ \ t\in[0,\infty[\,\big\},$$

α and β are real functions, $\gamma=\beta-\alpha$, $\gamma_0=\gamma(0)$, and τ_0, m, and k are physical constants. The boundary of Q_t is defined by

$$\Sigma_t=\bigcup_{0<t<\infty}\Big(\alpha(t)\times\beta(t),t\Big).$$

Medeiros at al [16] proved that the mixed problem associated with the equation (1.4) has local solutions. Besides, whether the equation (1.4) is considered over the action of a weak-internal damping $u_t(x,t)$, and the initial data are bounded, then the total energy of the system is asymptotically stable. All these results were got supposing Ω_t increasing.

The goal of this paper is to establish an analysis qualitative of the solutions of the equation (1.1) inside of a non-cylindrical domain Q_t in two different contexts. In the equation (1.1) we can see two linear internal damping, that is, $\sqrt{\rho}\delta u_t$ and νu_{xxxxt}. Motivated for this physical situation, firstly we neglect in the equation (1.1) the linear-strong damping, that is, we assume the aerodynamic constant ν is negligible. Thus, we have the weak-damped equation

(1.5)
$$u_{tt}(x,t) - \Big(\zeta_0 + \zeta_1 \int_{\Omega_t} |u_x(t)|^2 dx + \sigma \int_{\Omega_t} u_x(t) u_{xt}(t) dx\Big) u_{xx}(x,t) + \sqrt{\rho}\delta u_t(x,t) + u_{xxxx}(x,t) = 0 \text{ in } Q_t.$$

In the second situation we omit the linear-weak damping, assuming ρ is now negligible. This way, we have the strong damped equation

(1.6)
$$u_{tt}(x,t) - \Big(\zeta_0 + \zeta_1 \int_{\Omega_t} |u_x(t)|^2 dx + \sigma \int_{\Omega_t} u_x(t) u_{xt}(t) dx\Big) u_{xx}(x,t) + u_{xxxx}(x,t) + \nu u_{xxxxt}(x,t) = 0 \text{ in } Q_t.$$

We will make this choice to exalt the different difficulties technique caused by the effect of each damping when the equations are considerate in non-cylindrical domain Q_t. Perhaps, these difficulties happen due to the used method- Galerkin's method. But in this case, we believe that such method is the most natural, because the equations (2.4) and (3.10) obtained from the equations (1.5) and (1.6) respectively, through the variable change (1.7) have variable coefficients.

Comparing the equations (1.5) and (1.6) with the equation (1.4) we can conclude that ζ_0 and ζ_1 depend continuously on the lenght of domain Ω_t. Therefore, depend on the variable time t. But, to simplify our analysis we will assume ζ_0 and ζ_1 constants.

It will be considerate for the equations (1.5) and (1.6) the fully clamped boundary conditions

$$u(x,t) = u_x(x,t) = 0 \text{ on } \Sigma_t.$$

The strategy to obtain existence of solutions to the mixed problems associated with equations (1.5) and (1.6) in a non-cylindrical domain is to transform such problems into another, defined over a cylindrical domain whose sections are not time dependent. The technique that transfoms the equation from moving boundary into fixing boundary, which we will also use, was initialy utilized in nonlinear-hyperbolic equation by Ferrel-Medeiros [10] and Medeiros at al [16]. That is, we will use a suitable change of variable

$$v(y,t) = u(x,t), \tag{1.7}$$

where $y = \frac{x-\alpha(t)}{\gamma(t)}$. We can see from definition of Ω_t that $y \in (0,1)$.

The results of existence of solutions are obtained assuming the hypothesis

$$\alpha, \quad \beta \quad \text{belong to} \quad W^{3,1}([0, \infty[; \mathbb{R}). \tag{1.8}$$

Existence of global solutions within of fixed domain for the mixed problem associated with equation (1.1) was investigated by Hughes & Marsden [12]. With respect to asymptotic stability, Holmes & Marsden [11] supposing some restrictive hypotheses about the aerodynamic pressure ρ concluded that the derivative of solution is negative.

This paper is divided in two sections. The section 2 is devoted to investigate the existence and uniqueness of strong global solutions of the mixed problem associated with the equation (1.5). Furthermore, the asymptotic behavior of solutions is proved directly inside of non-cylindrical domain. In the section 3 we prove the existence and uniqueness of weak global solutions for a mixed problem associated with the equation (1.6), provided that the initial data are bounded. Besides, the exponential decay for energy, due to technical difficulties, initially will be shown for approximate solutions inside of the fixed domain Q. Made this, we will obtain the same result for the energy inside of the non-cylindrical domain Q_t.

2. Flutter panel with linear-weak damping

2.1 Existence and uniqueness of solutions

In this section we consider the initial data- and boundary- value problem

(2.1)
$$u_{tt}(x,t) - \Big(\zeta_0 + \zeta_1 \int_{\Omega_t} |u_x(t)|^2 dx + \sigma \int_{\Omega_t} u_x(t) u_{xt}(t) dx\Big) u_{xx}(x,t) + \sqrt{\rho}\delta u_t(x,t) + u_{xxxx}(x,t) = 0 \quad \text{in } Q_t,$$

(2.2)
$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega_0,$$

(2.3)
$$u(x,t) = u_x(x,t) = 0 \quad \text{on } \Sigma_t.$$

Tanking into account (1.7) into the system (2.1)-(2.3) yields

$$\begin{aligned} & v_{tt} + a_1(y,t) v_y + a_2(y,t) v_{yt} + a_3(y,t) v_{yy} + \\ & b_1(t) \Big(\int_\Omega |v_y|^2 dy \Big) v_{yy} + b_2(t) \Big(\int_\Omega v_y v_{yt} dy \Big) v_{yy} + \\ & \Big(\int_\Omega a_4(y,t) v_y v_{yy} dy \Big) v_{yy} + b_3(t) v_{yyyy} + \sqrt{\rho}\delta v_t = 0 \text{ in } Q, \end{aligned} \tag{2.4}$$

$$v(y,0) = v_0(y), \quad v_t(y,0) = v_1(y) \text{ with } 0 < y < 1, \tag{2.5}$$

$$v(0,t) = v_y(0,t) = 0, \quad v(1,t) = v_y(1,t) = 0 \text{ for all } t \geq 0, \tag{2.6}$$

where the coefficients of equation (2.4) are given by

$$\begin{aligned} & a_1(y,t) = \frac{1}{\gamma^2} \left\{ (2\gamma' - \sqrt{\rho}\delta\gamma)(\alpha' + y\gamma') - \gamma (\alpha'' + y\gamma'') \right\}, \\ & a_2(y,t) = -\frac{2}{\gamma}(\alpha' + y\gamma'), \quad a_3(y,t) = \frac{1}{\gamma^2}\Big\{ (\alpha' + y\gamma')^2 - \zeta_0 \Big\}, \\ & a_4(y,t) = \frac{\sigma}{\gamma^5}(\alpha' + y\gamma'), \quad b_1(t) = \frac{1}{\gamma^4}(\sigma\gamma' - \zeta_1\gamma), \\ & b_2(t) = -\frac{\sigma}{\gamma^3}, \quad b_3(t) = \frac{1}{\gamma^4}. \end{aligned} \tag{2.7}$$

Thanks to variable chance (1.7) the function u is a solution of the mixed problem (2.1)-(2.3) if, and only if v is a solution of the mixed problem (2.4)-(2.6). Thus, we have that the two following theorems are equivalents

Theorem 2.1. *Let α and β be satisfying the hypothesis (1.8). If the initial data u_0 belongs to $H_0^2(\Omega_0) \cap H^4(\Omega_0)$ and u_1 belongs to $H_0^2(\Omega_0)$, then the unique strong solution u of the mixed problem (2.1)-(2.3) lies in*

$$u \in L^{\infty}\left(0, \infty; H_0^2\left(\Omega_t\right) \cap H^4\left(\Omega_t\right)\right), \tag{2.8}$$

$$u_t \in L^{\infty}\left(0, \infty; H_0^2\left(\Omega_t\right)\right), \tag{2.9}$$

$$u_{tt} \in L^{\infty}\left(0, \infty; L^2\left(\Omega_t\right)\right), \tag{2.10}$$

and the equation (2.1) is verified in the sense of $L^{\infty}\left(0, \infty; L^2\left(\Omega_t\right)\right)$.

Theorem 2.2. *We assume α and β as in the Theorem 2.1, and*

$$-\frac{1}{8} \leq \frac{\gamma'(t)}{\gamma(t)} \leq \min\left\{\frac{1}{8}, \frac{2\zeta_1}{3\sigma}\right\} \text{ for all } t \geq 0. \tag{2.11}$$

Besides, if $\alpha'(0) = \beta'(0) = 0$, then v_0 belongs to $H_0^2(\Omega) \cap H^4(\Omega)$, v_1 belongs to $H_0^2(\Omega)$, there exists a unique strong solution v of the mixed problem (2.4)-(2.6) lies in

$$v \in L^{\infty}\left(0, \infty; H_0^2(\Omega) \cap H^4(\Omega)\right), \tag{2.12}$$

$$v_t \in L^{\infty}\left(0, \infty; H_0^2(\Omega)\right), \tag{2.13}$$

$$v_{tt} \in L^{\infty}(0, \infty; L^2\left(\Omega\right)), \tag{2.14}$$

and the equation (2.4) is verified in the sense of $L^{\infty}(0, \infty; L^2(\Omega))$.

To show the existence of solutions of the problem (2.4)-(2.6) we will use the Galerkin's methods. Thus, by $V_m = [w_1, w_2, w_3, \cdots, w_m]$ we will denote the subspace of $H_0^2(\Omega)$ spanned by the m first vectors of $(w_j)_{j\in\mathbb{N}}$, where $(w_j)_{j\in\mathbb{N}}$ is orthonormalized and complete in $L^2(\Omega)$. We assume that initial data in V_m satisfy

$$\begin{aligned} v_0^m(y,0) = v_0^m &\longrightarrow v_0 \text{ strongly in } H_0^2(\Omega) \cap H^4(\Omega), \\ v_{0t}^m(y,0) = v_1^m &\longrightarrow v_1 \text{ strongly in } H_0^2(\Omega). \end{aligned} \tag{2.15}$$

Proof of Theorem 2.2 - Existence. The approximate problem associated with (2.4)-(2.6) is given, for all $\phi \in V_m$, by the following system of ordinary differential equations

$$\begin{aligned} &(v_{tt}^m(t), \phi) + \left(a_1(y,t) v_y^m(t), \phi\right) + \left(a_2(y,t) v_{yt}^m(t), \phi\right) + \\ &\sqrt{\rho}\delta\left(v_t^m(t), \phi\right) + b_3(t)\left(v_{yy}^m(t), \phi_{yy}\right) + \\ &b_2(t)\left(\int_{\Omega} v_y^m(t) v_{yt}^m(t) dy\right)\left(v_{yy}^m(t), \phi\right) + \left(a_3(y,t) v_{yy}^m(t), \phi\right) + \\ &b_4(t)\left(\int_{\Omega} \left|v_y^m(t)\right|^2 dy\right)\left(v_{yy}^m(t), \phi\right) = 0, \\ &v^m(y,0) = v_0^m \text{ and } v_t^m(y,0) = v_1^m, \end{aligned} \tag{2.16}$$

where $b_4(t) = \frac{1}{2\gamma^5(t)}\left\{3\sigma\gamma'(t) - 2\zeta_1\gamma(t)\right\}$ was determinate by definition of the $b_1(t)$, $a_4(y,t)$ and of the identity $\int_\Omega a_4(y,t)v_y^m(t)v_{yy}^m(t)dy = \frac{\sigma\gamma'(t)}{2\gamma^5(t)}\int_\Omega \left|v_y^m(t)\right|^2 dy$.

Henceforth $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. Note that $b_4(t)$ is not a positive function in view of hypothesis (2.11).

Under the assumption (2.15), we have that the system (2.16) has a local solution v^m on the interval $[0,t_m[$. This interval will be extended to any interval $[0,\infty[$ thanks to estimates below. To simplify our analysis we will write, throughout this paper, v instead of v^m. Our next task is to determine estimative on v in order to take the limit in the system (2.16).

- **Estimate I.** Substituting ϕ by $2v_t(t)$ into $(2.16)_1$ and denoting by $\widehat{M}$ the function $\widehat{M}(\xi) = \int_0^\xi M(\tau)d\tau$ we get

$$\begin{aligned}
&\frac{d}{dt}\left|v_t(t)\right|^2 - b_4(t)\frac{d}{dt}\widehat{M}\Big(\left|v_y(t)\right|^2\Big) + b_3(t)\frac{d}{dt}\left|v_{yy}(t)\right|^2 + \\
&2\sqrt{\rho}\delta\left|v_t(t)\right|^2 + 2\left(a_2(y,t)v_{yt}(t), v_t(t)\right) - \\
&2b_2(t)\Big(\int_\Omega v_y(t)v_{yt}(t)dy\Big)^2 = \\
&-2\left(a_1(y,t)v_y(t), v_t(t)\right) - 2\left(a_3(y,t)v_{yy}(t), v_t(t)\right),
\end{aligned} \tag{2.17}$$

where $|\cdot|^2$ denotes the norm in $L^2(\Omega)$. Denoting by

$$b_5(t) = \sup_{y\in[0,1]}\left|a_1(y,t)\right|_{\mathbb{R}}, \quad b_6(t) = \sup_{y\in[0,1]}\left|a_3(y,t)\right|_{\mathbb{R}},$$

observing the relations

$$2(a_2(y,t)v_{yt}(t), v_t(t)) = \frac{2\gamma'(t)}{\gamma(t)}\left|v_t(t)\right|^2, \quad -2b_2(t)\left(\int_\Omega v_y(t)v_{yt}(t)dt\right)^2 \geq 0.$$

we get from (2.17) that

$$\begin{aligned}
&\frac{d}{dt}E_1(t) + 2\Big(\sqrt{\rho}\delta + \frac{\gamma'(t)}{\gamma(t)}\Big)|v_t(t)|^2 \leq \\
&|b_4'(t)|\,\widehat{M}\left(\left|v_y(t)\right|^2\right) + |b_3'(t)|\left|v_{yy}(t)\right|^2 + \\
&2b_5(t)\left|v_y(t)\right|\left|v_t(t)\right| + 2b_6(t)\left|v_{yy}(t)\right|\left|v_t(t)\right|,
\end{aligned} \tag{2.18}$$

where E_1 is the function

$$E_1(t) = \left|v_t(t)\right|^2 - b_4(t)\widehat{M}\left(\left|v_y(t)\right|^2\right) + b_3(t)\left|v_{yy}(t)\right|^2.$$

The two last terms of right-hand side of (2.18) satisfy

$$2b_5(t)\left|v_y(t)\right|\left|v_t(t)\right| \leq \frac{b_5^2(t)}{\sqrt{\rho}\delta}\left|v_{yy}(t)\right|^2 + \sqrt{\rho}\delta\left|v_t(t)\right|^2,$$

$$2b_6(t)\left|v_{yy}(t)\right|\left|v_t(t)\right| \leq \frac{b_6^2(t)}{\sqrt{\rho}\delta}\left|v_{yy}(t)\right|^2 + \sqrt{\rho}\delta\left|v_t(t)\right|^2.$$

Taking into account the last two inequalities into (2.18) yields

$$\frac{d}{dt}E_1(t) \le b_7(t)\left\{|v_t(t)|^2 + \widehat{M}\left(|v_y(t)|^2\right) + |v_{yy}(t)|^2\right\}, \tag{2.19}$$

where

$$b_7(t) = 2\frac{|\gamma'(t)|}{\gamma(t)} + |b_4'(t)| + |b_3'(t)| + \frac{1}{\sqrt{\rho}\delta}\left[b_5^2(t) + b_6^2(t))\right].$$

Integrating (2.19) from 0 to t and observing that $E_1(0)$ is bounded, yields

$$E_1(t) \le K_0 + \int_0^t b_7(s)\left\{|v_s(s)|^2 + \widehat{M}\left(|v_y(s)|^2\right) + |v_{yy}(s)|^2\right\}ds, \tag{2.20}$$

where K_0 is a real positive constant that depends only on the initial data. Using the hypotheses (1.8), (2.11), and definition of coefficients in (2.7) we obtain from (2.20) and by using of Gronwall's inequality that

$$\begin{aligned} &(v^m) \text{ is bounded in } L^\infty\left(0,\infty; H_0^2(\Omega)\right),\\ &(v_t^m) \text{ is bounded in } L^\infty\left(0,\infty; L^2(\Omega)\right). \end{aligned} \tag{2.21}$$

- **Estimate II.** Differentiating the equation $(2.16)_1$ with respect to $t,$ replacing ϕ by $2v_{tt}(t),$ using the identities

$$2\left(a_2(y,t)v_{ytt}(t), v_{tt}(t)\right) = \frac{2\gamma'(t)}{\gamma(t)}|v_{tt}(t)|^2,$$
$$2b_3'(t)\left(v_{yy}(t), v_{yytt}(t)\right) = 2b_3'(t)\frac{d}{dt}\left(v_{yy}(t), v_{yyt}(t)\right) - 2b_3'(t)|v_{yyt}(t)|^2,$$

and denoting by E_2 the function

$$E_2(t) = |v_{tt}(t)|^2 + 2b_3'(t)\left(v_{yy}(t), v_{yyt}(t)\right) + b_3(t)|v_{yyt}(t)|^2,$$

we get

$$\begin{aligned}
&\frac{d}{dt}E_2(t) + 2\Big(\sqrt{\rho}\delta + \frac{\gamma'(t)}{\gamma(t)}\Big)|v_{tt}(t)|^2 =\\
&2b_3''(t)\left(v_{yy}(t), v_{yyt}(t)\right) + 3b_3'(t)|v_{yyt}(t)|^2 -\\
&2\left(a_{1t}(y,t)v_y(t), v_{tt}(t)\right) - 2b_2(t)\left(\int_\Omega v_y(t)v_{ytt}(t)dy\right)\left(v_{yy}(t), v_{tt}(t)\right) -\\
&2\left(a_1(y,t)v_{yt}(t), v_{tt}(t)\right) - 2b_2(t)\left(\int_\Omega v_y(t)v_{yt}(t)dy\right)\left(v_{yyt}(t), v_{tt}(t)\right) -\\
&2\left(a_{2t}(y,t)v_{yt}(t), v_{tt}(t)\right) - 2b_4'(t)\left(\int_\Omega |v_y(t)|^2dy\right)\left(v_{yy}(t), v_{tt}(t)\right) -\\
&2\left(a_{3t}(y,t)v_{yy}(t), v_{tt}(t)\right) - 2b_4(t)\left(\int_\Omega |v_y(t)|^2dy\right)\left(v_{yyt}(t), v_{tt}(t)\right) -
\end{aligned}$$

$$2\left(a_3(y,t)v_{yyt}(t),v_{tt}(t)\right)-2b_2(t)\left(\int_\Omega |v_{yt}(t)|^2\,dy\right)(v_{yy}(t),v_{tt}(t))-$$
$$2\left\{2b_4(t)+b_2'(t)\right\}\left(\int_\Omega v_y(t)v_{yt}(t)dy\right)(v_{yy}(t),v_{tt}(t))\,.$$

Hence, proceeding as estimate I and using (2.21), hypothesis (1.8) and usual inequalities we obtain after some simplifications that

$$\frac{d}{dt}E_2(t)\le b_8(t)+b_9(t)\Big\{\,|v_{tt}(t)|^2+|v_{yyt}(t)|^2\,\Big\}, \tag{2.22}$$

where $b_8(t)$ and $b_9(t)$ are $L^1(0,\infty)-$ function defined from coefficients (2.7).

Defining by E_3 the function $E_3(t)=E_1(t)+E_2(t)$ and using (2.19), (2.22) yields

$$\begin{aligned}\frac{d}{dt}E_3(t)\le b_8(t)+(b_7(t)+b_9(t))\Big\{&|v_t(t)|^2+\widehat{M}\left(|v_y(t)|^2\right)+\\&|v_{yy}(t)|^2+|v_{tt}(t)|^2+|v_{yyt}(t)|^2\,\Big\}.\end{aligned} \tag{2.23}$$

Integrating (2.23) from 0 to t and observing that $v_{tt}(0)$ is bounded in $L^2(\Omega)$ we have, thanks to Gronwall's inequality, that there exists a real positive constant K_1 such that

$$E_3(t)\le K_1 \text{ for all } t\ge 0. \tag{2.24}$$

By usual inequality we get the following inequalities

$$\begin{aligned}&|2b_3'(t)\,(v_{yy}(t),v_{yyt}(t))|\le\frac{8|\gamma'(t)|}{\gamma^5(t)}\,|v_{yy}(t)|\,|v_{yyt}(t)|\le\\&\frac{4|\gamma'(t)|}{\gamma(t)}\Big(\frac{1}{\gamma^4(t)}\,|v_{yy}(t)|^2+\frac{1}{\gamma^4(t)}\,|v_{yyt}(t)|^2\,\Big)\le\\&\frac12 b_3(t)\,|v_{yy}(t)|^2+\frac12 b_3(t)\,|v_{yyt}(t)|^2,\end{aligned} \tag{2.25}$$

where we have used in the last inequality above the hypothesis (2.11), i.e., $\frac{4|\gamma'(t)|}{\gamma(t)}\le\frac12$. From definition of E_3 and (2.25) we have

$$\begin{aligned}&|v_t(t)|^2+b_4(t)\widehat{M}\left(|v_y(t)|^2\right)+|v_{tt}(t)|^2+\\&\frac12 b_3(t)\,|v_{yy}(t)|^2+\frac12 b_3(t)\,|v_{yyt}(t)|^2\le E_3(t)\text{ for all } t\ge 0.\end{aligned} \tag{2.26}$$

Thus, from (2.24) and (2.26) we obtain

$$\begin{aligned}&(v_t^m)\text{ is bounded in } L^\infty\left(0,\infty;H_0^2(\Omega)\right),\\&(v_{tt}^m)\text{ is bounded in } L^\infty\left(0,\infty;L^2(\Omega)\right).\end{aligned} \tag{2.27}$$

- **Limit of the approximate solutions.** From estimates obtained in (2.21) and (2.27) we can take the limit in the nonlinear system (2.16). In fact, from

(2.21) and (2.27) we have, in particular, that the sequences (v^m), (v_t^m) and (v_{tt}^m) are bounded in $L^2\left(0,T;H_0^1(\Omega)\right)$, $L^2\left(0,T;H_0^1(\Omega)\right)$ and $L^2\left(0,T;L^2(\Omega)\right)$, respectively. Thus, by compact injection of $H_0^1\left(\Omega\times]0,T[\right)$ into $L^2\left(\Omega\times]0,T[\right)$ it follows by Lions-Aubin's theorem that there exists a subsequence of $(v^m)_{m\in\mathbb{N}}$, which we denote as the original sequence, such that

$$\begin{aligned} v^m &\longrightarrow v \text{ strong in } L^2\left(0,T;H_0^1(\Omega)\right) \text{ as } m\longrightarrow\infty,\\ v_t^m &\longrightarrow v_t \text{ strong in } L^2\left(0,T;H_0^1(\Omega)\right) \text{ as } m\longrightarrow\infty. \end{aligned} \tag{2.28}$$

We still can obtain from (2.21) and (2.27) the following convergence

$$\begin{aligned} v^m &\longrightarrow v \text{ weak star in } L^\infty\left(0,\infty;H_0^2(\Omega)\right) \text{ as } m\longrightarrow\infty,\\ v_t^m &\longrightarrow v_t \text{ weak star in } L^\infty\left(0,\infty;H_0^2(\Omega)\right) \text{ as } m\longrightarrow\infty,\\ v_{tt}^m &\longrightarrow v_{tt} \text{ weak star in } L^\infty\left(0,\infty;L^2(\Omega)\right) \text{ as } m\longrightarrow\infty. \end{aligned} \tag{2.29}$$

The convergence (2.28) and $(2.29)_1$ are sufficient to take the limit in the nonlinear terms of the approximate system (2.16). Thus, there exists a strong solution v defined on $\Omega\times[0,\infty[$ with values in $\mathbb{R}$ satisfying (2.4). The statements (2.13) and (2.14) are given by convergence $(2.29)_{2,3}$. The regularity (2.12) is guaranteed by $(2.29)_1$ and by results on elliptic regularity (see, for instance, Nirenberg [17]) in view of

$$\begin{aligned} -v_{yyyy} &= \gamma^4\left\{v''+a_1v_y+a_2v_{yt}+a_3v_{yy}+\sqrt{\rho}\delta v_t\right\}+\\ &\gamma^4\left\{b_1\int_\Omega |v_y|^2dy+b_2\int_\Omega v_yv_{yt}dy+\int_\Omega a_4v_yv_{yyy}dy\right\}v_{yy}, \end{aligned}$$

to be obtained in the sense of $L^2\left(0,\infty;L^2(\Omega)\right)$.

• **Uniqueness.** Let $w=v-\tilde{v}$ be, where v and $\tilde{v}$ are two solutions of the problem (2.4)-(2.6) on $[0,\infty[$. Thus, for all $\phi\in H_0^2(\Omega)$, the solution w satisfies:

$$\begin{aligned} &(w_{tt}(t),\phi)+(a_1(y,t)w_y(t),\phi)+(a_2(y,t)w_{yt}(t),\phi)+b_3(t)\left(w_{yy}(t),\phi_{yy}\right)+\\ &b_2(t)\left(\int_\Omega w_y(t)v_{yt}(t)dy+\int_\Omega w_{yt}(t)\tilde{v}_y(t)dy\right)(w_{yy}(t),\phi)+\sqrt{\rho}\delta\left(w_t(t),\phi\right)+\\ &(a_3(y,t)w_{yy}(t),\phi)+b_4(t)\left(\int_\Omega |v_y(t)+\tilde{v}_y(t)|\,|w_y(t)|\,dy\right)(w_{yy}(t),\phi)=0,\\ &w(y,0)=w_t(y,0)=0 \text{ for all } y\in]0,1[,\\ &w(0,t)=w_y(0,t)=0 \text{ for all } t\geq 0,\\ &w(1,t)=w_y(1,t)=0 \text{ for all } t\geq 0. \end{aligned}$$

Replacing ϕ by w_t in the preceding system yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left|w_t(t)\right|^2+\frac{1}{2}b_3(t)\frac{d}{dt}\left|w_{yy}(t)\right|^2+\sqrt{\rho}\delta\left|w_t(t)\right|^2=-\\ &(a_1(y,t)w_y(t),w_t(t))-(a_2(y,t)w_{yt}(t),w_t(t))-(a_3(y,t)w_{yy}(t),w_t(t))-\\ &b_2(t)\left(\int_\Omega w_y(t)v_{yt}(t)dy+\int_\Omega w_{yt}(t)\tilde{v}_y(t)dy\right)(w_{yy}(t),w_t(t))-\\ &b_4(t)\left(\int_\Omega |v_y(t)+\tilde{v}_y(t)|\,|w_y(t)|\,dy\right)(w_{yy}(t),w_t(t)). \end{aligned}$$

Applying usual inequalities and $(2.21)_1$ in the right-hand side of preceding identity we obtain

$$\frac{1}{2}\frac{d}{dt}\left\{|w_t(t)|^2 + b_3(t)\,|w_{yy}(t)|^2\right\} + \sqrt{\rho}\delta\,|w_t(t)|^2 \leq |w_t(t)|^2 + b_{10}(t)\,|w_{yy}(t)|^2, \tag{2.30}$$

where $b_{10}(t)$ is a positive function defined on $[0,\infty[$, which depends on the coefficients of the equation (2.4). Integrating (2.30) from 0 to t, using the hypothesis (1.8), the initial-boundary conditions, and Gronwall's inequality we obtain $w = 0$ for all $t \in [0,\infty[$. Thus, we have concluded the proof of Theorem 2.2. Therefore, the Theorem 2.1 is got from Theorem 2.2 and (1.7) $\square$

2.2 Asymptotic behavior

Our goal is to show that the total energy of the system (2.1)-(2.3) decay exponentially as $t \to +\infty$. We will show this property directly in non-cylindrical domain Q_t. The structure of non-cylindrical domain has a particular importance in the prove of the asymptotic stability of the energy. This way, we assume that the non-cylindrical Q_t is increasing. It means, the functions α and β satisfy the following properties

$$\alpha,\ \beta \in W^{1,1}\big([0,\infty[;\mathbb{R}\big), \quad \alpha'(t) < 0 \ \text{ and } \ \beta'(t) > 0. \tag{2.31}$$

Besides, we suppose that the non-cylindrical domain is bounded, e.g., there exist real positive constants α_0 and β_0 such that

$$\alpha(t) > -\alpha_0 \ \text{ and } \ \beta(t) < \beta_0 \ \text{ for all } \ t \geq 0. \tag{2.32}$$

As $u(x,t) = u_x(x,t) = 0$ on Σ_t then we have, for all $t \geq 0$, the following relations to be used to proceed

$$\begin{aligned}
&u_t\left(\alpha(t),t\right) = -\alpha'(t)u_x\left(\alpha(t),t\right) = 0,\\
&u_t\left(\beta(t),t\right) = -\beta'(t)u_x\left(\beta(t),t\right) = 0,\\
&u_{xt}\left(\alpha(t),t\right) = -\alpha'(t)u_{xx}\left(\alpha(t),t\right),\\
&u_{xt}\left(\beta(t),t\right) = -\beta'(t)u_{xx}\left(\beta(t),t\right),\\
&\left(\int_{\Omega_t} u_x(t)u_{xt}(t)dx\right)^2 \geq 0.
\end{aligned} \tag{2.33}$$

Taking the scalar product on $L^2(\Omega_t)$ of both sides of (2.1) with u_t, integrating by parts the second and third term of left-hand side, observing the identities (2.33) and the Leibnitz's rule[1] yields

$$\frac{d}{dt}E(t) + \frac{1}{2}\beta'(t)u_{xx}^2(\beta(t),t) - \frac{1}{2}\alpha'(t)u_{xx}^2(\alpha(t),t) \leq -\sqrt{\rho}\delta\,|u_t(t)|^2 \ \text{ for all } \ t \geq 0,$$

[1]

$$\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(x,t)dx = \int_{\alpha(t)}^{\beta(t)} \frac{\partial}{\partial t}f(x,t)dx + f(\beta(t),t)\beta'(t) - f(\alpha(t),t)\alpha'(t).$$

where

$$E(t)=\frac{1}{2}\left\{|u_t(t)|^2+\zeta_0\,|u_x(t)|^2+\frac{\zeta_1}{2}\,|u_x(t)|^4+|u_{xx}(t)|^2\right\}. \tag{2.34}$$

From hypothesis $(2.31)_2$ we have

$$\frac{1}{2}\beta'(t)u_{xx}^2(\beta(t),t)-\frac{1}{2}\alpha'(t)u_{xx}^2(\alpha(t),t)\geq 0 \ \text{ for all } \ t\geq 0.$$

Thus, we obtain

$$\frac{d}{dt}E(t)\leq-\sqrt{\rho}\delta\,|u_t(t)|^2 \ \text{ for all } \ t\geq 0. \tag{2.35}$$

The asymptotic behavior of the energy is given by the following theorem

Theorem 2.3. *If the functions α and β satisfy the hypotheses (2.31) and (2.32), then the energy defined in (2.34) associated with the solution of the system (2.1)-(2.3) satisfies*

$$E(t)\leq K_2\exp(-\omega t) \ \textit{ for all } \ t\geq 0, \tag{2.36}$$

where $\omega=\dfrac{\epsilon}{2\vartheta_2}$ is a real-positive number, ϑ_2 is defined in (2.42), $K_2=E(0)+\epsilon F(0)$, the function $F(t)$ and the constant ϵ are defined by

$$\begin{aligned} F(t)&=\frac{1}{4}\left\{\sqrt{\rho}\delta\,|u(t)|^2+\frac{\sigma}{2}\,|u_x(t)|^4\right\}, \\ 0<\epsilon&=\min\left\{\frac{1}{\sqrt{\vartheta_1}},\ \sqrt{\rho}\delta\right\}. \end{aligned} \tag{2.37}$$

From hypothesis (2.32) the constant $\vartheta_1=\max_{t\geq 0}\gamma(t)$ there exists and represents the real-positive constant of immersion of $H_0^1(\Omega_t)$ into $L^2(\Omega_t)$. Note that ϑ_1 depend only of the length of the interval $\big[\alpha(t),\beta(t)\big]$.

Proof. For each $\epsilon>0$ we consider the auxiliary function

$$E_\epsilon(t)=E(t)+\frac{\epsilon}{2}\int_{\Omega_t}u_t(t)u(t)dx \ \text{ for all } \ t\geq 0. \tag{2.38}$$

As $H_0^1(\Omega_t)$ is continuously embedded in $L^2(\Omega_t)$ we obtain after application of usual inequalities that, there exist a constant ϑ_1

$$\frac{\epsilon}{2}\int_{\Omega_t}u_t(t)u(t)dx\leq\frac{1}{4}\,|u_t(t)|^2+\frac{\epsilon^2\vartheta_1}{4}\,|u_x(t)|^2. \tag{2.39}$$

Hence, and hypothesis $(2.37)_2$ we get

$$E_\epsilon(t)\leq\frac{3}{2}E(t) \ \text{ for all } \ t\geq 0.$$

On the other hand, we get from (2.38) and $(2.37)_2$ that

$$E_\epsilon(t) \geq E(t) - \frac{\epsilon}{2}\Big|\int_{\Omega_t} u_t(t)u(t)dx\Big| \geq \frac{1}{2}E(t) \text{ for all } t \geq 0.$$

Thus, the function E_ϵ satisfies the both inequalities

$$\frac{1}{2}E_\epsilon(t) \leq E(t) \leq \frac{3}{2}E_\epsilon(t) \text{ for all } t \geq 0. \tag{2.40}$$

Now, differentiating the function E_ϵ with respect to t yields

$$E_\epsilon'(t) = E'(t) + \frac{\epsilon}{2}\int_{\Omega_t} u_{tt}(t)u(t)dx + \frac{\epsilon}{2}|u_t(t)|^2 \text{ for all } t \geq 0.$$

Replacing u_{tt} by $\left\{\zeta_0 + \zeta_1\int_{\Omega_t}|u_x|^2dx + \int_{\Omega_t}u_xu_{xt}dx\right\}u_{xx} - u_{xxxx} - \sqrt{\rho}\delta u_t$ in the second term of right-hand side of the last identity above and using the definition of the function F yields, for all $t \geq 0$

$$E_\epsilon'(t) + \epsilon F'(t) = E'(t) + \frac{\epsilon}{2}|u_t(t)|^2 - \frac{\epsilon}{2}\left\{\zeta_0|u_x(t)|^2 + \zeta_1|u_x(t)|^4 + |u_{xx}(t)|^2\right\}.$$

Hence, (2.35) and $(2.37)_2$ we obtain

$$E_\epsilon'(t) + \epsilon F'(t) \leq -\frac{\epsilon}{2}E(t) \text{ for all } t \geq 0. \tag{2.41}$$

Using the definitions of the functions E, E_ϵ and F it is easy to see that there exists a real positive constant ϑ_2 dependent only of ϵ such that

$$E_\epsilon(t) + \epsilon F(t) \leq \vartheta_2 E(t) \text{ for all } t \geq 0. \tag{2.42}$$

Thus, from inequalities (2.41) and (2.42) there exists a suitable real positive constant ω dependent of ϵ and ϑ_2 such that

$$E_\epsilon'(t) + \epsilon F'(t) - \omega\{E_\epsilon(t) + \epsilon F(t)\} \leq 0 \text{ for all } t \geq 0.$$

Therefore,

$$E_\epsilon(t) + \epsilon F(t) \leq \{E_\epsilon(0) + \epsilon F(0)\}\exp(-\omega t) \text{ for all } t \geq 0.$$

Hence, and inequality (2.40) we can conclude that the inequality (2.36) holds. Thus, the demonstration of Theorem 2.3 is completed □

3. Flutter panel with linear-strong damping

3.1 Existence and uniqueness of solutions

To reach our goal mentioned previously we consider in this section the initial data- and boundary- value problem

(3.1)

$$u_{tt}(x,t) - \Big(\zeta_0 + \zeta_1\int_{\Omega_t}|u_x(t)|^2dx + \sigma\int_{\Omega_t}u_x(t)u_{xt}(t)dx\Big)u_{xx}(x,t) + u_{xxxx}(x,t) + \nu u_{xxxxt}(x,t) = 0 \text{ in } Q_t,$$

(3.2)

$$u(x,0) = u_0(x) \text{ and } u_t(x,0) = u_1(x) \text{ in } \Omega_0,$$

(3.3)

$$u(x,t) = u_x(x,t) = 0 \text{ on } \Sigma_t.$$

The global solutions of the system (3.1)-(3.3) will be obtained if the initial data are bounded. The structure of the non-cylindrical domain Q_t has an important perform in this section. We will fix to follow the hypotheses necessary to state the results associated with the problem (3.1)-(3.3). The functions $\alpha,\ \beta,\ \gamma,$ and its derivatives, satisfying for all $t\geq 0,$ the following conditions

$$\varepsilon=\max\big\{|\alpha'(t)|,\quad|\alpha''(t)|,\quad|\beta'(t)|,\ \ |\beta''(t)|\big\}. \tag{3.4}$$

$$\frac{\gamma'(t)}{\gamma(t)}\leq\frac{\zeta_1}{\sigma},\quad\frac{\gamma'(t)}{\gamma(t)}\leq\frac{1}{16\nu}. \tag{3.5}$$

As a consequence of (3.4) we have

$$|\gamma'(t)|\leq 2\varepsilon,\quad|\gamma''(t)|\leq 2\varepsilon. \tag{3.6}$$

We also assume that there exist real constants γ_0 and K_3 such that

$$0<\gamma_0\leq\gamma(t)\leq K_3. \tag{3.7}$$

$$\zeta_0K_3^4<\frac{1}{4}. \tag{3.8}$$

Finally, we choose ε small enough such that

$$\frac{\zeta_1}{4}\geq\varepsilon\Gamma_0,\ \ \frac{1}{16}\geq\varepsilon\Gamma_1,\ \ \frac{\nu}{8}\geq\varepsilon\Gamma_2\ \text{ and }\ \frac{3}{8}\geq\varepsilon\Gamma_3, \tag{3.9}$$

where $\Gamma_0,\ \Gamma_1,\ \Gamma_2$ and Γ_3 are defined in (3.37).

As done in the subsection 2.1, the function u is a solution of the mixed problem (3.1)-(3.3) if and only if v is a solution of the mixed problem

$$\begin{aligned}
&v_{tt}+c_1(y,t)v_y+c_2(y,t)v_{yt}+\\
&c_3(y,t)(t)v_{yy}+d_1(t)\Big(\int_\Omega|v_y|^2dy\Big)v_{yy}+\\
&d_2(t)\Big(\int_\Omega v_yv_{yt}dy\Big)v_{yy}+\Big(\int_\Omega c_4(y,t)v_yv_{yy}dy\Big)v_{yy}+\\
&d_3(t)v_{yyyy}+c_5(y,t)v_{yyyyy}+d_4(t)v_{yyyyt}=0\ \text{ in }\ Q,
\end{aligned} \tag{3.10}$$

$$v(y,0)=v_0(y),\quad v_t(y,0)=v_1(y)\ \text{ with }\ 0<y<1, \tag{3.11}$$

$$\begin{aligned}
&v(0,t)=v_y(0,t)=0\ \text{ for all }\ t\geq 0,\\
&v(1,t)=v_y(1,t)=0\ \text{ for all }\ t\geq 0,
\end{aligned} \tag{3.12}$$

where the coefficients of equation (3.10) are given by

$$\begin{aligned}
&c_1(y,t)=\frac{1}{\gamma^2}\left\{2\gamma'\left(\alpha'+y\gamma'\right)-\gamma\left(\alpha''+y\gamma''\right)\right\},\\
&c_2(y,t)=-\frac{2}{\gamma}\left(\alpha'+y\gamma'\right),\quad c_3(y,t)=\frac{1}{\gamma^2}\Big\{\left(\alpha'+y\gamma'\right)^2-\zeta_0\Big\},\\
&c_4(y,t)=\frac{\sigma}{\gamma^4}\left(\alpha'+y\gamma'\right),\quad c_5(y,t)=-\frac{\nu}{\gamma^5}\left(\alpha'+y\gamma'\right),\\
&d_1(t)=\frac{1}{\gamma^4}\left(\sigma\gamma'-\zeta_1\gamma\right),\quad d_2(t)=-\frac{\sigma}{\gamma^3},\\
&d_3(t)=\frac{1}{\gamma^5}\left(\gamma-4\nu\gamma'\right),\quad d_4(t)=\frac{\nu}{\gamma^4}.
\end{aligned} \tag{3.13}$$

Thus, we can state the first result of this section

Theorem 3.1. *Under the hypotheses (3.4)-(3.9). If the initial data* u_0, u_1 *belong to* $H_0^2(\Omega_0)\cap H^4(\Omega_0)$, $H_0^2(\Omega_0)$ *respectively, and*

$$\begin{aligned}\frac{\nu^2\gamma^4(0)}{32K_3^4}\left|u_{0xxxx}\right|^2+\frac{\nu^2}{32K_3^4}\left|u_{0xx}\right|^2+\frac{\sigma\gamma(0)}{4}\left|u_{0x}\right|^4+\\ \frac{1}{4}\left|\left[\alpha'(0)+\frac{\gamma'(0)}{\gamma(0)}\big(x-\alpha(0)\big)\right]u_{0x}+u_1\right|^2<\frac{1}{4K_4},\end{aligned} \tag{3.14}$$

then the unique weak solution u *of the mixed problem (3.1)-(3.3) lies in*

$$u\in L^\infty\left(0,\infty;H_0^2\left(\Omega_t\right)\cap H^4\left(\Omega_t\right)\right), \tag{3.15}$$

$$u_t\in L^\infty\left(0,\infty;H_0^2\left(\Omega_t\right)\right), \tag{3.16}$$

$$u_{tt}\in L^2\left(0,\infty;H^{-2}\left(\Omega_t\right)\right). \tag{3.17}$$

The equation (3.1) is verified in the sense of $L^2\left(0,\infty;H^{-2}\left(\Omega_t\right)\right)$. *The real positive constant* K_4 *is defined in (3.40).*

As done before, the Theorem 3.1 is equivalent to

Theorem 3.2. *Under the assumptions of Theorem 3.1, if* $\alpha'(0)=\beta'(0)=0$ *then* v_0, v_1 *belong to* $H_0^2(\Omega)\cap H^4(\Omega)$, $H_0^2(\Omega)$ *respectively,*

$$K_4E_5(0)<\frac{1}{4}, \tag{3.18}$$

and, in these conditions there exists a unique weak solution v *of the mixed problem (3.10)-(3.12) lies in*

$$v\in L^\infty\left(0,\infty;H_0^2(\Omega)\cap H^4(\Omega)\right), \tag{3.19}$$

$$v_t\in L^\infty\left(0,\infty;H_0^2(\Omega)\right), \tag{3.20}$$

$$v_{tt}\in L^2(0,\infty;H^{-2}\left(\Omega\right)). \tag{3.21}$$

The equation (3.10) is verified in the sense of $L^2(0,\infty;H^{-2}\left(\Omega\right))$ *and* $E_5(t)$ *is defined by*

$$\begin{aligned}E_5(t)=\frac{1}{4}\left|v_t(t)\right|^2+\frac{\sigma}{4\gamma^3(t)}\left|v_y(t)\right|^4+\\ \frac{\nu^2}{32K_3^4\gamma^4(t)}\left|v_{yy}(t)\right|^2+\frac{\nu^2}{32K_3^4\gamma^4(t)}\left|v_{yyyy}(t)\right|^2.\end{aligned} \tag{3.22}$$

Proof of Theorem 3.2 - Existence. We again will use the Galerkin's method. But, that time we are going to work with the special basis built with eigenfunctions of the operator $\mathbb{A}=\partial_y{}^4$ defined in $H_0^2(\Omega)\cap H^4(\Omega)$. Thus, by $V_m=[w_1,w_2,w_3,\cdots,w_m]$ we will denote the subspace of $H_0^2(\Omega)$ spanned by the m first eigenvectors of $\mathbb{A}$. We will assume the initial data in V_m satisfy

$$\begin{aligned}&v^m(y,0)=v_0^m\longrightarrow v_0\ \text{ strongly in }\ H_0^2(\Omega)\cap H^4(\Omega),\\ &v_t^m(y,0)=v_1^m\longrightarrow v_1\ \text{ strongly in }\ H_0^2(\Omega).\end{aligned} \tag{3.23}$$

For $v^m(t)$ belongs to V_m we have, for all $\phi \in V_m$, that the system of ordinary differential equations

$$
\begin{aligned}
&(v^m_{tt}(t),\phi) + \big(c_1(y,t)v^m_y(t),\phi\big) + \big(c_2(y,t)v^m_{yt}(t),\phi\big) + \\
&\big(c_3(y,t)v^m_{yy}(t),\phi\big) + d_1(t)\Big(\int_\Omega \left|v^m_y(t)\right|^2 dy\Big)\left(v^m_{yy}(t),\phi\right) + \\
&\Big(\int_\Omega c_4(y,t)v^m_y(t)v^m_{yy}(t)dy\Big)\left(v^m_{yy}(t),\phi\right) + \\
&d_2(t)\Big(\int_\Omega v^m_y(t)v^m_{yt}(t)dy\Big)\left(v^m_{yy}(t),\phi\right) + d_3(t)\left(v^m_{yyyy}(t),\phi\right) + \\
&\big(c_5(y,t)v^m_{yyyyy}(t),\phi\big) + d_4(t)\left(v^m_{yyyyt}(t),\phi\right) = 0, \\
&v^m(y,0) = v^m_0 \text{ and } v^m_t(y,0) = v^m_1,
\end{aligned}
\tag{3.24}
$$

has a local solution v^m provided that the hypothesis (3.23) holds. Again, throughout this subsection, we will write v instead of v^m. Our next task is to determine estimative on v sufficient to take the limit in the system (3.24).

• **Estimate I.** Substituting ϕ by $v_t(t)$ into $(3.24)_1$, integrating by parts from the third to the tenth terms we get the following relations

$$
(c_2(y,t)v_{yt}(t),v_t(t)) = \frac{2\gamma'(t)}{\gamma(t)}\left|v_t(t)\right|^2,
$$

$$
(c_3(y,t)v_{yy}(t),v_t(t)) = \Big(\frac{1}{\gamma^2(t)}\big(\alpha'(t)+y\gamma'(t)\big)^2 v_{yy}(t),v_t(t)\Big) + \frac{\zeta_0}{2\gamma^2(t)}\frac{d}{dt}\left|v_y(t)\right|^2,
$$

$$
\begin{aligned}
d_1(t)\Big(\int_\Omega \left|v_y(t)\right|^2 dy\Big)(v_{yy}(t),v_t(t)) + \Big(\int_\Omega c_4(y,t)v_y(t)v_{yy}(t)dy\Big)(v_{yy}(t),v_t(t)) = \\
-\frac{1}{4}\Big(-\frac{\zeta_1\gamma(t)}{\gamma^4(t)} + \frac{\sigma\gamma'(t)}{2\gamma^4(t)}\Big)\frac{d}{dt}\left|v_y(t)\right|^4,
\end{aligned}
$$

$$
d_2(t)\Big(\int_\Omega v_y(t)v_{yt}(t)dy\Big)(v_{yy}(t),v_t(t)) = \frac{\sigma}{\gamma^3(t)}\Big(\int_\Omega v_y(t)v_{yt}(t)dy\Big)^2 \geq 0,
$$

$$
\begin{aligned}
d_3(t)\left(v_{yyyy}(t),v_t(t)\right) + d_4(t)\left(v_{yyyyt}(t),v_t(t)\right) = \\
\frac{1}{2\gamma^5(t)}\Big(\gamma(t) - 4\nu\gamma'(t)\Big)\frac{d}{dt}\left|v_{yy}(t)\right|^2 + \frac{\nu}{\gamma^4(t)}\left|v_{yyt}(t)\right|^2,
\end{aligned}
$$

$$
(c_5(y,t)v_{yyyyy}(t),v_t(t)) = -\bigg(v_{yyyy}(t),\frac{\nu\gamma'(t)}{\gamma^5(t)}v_t(t) + \frac{\nu}{\gamma^5(t)}\big(\alpha'(t)+y\gamma'(t)v_{yt}(t)\big)\bigg).
$$

Thus, using the six precedents relations yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|v_t(t)\right|^2 + \frac{\zeta_0}{2\gamma^2(t)}\frac{d}{dt}\left|v_y(t)\right|^2 + \frac{1}{4}\Big(\frac{\zeta_1}{\gamma^3(t)} - \frac{\sigma\gamma'(t)}{2\gamma^4(t)}\Big)\frac{d}{dt}\left|v_y(t)\right|^4 + \\
&\frac{1}{2\gamma^5(t)}\Big(\gamma(t) - 4\nu\gamma'(t)\Big)\frac{d}{dt}\left|v_{yy}(t)\right|^2 + \frac{\nu}{\gamma^4(t)}\left|v_{yyt}(t)\right|^2 \leq \\
&\frac{2|\gamma'(t)|}{\gamma(t)}\left|v_t(t)\right|^2 + (c_1(y,t)v_y(t),v_t(t)) + \\
&\Big(\frac{1}{\gamma^2(t)}\big(\alpha'(t)+y\gamma'(t)\big)v_{yy}(t),v_t(t)\Big) + \\
&\bigg(v_{yyyy}(t),\frac{\nu\gamma'(t)}{\gamma^5(t)}v_t(t) + \frac{\nu}{\gamma^5(t)}\big(\alpha'(t)+y\gamma'(t)v_{yt}(t)\big)\bigg).
\end{aligned}
\tag{3.25}
$$

Using the hypotheses (3.4), (3.5), (3.7) and usual inequalities we get estimates for the right-hand side of (3.25). Thus, denoting by $E_1(t)$ the function

(3.26)
$$E_1(t) = \frac{1}{2}\left|v_t(t)\right|^2 + \frac{\zeta_0}{2\gamma^2(t)}\left|v_y(t)\right|^2 + \\ \frac{1}{4}\Big(\frac{\zeta_1}{\gamma^3(t)} - \frac{\sigma\gamma'(t)}{2\gamma^4(t)}\Big)\left|v_y(t)\right|^4 + \frac{1}{2\gamma^5(t)}\Big(\gamma(t) - 4\nu\gamma'(t)\Big)\left|v_{yy}(t)\right|^2,$$

yields

(3.27)
$$\frac{d}{dt}E_1(t) + \frac{1}{\gamma^4(t)}\Big(\frac{\nu}{2} - 4\varepsilon K_3^3\Big)\left|v_{yyt}(t)\right|^2 \leq \\ \frac{\varepsilon C_0}{\gamma^2(t)}\left|v_y(t)\right|^4 + \frac{\varepsilon C_1}{\gamma^4(t)}\left|v_{yy}(t)\right|^2 + \frac{10\nu\varepsilon^2}{\gamma^6(t)}\left|v_{yyyy}(t)\right|^2,$$

where C_0 and C_1 depends on ζ_1, K_3, σ, ϵ, γ_0 and ζ_0, K_3, ν, ϵ, γ_0, respectively.

- **Estimate II.** Substituting ϕ by $v(t)$ into $(3.24)_1$ and proceeding as in the estimate I and denoting by $E_2(t)$ the function

(3.28)
$$E_2(t) = (v_t(t), v(t)) + \frac{\nu}{2\gamma^4(t)}\left|v_{yy}(t)\right|^2 + \frac{\sigma}{4\gamma^3(t)}\left|v_y(t)\right|^4,$$

we can write

$$\frac{d}{dt}E_2(t) + \frac{\zeta_0}{\gamma^2(t)}\left|v_y(t)\right|^2 + \frac{1}{\gamma^4(t)}\left|v_{yy}(t)\right|^2 + \frac{\zeta_1}{\gamma(t)}\left|v_y(t)\right|^4 = \\ \left|v_t(t)\right|^2 + \frac{6\nu\gamma'(t)}{\gamma^5(t)}\left|v_{yy}(t)\right|^2 + \frac{\sigma\gamma'(t)}{4\gamma^4(t)}\big(4\gamma^2(t)+1\big)\left|v_y(t)\right|^4 -$$

(3.29)
$$(c_2(y,t)v_{yt}(t), v(t)) - \Big(\frac{1}{\gamma^2(t)}\big(\alpha'(t) + y\gamma'(t)\big)^2 v_{yy}(t), v(t)\Big) - \\ \Big(v_{yyyy}(t), \frac{\nu\gamma'(t)}{\gamma^5(t)}v(t) + \frac{\nu}{\gamma^5(t)}\big(\alpha'(t) + y\gamma'(t)\big)v_y(t)\Big).$$

Using the hypotheses (3.4), (3.6), (3.7) and Poincarè's inequality we obtain estimates for the right-hand side of (3.29) yielding

$$\frac{d}{dt}E_2(t) + \frac{\zeta_0}{\gamma^2(t)}\left|v_y(t)\right|^2 + \frac{1}{\gamma(t)}\Big(\zeta_1 - \varepsilon C_2\Big)\left|v_y(t)\right|^4 + \\ \frac{1}{\gamma^4(t)}\Big(\frac{1}{2} - \varepsilon C_3\Big)\left|v_{yy}(t)\right|^2 \leq \\ \big(1 + 2\varepsilon^2 K_3^2\big)\left|v_{yyt}(t)\right|^2 + \frac{1}{4\gamma^4(t)}\left|v_{yyyy}(t)\right|^2. \tag{3.30}$$

where C_2 and C_3 depends on K_3, σ, γ_0 and K_3, ν, ϵ, γ_0, respectively.

- **Estimate III.** Substituting ϕ by $v_{yyyy}(t)$ into $(3.24)_1$, proceeding as in the estimates I and II and denoting by $E_3(t)$ the function

$$E_3(t) = (v_t(t), v_{yyyy}(t)) + \frac{\nu}{2\gamma^4(t)} |v_{yyyy}(t)|^2 , \tag{3.31}$$

we obtain

$$\begin{aligned} &\frac{d}{dt}E_3(t) + \frac{1}{\gamma^4(t)} |v_{yyyy}(t)|^2 = \\ &|v_{yyt}(t)|^2 + \frac{11\nu\gamma'(t)}{2\gamma^5(t)} |v_{yyyy}(t)|^2 - (c_1(y,t)v_y(t), v_{yyyy}(t)) - \\ &(c_2(y,t)v_{yt}(t), v_{yyyy}(t)) - (c_3(y,t)v_{yy}(t), v_{yyyy}(t)) - \\ &d_1(t)\Big(\int_\Omega |v_y(t)|^2\,dy\Big)(v_{yy}(t), v_{yyyy}(t)) - \\ &\Big(\int_\Omega c_4(y,t)v_y(t)v_{yy}(t)dy\Big)(v_{yy}(t), v_{yyyy}(t)) - \\ &d_2(t)\Big(\int_\Omega v_y(t)v_{yt}(t)dy\Big)(v_{yy}(t), v_{yyyy}(t)) . \end{aligned} \tag{3.32}$$

Again, using the hypotheses (3.4), (3.6), (3.7), Poincarè's inequality we obtain estimates for the right-hand side of (3.34), and this way yields

$$\begin{aligned} &\frac{d}{dt}E_3(t) + \frac{1}{\gamma^4(t)} |v_{yyyy}(t)|^2 \le \\ &\Big[2\varepsilon\Big(17\varepsilon^2 + 8\varepsilon K_3 + K_3^2 + 2\zeta_0\Big) + 2\zeta_0^2\Big] |v_{yy}(t)|^2 + \\ &\Big(1 + 8\varepsilon^2 K_3^2\Big) |v_{yyt}(t)|^2 + \frac{\varepsilon}{\gamma^4(t)}\Big(\frac{11\nu}{\gamma_0} + \frac{3}{4\gamma_0}\Big) |v_{yyyy}(t)|^2 + \\ &\frac{1}{\gamma^4(t)}\Big[\varepsilon\big(\zeta_1 + \sigma\big) |v_{yy}(t)|\,|v_{yyyy}(t)| + \sigma K_3 |v_t(t)|\,|v_{yyyy}(t)|\Big] |v_{yy}(t)|^2 . \end{aligned}$$

Applying Cauchy-Schwartz's inequality in the last term of precedent inequality and making some modifications yields

$$\begin{aligned} &\frac{d}{dt}E_3(t) + \frac{1}{\gamma^4(t)}\Big[1 - \varepsilon\Big(\frac{11\nu}{\gamma_0} + \frac{3}{4\gamma_0}\Big)\Big] |v_{yyyy}(t)|^2 \le \\ &\Big(\varepsilon C_4 + 2\zeta_0^2\Big) |v_{yy}(t)|^2 + \Big(1 + 8\varepsilon^2 K_3^2\Big) |v_{yyt}(t)|^2 + \\ &\frac{1}{2\gamma^4(t)}\Big[\, |v_t(t)|^2 + \varepsilon\big(\zeta_1 + \sigma\big) |v_{yy}(t)|^2 + C_5 |v_{yyyy}(t)|^2 \Big] |v_{yy}(t)|^2 , \end{aligned} \tag{3.33}$$

where C_4 and C_5 depende on ζ_0, K_3, ϵ, and ζ_1, K_3, σ, ϵ, respectively.

Multiplying the inequalities (3.30) and (3.33) for $\dfrac{\nu}{8K_3^4}$ and, to the result it is added to the (3.27) yields

$$\begin{aligned} &\frac{d}{dt}E_4(t) + \frac{\nu}{8K_3^6}\Big[\frac{\zeta_1}{2} - \varepsilon\Big(C_2 + \frac{8K_3^6 C_0}{\nu\gamma_0^2}\Big)\Big] |v_y(t)|^4 + \\ &\frac{\nu}{8K_3^4}\frac{1}{\gamma^4(t)}\Big[\frac{1}{8} - 2\zeta_0^2 K_3^4 + \frac{1}{8} - \varepsilon\Big(C_3 + C_4 + \frac{8K_3^4 C_1}{\nu\gamma_0^4}\Big)\Big] |v_{yy}(t)|^2 + \end{aligned}$$

(3.34)
$$\frac{1}{\gamma^4(t)}\Big[\frac{\nu}{4}-\frac{\varepsilon}{K_3^2}\Big(4K_3^5+\nu\varepsilon\Big)\Big]\left|v_{yyt}(t)\right|^2+$$
$$\frac{\nu}{8K_3^8}\Big[\frac{3}{4}-\varepsilon\Big(\frac{11\nu}{\gamma_0}+\frac{3}{4\gamma_0}+\frac{80\varepsilon K_3^4}{\gamma_0^6}\Big)\Big]\left|v_{yyyy}(t)\right|^2+$$
$$\frac{\nu}{8K_3^4}\frac{1}{\gamma^4(t)}\Big(\frac{1}{4}-\Lambda(t)\Big)\left|v_{yy}(t)\right|^2\leq 0,$$

where

(3.35)
$$E_4(t)=E_1(t)+\frac{\nu}{8K_3^4}E_2(t)+\frac{\nu}{8K_3^4}E_3(t),$$

(3.36)
$$\Lambda(t)=\frac{1}{2}\left\{\left|v_t(t)\right|^2+\frac{\varepsilon}{\gamma_0^4}\left(\zeta_1+\sigma\right)\left|v_{yy}(t)\right|^2+C_5\left|v_{yyyy}(t)\right|^2\right\}.$$

Denoting by

(3.37)
$$\Gamma_0=C_2+\frac{8K_3^6C_0}{\nu\gamma_0^2},\quad \Gamma_1=C_3+C_4+\frac{8K_3^4C_1}{\nu\gamma_0^4}$$
$$\Gamma_2=\frac{1}{K_3^2}\Big(4K_3^5+\nu\varepsilon\Big),\quad \Gamma_3=\frac{11\nu}{\gamma_0}+\frac{3}{4\gamma_0}+\frac{80\varepsilon K_3^4}{\gamma_0^6},$$

and observing the hypotheses (3.8) and (3.9) we get from (3.24) that

(3.38)
$$\frac{d}{dt}E_4(t)+\frac{\nu}{8K_3^4}\frac{1}{\gamma^4(t)}\Big(\frac{1}{4}-\Lambda(t)\Big)\left|v_{yy}(t)\right|^2\leq 0.$$

Our next task is to prove that

(3.39)
$$E_4(t)\leq E_4(0)\quad\text{for all}\quad t\geq 0.$$

In fact, from (3.22), (3.35) and (3.36) is not hard to see that there exist real positive constants K_4 and K_5 such that

(3.40)
$$\Lambda(t)\leq K_5E_4(t)\leq K_4E_5(t)\ \text{ for all }\ t\geq 0.$$

From (3.18) and (3.40) yields $\Lambda(0)<\frac{1}{4}$. Hence, as $\Lambda(t)$ is continuous on $[0,\infty[$ then there exists a positive t_m such that

(3.41)
$$\Lambda(t)<\frac{1}{4}\ \text{ for all }\ t\in[0,t_m[.$$

From (3.41) we affirm that there exists t^* such that

(3.42)
$$\Lambda(t)<\frac{1}{4}\ \text{ for all }\ t\in[0,t^*[.$$

Besides, we suppose that

(3.43)
$$\Lambda(t^*)=\frac{1}{4}.$$

Thus, integrating (3.38) from 0 to t^* and in addition observing (3.42) yields

$$E_4(t^*) \leq E_4(0). \tag{3.44}$$

From (3.40), (3.44) and (3.18) yields

$$\Lambda(t^*) \leq K_5 E_4(t^*) \leq K_5 E_4(0) \leq K_4 E_5(0) < \frac{1}{4}. \tag{3.45}$$

Thus, from (3.43) and (3.45) we obtain a contradiction $\rightarrow \mid \leftarrow$. Therefore, we have that (3.39) is true.

On the other hand, the constant $\dfrac{\nu}{8K_3^4}$ is such that

$$\begin{aligned}
&\frac{1}{4}\left|v_t(t)\right|^2 + \frac{\nu^2}{32K_3^4\gamma^4(t)}\left|v_{yy}(t)\right|^2 + \frac{\nu^2}{32K_3^4\gamma^4(t)}\left|v_{yyyy}(t)\right|^2 \leq \\
&\frac{1}{2}\left|v_t(t)\right|^2 + \frac{\nu}{8K_3^4}\left(v_t(t), v(t)\right) + \frac{\nu}{8K_3^4}\left(v_t(t), v_{yyyy}(t)\right) + \\
&\frac{\nu^2}{16K_3^4\gamma^4(t)}\left|v_{yy}(t)\right|^2 + \frac{\nu^2}{16K_3^4\gamma^4(t)}\left|v_{yyyy}(t)\right|^2 .
\end{aligned} \tag{3.46}$$

In fact, using the hypothesis (3.7), the inequality (3.46) is a consequence of

$$\frac{\nu}{8K_3^4}\left(v_t(t), v(t)\right) \geq -\frac{1}{8}\left|v_t(t)\right|^2 - \frac{\nu^2}{32K_3^4\gamma^4(t)}\left|v_{yy}(t)\right|^2 ,$$

$$\frac{\nu}{8K_3^4}\left(v_t(t), v_{yyyy}(t)\right) \geq -\frac{1}{8}\left|v_t(t)\right|^2 - \frac{\nu^2}{32K_3^4\gamma^4(t)}\left|v_{yyyy}(t)\right|^2 .$$

Using the left-hand side of (3.46) we define the function $E_6(t)$ by

$$E_6(t) = \frac{1}{4}\left|v_t(t)\right|^2 + \frac{\nu^2}{32K_3^4\gamma^4(t)}\left|v_{yy}(t)\right|^2 + \frac{\nu^2}{32K_3^4\gamma^4(t)}\left|v_{yyyy}(t)\right|^4 . \tag{3.47}$$

From (3.47), (3.46), (3.35), (3.22), (3.5) and neglecting some positive terms we obtain

$$E_4(t) \geq E_5(t) \geq E_6(t). \tag{3.48}$$

Thus, from (3.39) and (3.48) yields

$$\begin{aligned}
&(v^m) \text{ is bounded in } L^\infty\left(0,\infty; H^4(\Omega)\right), \\
&(v_t^m) \text{ is bounded in } L^\infty\left(0,\infty; L^2(\Omega)\right).
\end{aligned} \tag{3.49}$$

Integrating with respect to t the inequalities (3.34) we can see without difficulty that

$$(v_t^m) \text{ is bounded in } L^2\left(0,\infty; H^2(\Omega)\right). \tag{3.50}$$

From (3.49) and (3.50) we can take the limit in the approximate solutions of the system (3.24). It is made by a similar way of subsection 2.1, and the limit solution v is got in the weak sense of distributions. Still, from $(3.49)_1$ and (3.50) we can affirm that v_{yyyyyy} and v_{yyyyt} belong to $L^2\left(0,\infty; H^{-1}(\Omega)\right)$ and $L^2\left(0,\infty; H^{-2}(\Omega)\right)$ respectively. Hence, we can conclude that (3.10) is verified in the sense of $L^2\left(0,\infty; H^{-2}(\Omega)\right)$. Consequently (3.21) is verified, and (3.19), (3.20) are guaranteed by $(3.49)_1$ and (3.50) respectively.

The uniqueness is obtained by energy's methods. This way the demonstration of Theorem 3.2 is concluded. Finally, as said before, the demonstration of Theorem 3.1 is established via Theorem 3.2 and (1.7) $\square$

3.2 Asymptotic behavior

We start this subsection pointing out the difficulty to establish the asymptotic behavior of energy directly inside of the non-cylindrical domain. In fact, taking the scalar product on $L^2(\Omega_t)$ of both sides of (3.1) with u_t, integrating by parts, assuming the hypothesis $(2.31)_2$, and observing the boundary conditions we get

$$\frac{d}{dt}E(t) \le u_{xxt}u_{xt}\Big|_{\alpha(t)}^{\beta(t)} - \nu\,|u_{xxt}(t)|^2 \quad \text{for all } \; t \ge 0, \tag{3.51}$$

where

$$E(t) = \frac{1}{2}\left\{|u_t(t)|^2 + \zeta_0\,|u_x(t)|^2 + \frac{\zeta_1}{2}\,|u_x(t)|^4 + |u_{xx}(t)|^2\right\}. \tag{3.52}$$

Note that the term of greater order in the energy has two spacial derivatives, while the pointwise term in (3.51) has three. Thus, is not easy to build a Lyapunov functional as made in the subsection 2.2. Therefore, we initially will show the exponential decay of the energy associated with the solution v guaranteed by Theorem 3.2.

Lemma 3.1. *If v is the solution of the system (3.10)-(3.12), then*

$$E_5(t) \le K_6 \exp\Big\{-\frac{t}{\Upsilon}\Big\} \quad \textit{for all } \; t \ge 0, \tag{3.53}$$

where $E_5(t)$ is defined in (3.22), $K_6 = E_4(0)$, and Υ is real positive constant determinate in (3.61).

Proof. The inequalities (3.53) will be obtained as limit of the approximate solutions v^m. The convergence obtained in the proof of Theorem 3.2 allow us to conclude such result. Therefore, we will also denote henceforth the function v^m by v. From (3.34) and (3.9) we have for all $t \ge 0$ that

$$\begin{aligned}\frac{d}{dt}E_4(t) + \frac{\nu\zeta_1}{32K_3^6}\,|v_y(t)|^4 + \frac{\nu}{128K_3^4\gamma^4(t)}\,|v_{yy}(t)|^2 + {} \\ \frac{\nu}{8\gamma^4(t)}\,|v_{yyt}(t)|^2 + \frac{3\nu}{64K_3^8}\,|v_{yyyy}(t)|^2 \le 0.\end{aligned} \tag{3.54}$$

On the other hand, observing that $\dfrac{\nu}{128K_3^4\gamma^4(t)} = \dfrac{\nu}{256K_3^4\gamma^4(t)} + \dfrac{\nu}{256K_3^4\gamma^4(t)}$, and by Poicarè's inequality we get the following inequalities

$$\frac{1}{2}\,|v_t(t)|^2 \le \Upsilon_0\frac{\nu}{8\gamma^4(t)}\,|v_{yyt}(t)|^2 \quad \text{where } \; \Upsilon_0 = \frac{4K_3^4}{\nu}. \tag{3.55}$$

$$\frac{\zeta_0}{2\gamma^2(t)}\,|v_y(t)|^2 \le \Upsilon_1\frac{\nu}{256K_3^4\gamma^4(t)}\,|v_{yy}(t)|^2 \quad \text{and } \; \Upsilon_1 = \frac{128\zeta_0K_3^6}{\nu}. \tag{3.56}$$

$$\Big(\frac{\zeta_1}{4\gamma^3(t)} - \frac{\sigma\gamma'(t)}{8\gamma^4(t)} + \frac{\nu}{8K_3^4}\frac{\sigma}{4\gamma^3(t)}\Big)\,|v_y(t)|^4 \le \Upsilon_2\frac{\nu\zeta_1}{32K_3^6}\,|v_y(t)|^4, \tag{3.57}$$

where $\Upsilon_2 = \dfrac{K_3^3}{\nu\zeta_1\gamma_0^4}\Big(8\zeta_1 K_3^4 + 4K_3^3\sigma\varepsilon_0 + \nu\sigma\Big)$ and $\varepsilon_0 = \inf\limits_{t\geq 0}|\gamma'(t)|$.

$$
\begin{aligned}
&\Big[\frac{\nu^2}{16K_3^4\gamma^4(t)} + \frac{1}{2\gamma^5(t)}\Big(\gamma(t) - 4\nu\gamma'(t)\Big)\Big]\,|v_{yy}(t)|^2 \leq \\
&\Upsilon_3\frac{\nu}{256K_3^4\gamma^4(t)}\,|v_{yyy}(t)|^2 \quad \text{where} \\
&\Upsilon_3 = \frac{K_3}{\nu\gamma_0}\Big(\nu^2 + 8K_3^4 + 32\nu\varepsilon_0K_3^3\Big).
\end{aligned}
\tag{3.58}
$$

From similar way we can affirm that there exist real positive constants Υ_4 and Υ_5 such that

$$
\begin{aligned}
\frac{\nu}{8K_3^4}\Big[\,(v_t(t),v(t)) + (v_t(t),v_{yyyy}(t)) + \frac{\nu}{\gamma^4(t)}\,|v_{yyyy}(t)|^2\,\Big] \leq \\
\Upsilon_4\,|v_{yyt}(t)|^2 + \Upsilon_5\frac{3\nu}{64K_3^2}\,|v_{yyyy}(t)|^2
\end{aligned}
\tag{3.59}
$$

From (3.55)-(3.59) and (3.35) we can write

$$
\begin{aligned}
E_4(t) \leq \Upsilon\frac{\nu\zeta_1}{32K_3^6}\,|v_y(t)|^4 + \Upsilon\frac{\nu}{128K_3^4\gamma^4(t)}\,|v_{yy}(t)|^2 + \\
\Upsilon\frac{\nu}{8\gamma^4(t)}\,|v_{yyt}(t)|^2 + \Upsilon\frac{3\nu}{64K_3^8}\,|v_{yyyy}(t)|^2,
\end{aligned}
\tag{3.60}
$$

where

$$
\Upsilon = \max\big\{\Upsilon_0,\ \Upsilon_1,\ \Upsilon_2,\ \Upsilon_3,\ \Upsilon_4,\ \Upsilon_5,\big\}.
\tag{3.61}
$$

From (3.54) and (3.60) we obtain

$$
\frac{d}{dt}E_4(t) + \frac{1}{\Upsilon}E_4(t) \leq 0 \ \text{ for all } \ t \geq 0.
$$

It implies that

$$
E_4(t) \leq E_4(0)\exp\Big\{-\frac{t}{\Upsilon}\Big\} \ \text{ for all } \ t \geq 0.
\tag{3.65}
$$

Using (3.46) and the definitions of $E_4(t)$ and $E_5(t)$ in (3.35) and (3.22) respectively, we obtain

$$
E_4(t) \geq E_5(t) \ \text{ for all } \ t \geq 0.
$$

Hence, we conclude (3.53) $\square$

To conclude this subsection we will prove that the energy of the system (3.1)-(3.3) given by (3.52) has the same behavior of the function $E_5(t)$, i.e.,

Theorem 3.3. *If u is the solution of the system (3.1)-(3.3), then*

$$
E(t) \leq K_7\exp\Big\{-\frac{t}{\Upsilon}\Big\} \ \textit{ for all } \ t \geq 0,
\tag{3.63}
$$

where $E(t)$, K_7 *and* Υ *are defined in (3.52), (3.69) and (3.61) respectively.*

Proof. From (1.7) we have

$$u_t = v_t - \frac{1}{\gamma}(\alpha' + y\gamma')v_y, \quad \partial_x^i u = \frac{1}{\gamma^i}\partial_y^i v \text{ for } i \in \mathbb{N}. \tag{3.64}$$

From $(3.64)_1$ we obtain

$$|u_t(x,t)|^2_{\mathbb{R}} \leq 2\,|v_t(y,t)|^2_{\mathbb{R}} + \frac{2}{\gamma^2(t)}\big(\,|\alpha'(t)| + |\gamma'(t)|\,\big)^2\,|u_y(y,t)|^2_{\mathbb{R}}\,.$$

Integrating this inequalities from $\alpha(t)$ to $\beta(t)$ and observing that $y = \frac{x-\alpha(t)}{\gamma(t)}$ and $dy = \frac{1}{\gamma(t)}dx$ then we get by using (3.4), (3.6) and (3.7) that

$$|u_t(t)|^2_{L^2(\Omega_t)} \leq 2K_3\,|v_t(t)|^2_{L^2(0,1)} + \frac{18\varepsilon^2}{\gamma_0^2}\,|v_y(t)|^2_{L^2(0,1)}\,. \tag{3.65}$$

Using $(3.64)_2$ and repeating the same arguments above we get

$$\left|\partial_x^i u\right|^2_{L^2(\Omega_t)} \leq \frac{1}{\gamma_0^i}\,\left|\partial_y^i v\right|^2_{L^2(0,1)} \quad \text{for } i \in \mathbb{N}. \tag{3.66}$$

From (3.5), (3.7) and definition of $E_5(t)$ we obtain

$$\begin{aligned} E_5(t) \geq \frac{1}{4}\,|v_t(t)|^2 + \frac{\sigma}{4K_3^3}\,|v_y(t)|^2 + \\ \frac{\nu^2}{64K_3^8}\,|v_y(t)|^4 + \frac{\nu^2}{64K_3^8}\,|v_{yy}(t)|^2\,. \end{aligned} \tag{3.67}$$

Using (3.65), (3.66) and observing the definition of $E(t)$ in (3.52) yields

$$\begin{aligned} E(t) \leq K_3\,|v_t(t)|^2 + \Big(\frac{9\varepsilon^2}{\gamma_0^2} + \frac{\zeta_0}{2\gamma_0}\Big)\,|v_y(t)|^2 + \\ \frac{\zeta_1}{4\gamma_0^3}\,|v_y(t)|^4 + \frac{1}{2\gamma_0^3}\,|v_{yy}(t)|^2\,. \end{aligned} \tag{3.68}$$

From (3.53), (3.67) and (3.68) we obtain (3.63) where

$$K_7 = K_6\Big[4K_3 + \Big(\frac{9\varepsilon^2}{\gamma_0^2} + \frac{\zeta_0}{2\gamma_0}\Big)\frac{4K_3^4}{\sigma} + \frac{16\zeta_1 K_3^5}{\nu^2} + \frac{32K_3^5}{\nu^2}\Big]. \tag{3.69}$$

Thus, the proof of Theorem 3.3 is completed $\square$

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ON THE ZEROS OF THE RIEMANN ξ-FUNCTION

George Csordas and Chung-Chun Yang

Abstract. Necessary and sufficient conditions for the validity of the Riemann Hypothesis are established in terms of the sum of the reciprocal squares of the zeros of the Riemann ξ-function. Numerical work, in conjunction with density estimates, involving the first 100,000 zeros of the Riemann ξ-function are presented.

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Key Words and Phrases. Zeros of the Riemann ξ-function, Riemann Hypothesis

1. Introduction. This paper is motivated by a recent result of P. C. Hu dealing with the Riemann Hypothesis. After a brief review of the standard terminology and notation adopted in the theory of the Riemann zeta function (Section 2), we proffer a new proof of Hu's theorem (Theorem 2.2). This theorem provides a necessary and sufficient condition for the validity of the Riemann Hypothesis in terms of the sum of the reciprocal squares of the zeros of the Riemann ξ-function. We also prove an analogous theorem (Theorem 2.3) using the Fourier integral representation of the Riemann ξ-function. In Section 3, we elucidate the aforementioned theoretical results by means of concrete examples involving the first 100,000 zeros the Riemann ξ-function. With the aid of a refinement of the Riemann–von Mangoldt formula (see Backlund's density theorem, Theorem 3.1), we are able to estimate the tail end of the series of the sum of the reciprocal squares of the zeros (Lemma 3.2 and Proposition 3.3). The paper ends with an open problem pertaini! ng to the distribution of zeros of a a real entire function related to the Riemann ξ-function.

2. The zeros of the Riemann ξ-function. The function, $\zeta(s)$, is defined for $\sigma = \operatorname{Re} s > 1$ by the simplest of Dirichlet series, the series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (s = \sigma + it). \tag{2.1}$$

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This Dirichlet series is convergent for $\operatorname{Re} s = \sigma > 1$ and is uniformly convergent for $\sigma \geq \sigma_0 > 1$. The function $\zeta(s)$ admits analytic (meromorphic) continuation to the entire complex plane where its only singularity is a simple pole at $s = 1$ with residue 1 ([15, p. 13]) For general information about $\zeta(s)$ we refer the reader to the excellent monographs [10], [11], [12], [14] and [15]. The functional equation (for the zeta function) can be expressed in a number of different ways (cf. [15, p. 13, p. 16] or [11, p. 8]):

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \qquad (s \in \mathbb{C}) \tag{2.2}$$

or, for example, in the symmetric form

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{2.3}$$

The Riemann ξ-function defined by the formula

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{2.4}$$

satisfies the relation $\xi(s) = \xi(1-s)$ which is equivalent to (2.3). Now it is known that $\xi(s)$ is an entire function of order 1 and of maximal type ([15, p. 29] or [12, p. 20].

The non-real (non-trivial) zeros of $\zeta(s)$, $\{\rho_k\}_{k=1}^{\infty}$, are the zeros of $\xi(s)$ and these zeros all lie in the *critical strip* $0 \leq \sigma \leq 1$. (The "trivial zeros" of $\zeta(s)$, $-2, -4, -6, \cdots$, are not zeros of $\xi(s)$ since they are cancelled by the poles of $\Gamma(s/2)$.) Now, $\zeta(s)$ (and *a fortiori* $\xi(s)$) has an infinite number of zeros in the critical strip ([15, p. 30] and the Riemann Hypothesis is the assertion that the non-trivial zeros of $\zeta(s)$ (that is, the zeros of $\xi(s)$) all lie on the*critical line* $\operatorname{Re} s = \sigma = 1/2$. Since $\zeta(s)$ is real on the real axis, $\overline{\xi(s)} = \xi(\overline{s})$, and whence $\overline{\rho}$ is a zero of $\xi(s)$ whenever ρ is. Also, by the functional equation $\xi(s) = \xi(1-s)$, if ρ is a zero of $\xi(s)$, then $1-\rho$ and $1-\overline{\rho}$ are also zeros of $\xi(s)$. Thus, the zeros of $\xi(s)$ are situated symmetrically with respect to the lines $\operatorname{Im} s = 0$ and $\operatorname{Re} s = \frac{1}{2}$.

Since $\xi(s)$ is an entire function of order 1, and $\xi(0) = -\zeta(0) = \frac{1}{2}$, the Hadamard factorization theorem gives (cf. [15, p. 30]) for all $s \in \mathbb{C}$

$$\xi(s) = \frac{1}{2}e^{B_0 s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}, \qquad \left(\sum_{\rho}\frac{1}{|\rho|^2} < \infty\right), \tag{2.5}$$

where the product extends over all the zeros of $\xi(s)$ and B_0 is the constant

$$B_0 := \frac{1}{2}\log 4\pi - 1 - \frac{1}{2}\gamma, \quad \text{where} \quad \gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) \tag{2.6}$$

and where γ is Euler's constant ($\gamma = -\Gamma'(1) = 0.577\ 215\ 664\cdots$).

Remark 2.1. *In 1896, Hadamard and de la Vallée-Poussin proved independently that $\zeta(s)$ has no zeros on the line* $\operatorname{Re} s = \sigma = 1$ *(*[15, p. 45] *or* [12, p. 25]*). Moreover, Hardy proved in 1914 that $\zeta(s)$ has an infinite number of zeros on the critical line* $\operatorname{Re} s = \sigma = \frac{1}{2}$. *Thus, the zeros of $\xi(s)$ all lie in the open critical strip* $0 < \sigma < 1$ *and $\xi(s)$ has an infinite number of zeros on the critical line* $\sigma = \frac{1}{2}$.

Theorem 2.2 (Hu [9, Theorem 2]). *Let $\rho_k = \frac{1}{2} + i\gamma_k$, $(k = 1, 2, 3 \dots)$, $0 < \gamma_k \le \gamma_{k+1} \le \cdots$, denote the zeros of $\xi(s)$ (that is, the nontrivial zeros of $\zeta(s)$ on the critical line in the upper half-plane $\operatorname{Im} s > 0$). Then the Riemann Hypothesis is true if and only if*

$$\sum_{k=1}^{\infty} \frac{1}{\frac{1}{4} + \gamma_k^2} = 1 + \frac{\gamma}{2} - \frac{1}{2}\log(4\pi), \tag{2.7}$$

where γ (without a subscript) denotes Euler's constant.

Proof. In view of the Hadamard factorization of $\xi(s)$ ((2.5)), logarithmic differentiation yields

(2.8)

$$\frac{\xi'(s)}{\xi(s)} = B_0 + \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) \quad \text{and} \quad -\frac{\xi'(1-s)}{\xi(1-s)} = -B_0 - \sum_{\rho} \left(\frac{1}{1-s-\rho} + \frac{1}{\rho} \right),$$

where B_0 is defined by (2.6), ρ extends over all the zeros of $\xi(s)$, and the series converge uniformly on compact subsets of the plane punctured at the zeros of $\xi(s)$. Since $\xi(s) = \xi(1-s)$, $\dfrac{\xi'(s)}{\xi(s)} = -\dfrac{\xi'(1-s)}{\xi(1-s)}$ and so by (2.8) we have

$$\sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) + \sum_{\rho} \left(\frac{1}{1-s-\rho} + \frac{1}{\rho} \right) = -2B_0. \tag{2.9}$$

Setting $s = 0$ in (2.9), yields

$$\sum_{\rho} \left(\frac{1}{1-\rho} + \frac{1}{\rho} \right) = -2B_0. \tag{2.10}$$

Since the zeros of $\xi(s)$ are situated symmetrically with respect to the lines $\operatorname{Im} s = 0$ and $\operatorname{Re} s = 1/2$, we can express (2.10) in the form

$$\begin{aligned} -2B_0 &= \sum_{\operatorname{Im}\rho > 0} \left(\frac{1}{1-\rho} + \frac{1}{\rho} + \frac{1}{1-\bar{\rho}} + \frac{1}{\bar{\rho}} \right) = 2 \sum_{\operatorname{Im}\rho > 0} \left(\frac{1-\operatorname{Re}\rho}{|1-\rho|^2} + \frac{\operatorname{Re}\rho}{|\rho|^2} \right) \\ &= 2 \sum_{\operatorname{Re}\rho = 1/2,\, \operatorname{Im}\rho > 0} \left(\frac{1-\operatorname{Re}\rho}{|1-\rho|^2} + \frac{\operatorname{Re}\rho}{|\rho|^2} \right) + 2 \sum_{\operatorname{Re}\rho \ne 1/2,\, \operatorname{Im}\rho > 0} \left(\frac{1-\operatorname{Re}\rho}{|1-\rho|^2} + \frac{\operatorname{Re}\rho}{|\rho|^2} \right) \\ &= 2 \sum_{k=1}^{\infty} \frac{1}{\frac{1}{4} + \gamma_k^2} + 2 \sum_{\operatorname{Re}\rho \ne 1/2,\, \operatorname{Im}\rho > 0} \left(\frac{1-\operatorname{Re}\rho}{|1-\rho|^2} + \frac{\operatorname{Re}\rho}{|\rho|^2} \right). \end{aligned}$$

Thus, if we set $S := \{\rho \,|\, \xi(\rho) = 0, \operatorname{Re}\rho \ne \frac{1}{2}, \operatorname{Im}\rho > 0\}$, we have (using (2.6)),

$$\sum_{k=1}^{\infty} \frac{1}{\frac{1}{4} + \gamma_k^2} + \sum_{\rho \in S} \left(\frac{1-\operatorname{Re}\rho}{|1-\rho|^2} + \frac{\operatorname{Re}\rho}{|\rho|^2} \right) = 1 + \frac{\gamma}{2} - \frac{1}{2}\log(4\pi), \tag{2.11}$$

where we adopt the usual convention that any sum over the empty set is zero. Now, if the Riemann Hypothesis is valid, then $S = \emptyset$ in (2.11) and so (2.7) holds.

Conversely, suppose that (2.7) holds. If the Riemann Hypothesis is false, then $S \neq \emptyset$. But then, since $\operatorname{Re}\rho > 0$ and $1 - \operatorname{Re}\rho > 0$ for $\rho \in S$ (cf. Remark 2.1), the second summand on the left-hand side of (2.11) is positive. This contradicts (2.7) and thus, if (2.7) holds, then the Riemann Hypothesis is true. □

In the foregoing proof, it was convenient to evaluate the sum in (2.9) at $s = 0$. We remark that any other fixed value of s could have been used to carry out the proof.

We next establish another necessary and sufficient condition, analogous to Theorem 2.2, for the validity of the Riemann Hypothesis. To this end, we recall that the Riemann Ξ-function admits the Fourier integral representation (see, for example, [15, p. 255] or [3])

$$H(x) := \frac{1}{8}\Xi\left(\frac{x}{2}\right) = \int_0^\infty \Phi(t)\cos(xt)\,dt \tag{2.12}$$

where $\Xi(x) := \xi(\frac{1}{2} + ix)$ ([15, p. 16]) and

$$\Phi(t) := \sum_{n=1}^{\infty} \left(2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t}\right) \exp\left(-\pi n^2 e^{4t}\right) \quad (t \in \mathbb{R}). \tag{2.13}$$

The Riemann Hypothesis is the statement that all the zeros of $H(x)$ are real. (We mention here parenthetically that Riemann stated his hypothesis in terms of the reality of the zeros of the Ξ-function, see, for example, [10, Appendix, pp 299–305] for a translation of Riemann's famous memoir *"Ueber die Anzahl der Primzahlen unter einer gegebene Grösse".*) For the various properties of $\Phi(t)$ we refer the reader to [3, Theorem A] and [7, Theorem 2.1]. Now, $H(x)$ is an *even*, *real* entire function of order 1 (and of maximal type) all of whose zeros lie in the horizontal strip $|\operatorname{Im} z| < 1$. Thus, if $z = \alpha + i\beta$ ($|\operatorname{Im}\beta| < 1$) is a zero of $H(x)$, then $-\alpha + i\beta$, $-\alpha - i\beta$ and $\alpha - i\beta$ are also zeros of $H(x)$. Since $H(x) = \frac{1}{8}\Xi\left(\frac{x}{2}\right) = \frac{1}{8}\xi(\frac{1}{2} + i\frac{x}{2})$, we see that the zeros of $\xi(s)$ on the critical line are precisely of the form $\dfrac{1}{2} + i\dfrac{x_j}{2}$, where x_j is a real zero of $H(x)$. Hence, if x_j is a real zero of $H(x)$ in the right half-plane, then $x_j = 2\gamma_j$, where $\rho_j = \frac{1}{2} + i\gamma_j$ is a zero of $\xi(s)$ on the critical line in the upper half-plane, $\operatorname{Im} s > 0$. Finally, we also note that the Taylor series expansion of $H(x)$ about the origin can be written in the form

$$H(x) = \sum_{k=0}^{\infty} \frac{(-1)^k b_k}{(2k)!} x^{2k}, \tag{2.14}$$

where the b_k's are the moments defined by

$$b_k = \int_0^\infty t^{2k}\Phi(t)\,dt, \quad k = 0, 1, 2, \ldots. \tag{2.15}$$

Preliminaries aside, we are now in position to prove the following analog of Theorem 2.2.

Theorem 2.3. *Let* $\{x_j\}_{j=1}^{\infty}$, $0 < x_1 \leq x_2 \leq \cdots$, *denote the real zeros of* $H(x)$ *in the right half-plane. Then the Riemann Hypothesis is true if and only if*

$$\sum_{j=1}^{\infty} \frac{1}{x_j^2} = \frac{b_1}{2b_0} \qquad \text{or} \qquad \sum_{j=1}^{\infty} \frac{1}{\gamma_j^2} = \frac{2b_1}{b_0} \tag{2.16}$$

where b_k *denotes the* k^{th} *moment of* $\Phi(t)$ *defined in (2.15) and* $x_j = 2\gamma_j$ $(j = 1, 2, 3 \cdots)$.

Proof. By the Hadamard factorization theorem $H(z)$ can be represented in the form

$$H(z) = H(0) \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{z_k^2}\right), \qquad \left(\sum_{k=1}^{\infty} \frac{1}{|z_k|^2} < \infty\right), \tag{2.17}$$

where $H(0) > 0$ (since $\Phi(t) > 0$ for all $t \in \mathbb{R}$) and where the zeros of $H(z)$ are numbered according to increasing moduli, i.e., $0 < |z_1| \leq |z_2| \leq \cdots$. Now let $S := \{z_k = \alpha_k + i\beta_k \mid H(z_k) = 0, \operatorname{Im} \beta_k \neq 0\}$ and let $\{x_j\}_{j=1}^{\infty}$, denote the real zeros of $H(x)$ in the right half-plane. Then (2.17) can be expressed in the form

$$H(z) = H(0) \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{x_j^2}\right) \prod_{z_k \in S} \left(1 - \frac{z^2}{z_k^2}\right), \tag{2.18}$$

where the product over the zeros in S is defined to be 1, if S is the empty set. Now using (2.18), a calculation shows that the derivative of the logarithmic derivative of $H(z)$ is

$$\frac{H(z)H''(z) - (H'(z))^2}{(H(z))^2} = -\sum_{j=1}^{\infty} \left(\frac{1}{(z - x_j)^2} + \frac{1}{(z + x_j)^2}\right) - \sum_{z_k \in S} \left(\frac{1}{(z - z_k)^2} + \frac{1}{(z + z_k)^2}\right). \tag{2.19}$$

Next, using (2.15), (2.19) and the fact that $H(z)$ is an even function (so that $H'(0) = 0$), we find that

$$\left(\frac{H'}{H}\right)'(0) = \frac{H''(0)}{H(0)} = -\frac{b_1}{b_0}. \tag{2.20}$$

On the other hand, setting $z = 0$ in (2.19), we obtain

$$\left(\frac{H'}{H}\right)'(0) = -2 \sum_{j=1}^{\infty} \frac{1}{x_j^2} - 2 \sum_{\substack{\alpha_k + i\beta_k \in S \\ \operatorname{Im} \beta_k > 0}} \left(\frac{\alpha_k^2 - \beta_k^2}{|\alpha_k + i\beta_k|^4}\right)$$

or (using (2.20))

$$\sum_{j=1}^{\infty} \frac{1}{x_j^2} + \sum_{\substack{\alpha_k + i\beta_k \in S \\ \operatorname{Im} \beta_k > 0}} \frac{\alpha_k^2 - \beta_k^2}{|\alpha_k + i\beta_k|^4} = \frac{b_1}{2b_0}. \tag{2.21}$$

But if $S \neq \emptyset$, then the second summand on the left-hand side of (2.21) is positive since all the zeros of $H(z)$ lie in the strip $|Im\, z| < 1$ and $H(z)$ has no zeros ([15, p. 389] or see Section 3 below) in the closed unit square centered at the origin. Thus, we infer from (2.21) that the Riemann Hypothesis is true if and only if (2.16) holds. $\square$

3. Some numerical calculations and density considerations.

With the advent of fast computers there has been a fruitful interplay between mathematical analysis and high-precision computations. In this section our goal is merely to indicate some numerical work pertaining to Theorem 2.2 and Theorem 2.3. All the calculations reported here have been carried out on a Sun Ultra 5 workstation using Mathematica 4.0.1. As before, we let $0 < \gamma_1 \leq \gamma_2 \leq \cdots$ denote the ordinates of the zeros of $\xi(s)$ on the critical line in the upper half-plane $\mathrm{Re}\, s > 0$. Now it is known that the first 1.5×10^9 zeros of $\xi(s)$ lie on the critical line [13]. Thanks to Dr. A. M. Odlyzko, the first $100,000$ zeros of the Riemann zeta function, accurate to within 3×10^{-9}, are available on the internet (*http : //www.dtc.umn.edu/ odlyzko/zeta_tables/*). Also, we gratefully acknowledge Dr. H. J. J. te Riele for providing one of the authors with the first 500 moments (see the b_k's defined by (2.15)) accurate to within 200 places. (These moments have also been calculated in the Department of Mathematical sciences at Kent State University, using extended-precision floating-point numbers and 110 digits of precision [3, Table 4.1].)

To begin with, it may be of interest to list here a few values of the γ's: $\gamma_1 = 14.134\,725\cdots$, $\gamma_{10,000} = 9877.782\,654\cdots$, $\gamma_{20,000} = 18046.464\,296\cdots$, $\gamma_{100,000} = 74920.827\,498\cdots$. Next, set

$$s_n := \sum_{k=1}^{n} \frac{1}{\frac{1}{4} + \gamma_k^2} \qquad \text{and} \qquad \tau_n := \sum_{j=1}^{n} \frac{1}{\gamma_j^2}.$$

Thus, by Theorem 2.2, if the Riemann Hypothesis is valid, then

$$s_\infty = 1 + \frac{\gamma}{2} - \frac{1}{2}\log(4\pi) = 0.023\,095\,708\cdots. \tag{3.1}$$

Also, by Theorem 2.3, if the Riemann Hypothesis is valid, then

$$\tau_\infty = 4\left(\frac{b_1}{2b_0}\right) = \left(\frac{2b_1}{b_0}\right) = 0.023\,104\,993\cdots, \tag{3.2}$$

where we recall that the x_j's in Theorem 2.3 are related to the γ_j's by $x_j = 2\gamma_j$ and where (cf. [5, Lemma 4.1])

$$b_0 = 0.023\,073\,645\cdots \qquad \text{and} \qquad b_1 = 0.000\,717\,873\cdots.$$

On the other hand, using the values of the γ_k's, $1 \leq k \leq 100,000$, made available by Dr. Odlyzko, we find that

$$s_{100,000} = 0.023\,073\,645\cdots \qquad \text{and} \qquad \tau_{100,000} = 0.023\,082\,929\cdots, \tag{3.3}$$

where we have used several "flags" and procedures (which we will not detail here) in an attempt to control the round-off errors. Thus, with the remainders defined by

$$R_n(s) := s_\infty - s_n \qquad \text{and} \qquad R_n(\tau) := \tau_\infty - \tau_n, \tag{3.4}$$

we find (using (3.1), (3.2) and (3.3)) that

$$R_{100,000}(s) = 2.206\,306\cdots \times 10^{-5} \qquad \text{and} \qquad R_{100,000}(\tau) = 3.134\,776\cdots \times 10^{-5}. \tag{3.5}$$

It may be of interest to combine the foregoing numerical results with certain theoretical estimates involving the density of the (non-trivial) zeros of Riemann zeta function. Let $N(T)$ $(T > 0)$ denote the number of zeros of the Riemann zeta function $\zeta(s)$, where $s = \sigma + it$, in the rectangle $0 \leq \sigma \leq 1$ and $0 \leq t \leq T$. The distribution of zeros can then be studied with the aid of formulae involving $N(T)$. Perhaps the best known such formula is the Riemann–von Mangoldt formula, that is, $N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log T)$ (as $T \to \infty$). We will use the following refinement of the Riemann–von Mangoldt formula that was proved by Backlund in 1918 [10, p. 134] or [4, Theorem C].

Theorem 3.1. (Backlund's Density Theorem, [1]) $N(T)$ satisfies

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8} + e(T), \tag{3.6}$$

where

$$|e(T)| < U(T) := 0.137 \log T + 0.443 \log\log T + 4.35 \qquad (T \geq 2). \tag{3.7}$$

Lemma 3.2. *Let*

$$q(T) := \frac{1}{2\pi}\left[(T+1)\log\left(1 + \frac{1}{T}\right) - 1 - \log 2\pi\right]. \tag{3.8}$$

Then for all $T \geq T_0 \geq 2$

$$N(T+1) - N(T) \leq \left[\frac{1}{2\pi} + \frac{q(T_0)}{\log T} + \frac{2U(T_0+1)}{\log T_0}\right] \log T, \tag{3.9}$$

where $N(T)$ *and* $U(T)$ *are defined by (3.6) and (3.7) respectively.*

Proof. Let

$$f(T) := \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8}.$$

Then algebraic simplifications yield

$$f(T+1) - f(T) = \frac{\log T}{2\pi} + q(T).$$

Since $U(T) > 0$ and $U(T)$ is increasing for all $T \geq T_0 \geq 2$, we obtain the following upper estimate (with $e(T)$ given by (3.6))

$$N(T+1) - N(T) = f(T+1) - f(T) + e(T+1) - e(T) \leq \frac{\log T}{2\pi} + q(T) + 2U(T+1). \tag{3.10}$$

Now elementary, albeit somewhat involved, considerations show that the function $U(T+1)/\log T$ is a decreasing function for $T \geq 2$. Also, since $q(T) < 0$ and $q'(T) < 0$, using (3.10) we obtain the desired inequality (3.9). $\square$

Proposition 3.3. *Let $\rho_k = \frac{1}{2} + i\gamma_k$, $(k = 1, 2, 3 \ldots)$, $0 < \gamma_k \leq \gamma_{k+1} \leq \cdots$, denote the zeros of $\xi(s)$ on the critical line in* Im $s > 0$. *Fix a positive integer m and let $T_0 = [\gamma_m]$, where $[x]$ denotes the greatest integer $\leq x$. Then*

$$\sum_{j=m}^{\infty} \frac{1}{\gamma_j^2} \leq C_1 \frac{1 + \log(T_0 - 1)}{T_0 - 1} + \frac{C_2}{T_0 - 1}, \tag{3.11}$$

where

$$C_1 := C_1(T_0) = \frac{1}{2\pi} + \frac{2U(T_0 + 1)}{\log T_0} \quad \text{and} \quad C_2 := C_2(T_0) = q(T_0), \tag{3.12}$$

and the functions U and q are defined by (3.7) and (3.8), respectively.

Proof. By Lemma 3.2 and the integral test, we obtain the estimates

$$\begin{aligned}
\sum_{j=m}^{\infty} \frac{1}{\gamma_j^2} \leq & \sum_{j=T_0}^{\infty} \sum_{j \leq \gamma_k < j+1} \frac{1}{\gamma_k^2} \\
\leq & \sum_{j=T_0}^{\infty} \frac{N(j+1) - N(j)}{j^2} \\
\leq & \sum_{j=T_0}^{\infty} \frac{\log j}{j^2} \left(C_1 + \frac{C_2}{\log j} \right) \\
\leq & C_1 \int_{T_0 - 1}^{\infty} \frac{\log v}{v^2}\, dv + C_2 \int_{T_0 - 1}^{\infty} \frac{dv}{v^2} \\
= & C_1 \frac{1 + \log(T_0 - 1)}{T_0 - 1} + \frac{C_2}{T_0 - 1}.
\end{aligned}$$

□

In light of Theorem 2.3, it would be of interest to evaluate the sum $\sum_{j=1}^{m} \frac{1}{\gamma_j^2}$ (3.11) for various "large" values of m, say, $m = 10^6$ or $m = 10^8$. Here, we will report the results for $m = 100,000$. Referring back to the beginning of this section, we know that $\gamma_{100,000} = 74920.827\ 498 \cdots$. Thus, with $T_0 := 74920$, a calculation shows that

$$C_1(T_0) = 1.399\ 142\ 523 \cdots \qquad \text{and} \qquad C_2(T_0) = 0.292\ 506\ 157 \cdots$$

and so in this case the upper bound is

$$C_1 \frac{1 + \log(T_0 - 1)}{T_0 - 1} + \frac{C_2}{T_0 - 1} = 0.000\ 224\ 386 \cdots. \tag{3.13}$$

If $m := 1,048,449,114$, then it is known that the m^{th} zero $\frac{1}{2} + i\gamma_m$ has imaginary part $\gamma_m = 3.888\ 588\ 853 \cdots \times 10^8$ (see [4, (2.8)] or [6]). Therefore, if we set $T_0 :=$ 388,858,885, then the upper bound in (3.11) becomes

$$C_1 \frac{1 + \log(T_0 - 1)}{T_0 - 1} + \frac{C_2}{T_0 - 1} = 5.304\ 193\ \cdots \times 10^{-8},$$

which, as expected, is a substantial improvement over the upper bound obtained in (3.13).

We conclude this paper with an open problem that bears relevance to the investigation of the distribution of zeros of $H(x)$ defined by (2.12). For $R > 0$, let

$$H_R(x) := \int_0^R \Phi(t)\cos(xt)\,dt, \tag{3.14}$$

where the kernel $\Phi(t)$ is defined by (2.13). Then, $H_R(x)$ is an even, real entire function of order 1 (and of exponential type). Since the zeros of $H(x) := H_\infty(x)$ all lie in the horizontal strip $|\operatorname{Im} z| < 1$, it follows from Hurwitz' theorem that for all R, sufficiently large, the zeros of $H_R(x)$ also lie in this strip. We can establish a result analogous to Theorem 2.3, where now the moments are

$$b_k(R) = \int_0^R t^{2k}\Phi(t)\,dt, \quad k = 0, 1, 2, \ldots. \tag{3.15.}$$

It is known that the moments $b_k(R)$ satisfy the *Turán inequalities* (cf. [2, Corollary 2.9] and [7]); that is,

$$b_k(R)^2 \geq \frac{2k-1}{2k+1} b_{k-1}(R) b_{k+1}(R), \qquad (k \geq 1)$$

These are the simplest necessary conditions for $H_R(x)$ to have only real zeros. If R is "small", $R \in (0, 0.11]$, then it is known that $H_R(x)$ has only real zeros [8, Theorem 3.6]. Does $H_R(x)$ have only real zeros for other values of R?

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NONLINEAR CONTROLLERS FOR A PLANAR ROBOT ARM IN A CONSTRAINED ENVIRONMENT

Jito Vanualailai and Bibhya Sharma

Abstract. In this article, we explore the fundamentals of an emerging technique applicable, at least, in principle, to robot navigation, or motion planning. Termed the Second Method of Lyapunov, it is currently a powerful mathematical technique used to study the qualitative behaviour of natural or man-made systems that could be modeled, in an approximate way, by differential equations. We review the Lyapunov method and then in a simple and direct way, we use it to propose a theoretical technique to control the motion of a planar arm in a constrained environment. The controllers are mathematical entities which are nonlinear in nature. Computer simulations are used to illustrate the effectiveness of the proposed controllers.

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Key Words and Phrases. Lyapunov Stability, Lyapunov Function, Robot Dynamics and Control, Findpath Problem, Motion Planning

1. Introduction

In [1], Meyer cleverly brings within the grasp of mathematicians the interesting mathematical and physical concepts associated with planar robot arms or manipulators.

An interesting two-dimensional geometric problem arising from Meyer's work is as follows: *Given a robot and a description of its working space or* workspace, *which could be cluttered with stationary or mobile solid objects or obstacles, propose a collision-free path that will lead the robot from the desired starting point to the desired location or target.* This is known as the *findpath problem*, the quest for the solutions of which is, at present, one of the most interesting theoretical undertakings in robotics research. For a review of various findpath schemes, see, for example, Sheu and Xue [2]. A more recent review is by Kumar et al. [3].

Meyer proposed a findpath scheme based on the velocities of the various components of the arm. In this article, we build on the work of Meyer by applying

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a relatively new scheme that is based on acceleration. The scheme gives us the advantage of including constraints that might affect the operation of the arm. First proposed in 1990 by Stonier [4] and then elaborated on in 1995 and 1998 by Vanualailai et al. [5],[6], with follow-ups by Ha and Shim in 2000 [7] and 2001 [8], the scheme is based on elementary differentiation, and thus, is within the grasp of the college sophomore. However, at the heart of the control scheme is a powerful mathematical technique, called the *Direct Method of Lyapunov* [9] or simply, the method of Lyapunov, that requires some understanding of the nature of solutions of autonomous systems of first-order ordinary differential equations.

We start by re-looking at the equations used by Meyer to control the arm via velocity.

2. Velocity Control

Figure 1 shows a schematic represenation of the simplified robot arm considered by Meyer.

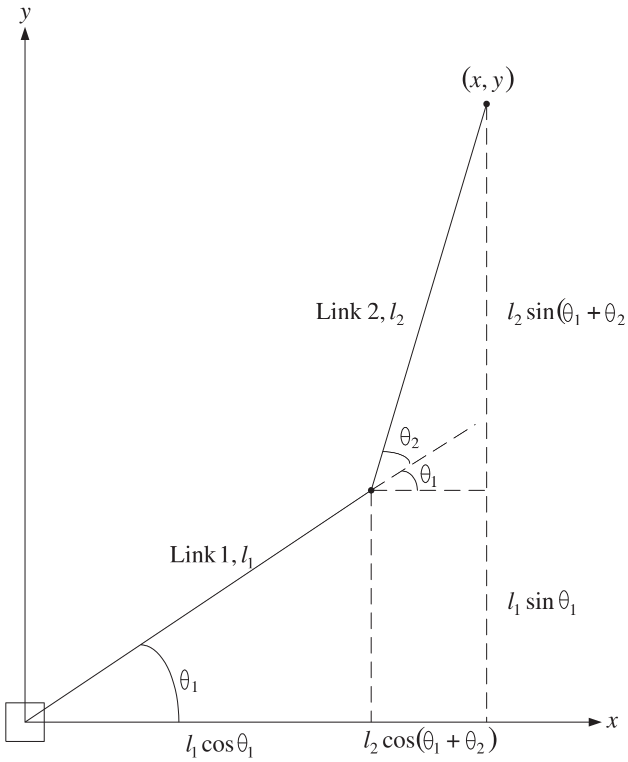

Figure 1: Schematic representation of the planar robot arm.

The position $(x(t), y(t))$ of the end-effector at time t is given by the forward kinematic equations

$$\begin{cases} x(t) = l_1 \cos\theta_1(t) + l_2 \cos(\theta_1(t) + \theta_2(t))\,, \\ y(t) = l_1 \sin\theta_1(t) + l_2 \sin(\theta_1(t) + \theta_2(t))\,. \end{cases} \tag{1}$$

The instantaneous velocity $(x'(t), y'(t))$ can be obtained from the equations

$$(2) \qquad \begin{cases} x'(t) = -y\theta_1'(t) - l_2 \sin(\theta_1(t) + \theta_2(t))\theta_2'(t)\,, \\ y'(t) = x\theta_1'(t) + l_2 \cos(\theta_1(t) + \theta_2(t))\theta_2'(t)\,. \end{cases}$$

One way to guide the end-effector is to control the velocities, (x', y') and θ_1' and θ_2'. If we want to maintain a constant velocity, say, $(x'(t), y'(t)) = (v_1, v_2)$ for all time $t \geq 0$, then our *velocity control scheme* is as follows:

Step 1. For the end-effector, fix the desired velocity, say, $(x'(t), y'(t)) = (v_1, v_2)$ at all time $t \geq 0$.

Step 2. Record the current position of the end-effector, say $(x, y) = (p_1, p_2)$, with the left-hand-side of (1) and then derive the angular positions θ_1 and θ_2.

Step 3. Insert $(x, y) = (p_1, p_2)$, $(x', y') = (v_1, v_2)$, θ_1 and θ_2 into (2) and then derive the angular velocities θ_1' and θ_2'.

The computer controlling the robot then moves the joints at the angular velocities θ_1' and θ_2' while maintaining the constant velocity (v_1, v_2) of the end-effector at the point $(x, y) = (p_1, p_2)$.

Our scheme to be proposed essentially allows one to control the velocities $\theta_1'(t)$ and $\theta_2'(t)$ at every time t by controlling the instantaneous angular accelerations, $\theta_1''(t)$ and $\theta_2''(t)$. But, first, let us briefly look at the Direct Method of Lyapunov on which our scheme will be based.

3. The Direct Method of Lyapunov

In 1892, Lyapunov [9] established a qualitative method of analyzing solutions of autonomous systems of first-order differential equations of the form

$$(3) \qquad \begin{cases} x_1'(t) = f_1(x_1(t), x_2(t), \dots, x_n(t))\,, \\ \qquad \vdots \\ x_n'(t) = f_n(x_1(t), x_2(t), \dots, x_n(t))\,, \end{cases}$$

with the known solution, $(x_1(t_0), \dots, x_n(t_0))$, at some time $t = t_0 \geq 0$. This known solution is referred to as the *initial state* of system (3). The basic assumption is that the solutions of (3) exist at any given time t and are unique, with each solution depending on the initial state and continuable to $+\infty$. That is, we have unique and continuous solutions of the form $x_1(t) = x_1(t; t_0, x_1(t_0))$, $\cdots$, $x_n(t) = x_n(t; t_0, x_n(t_0))$, which may be regarded either as a curve in the space of the $n+1$ variables $x_1, x_2, \dots, t$ or as a curve in the space of n variables $x_1, x_2, \dots, x_n$ with t regarded as a parameter. In the latter case (which will be our principal concern), the curve is called a *trajectory* in the *phase space* – the space of n variables, $x_1, x_2, \dots, x_n$. If at a particular point $(p_1, \dots, p_n)$ in the phase space, we have that $(f_1(p_1, \dots, p_n), \dots, f_n(p_1, \dots, p_n)) \equiv (0, \dots, 0)$ for all time $t \geq 0$, then we say that the point is an *equilibrium point* of system (3). We will henceforth assume that at least one such point exists.

A question that the Lyapunov method can answer is as follows: *Let $(p_1, \dots, p_n)$ be an equilibrium point of system (3) in a region of the phase space. If we start from the initial state within the region and close to the equilibrium point, does the trajectory $(x_1(t), \dots, x_n(t))$ remain within the region and close to $(p_1, \dots, p_n)$ for all time t?*

If the answer is yes, then we say that the point $(p_1, \dots, p_n)$ is *stable*.

Precisely, if we let $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$, and $\mathbf{x}^* = (p_1, \ldots, p_n)$, then $\mathbf{x}^*$ is a stable equilibrium point of (3) if for all $t_0 \geq 0$ and $\epsilon > 0$, there is $\delta(t_0, \epsilon) > 0$ such that

$$\|\mathbf{x}(t_0) - \mathbf{x}^*\| < \delta(t_0, \epsilon)\,,$$

implies

$$\|\mathbf{x}(t; t_0, \mathbf{x}(t_0)) - \mathbf{x}^*\| < \epsilon\,,$$

for all $t \geq t_0$. The beautiful aspect of the Lyapunov method, as revealed in the theorem below, is that we do not need the explicit form of the solutions to establish stability.

Theorem 1 (Lyapunov [9]**).** *Let $\mathbf{x}^*$ be an equilibrium point of system (3). If, in an open neighborhood $\mathbb{D}$ of the equilibrium point, there exists a real scalar function V such that* (a) *$V(\mathbf{x})$ is continuous on $\mathbb{D}$ and has continuous first partial derivatives with respect to $\mathbf{x}$,* (b) *$V(\mathbf{x}^*) = 0$, $\mathbf{x}^* \in \mathbb{D}$,* (c) *$V(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{D}$, $\mathbf{x} \neq \mathbf{x}^*$, and* (d) *the time-derivative of V with respect to system (3) is*

$$\frac{d}{dt}\,[V(\mathbf{x})]_{(3)} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\,[V(\mathbf{x})]\, f_i(\mathbf{x}) \leq 0$$

for all $\mathbf{x} \in \mathbb{D}$, then $\mathbf{x}^*$ is a stable equilibrium point of system (3).

The scalar function V in Theorem 1 is called a *Lyapunov function for system (3) on* $\mathbb{D}$. To understand therefore the behaviour of trajectories in the neighbourhood of the equilibrium hinges only on discovering a Lyapunov function without the need to find the explicit solutions, which, particularly for nonlinear systems, are usually difficult to formulate in closed form. This single aspect of the Lyapunov method makes it a powerful tool in the stability analysis of nonlinear systems. The Lyapunov method is now a critical component in specializations such as control engineering, power system engineering, robotics, neural networks, chaos and economics, to name but a few. A good review of the method, including recent advances and several applications can be found in Sastry [10].

For our application, where we desire the convergence of trajectories to equilibrium points, Theorem 1 tells us that stability ensures only boundedness of solutions in a neighbourhood of $\mathbf{x}^*$. However, it is known that if in addition $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})$ is bounded for $\mathbf{x}$ bounded, then whenever $d[V(\mathbf{x})]/dt < 0$ for $\mathbf{x} \neq \mathbf{x}^*$ and $d[V(\mathbf{x}^*)]/dt = 0$ in Theorem 1, the equilibrium point $\mathbf{x}^*$ is not only stable, but also attracts trajectories to it. That is, $\mathbf{x}^*$ is *asymptotically stable*, which is clearly more desirable than stability from a practical point of view. In this paper, to simplify our discussion, it is suffice to guarantee stability. The computer is then used to find initial conditions that guarantee attractiveness.

4. Acceleration Control

As mentioned earlier, we intend to control the end-effector by controlling the angular accelerations, $\theta_1''(t)$ and $\theta_2''(t)$. If we are to achieve this via the method of Lyapunov, then, first, we need to have a system of first-order differential equations describing the motion of the planar robot arm. Thus, if we let

$$\begin{aligned}
x_1(t) &= \text{ the } x\text{-component of the position of the end-effector,}\\
x_2(t) &= \text{ the } y\text{-component of the position of the end-effector,}\\
x_3(t) &= \text{ the angular position, } \theta_1(t), \text{ of Link 1 },\\
x_4(t) &= \text{ the angular position, } \theta_2(t), \text{ of Link 2 },\\
x_5(t) &= \text{ the angular velocity, } \theta_1'(t), \text{ of Link 1 },\\
x_6(t) &= \text{ the angular velocity, } \theta_2'(t), \text{ of Link 2 },\\
u_1(t) &= \text{ the angular acceleration, } \theta_1''(t), \text{ of Link 1 },\\
u_2(t) &= \text{ the angular acceleration, } \theta_2''(t), \text{ of Link 2,}
\end{aligned}$$

at time t, then using equations (2), we have (on suppressing t),

$$\left\{\begin{aligned} x_1' &= -x_2x_5 - l_2\sin(x_3+x_4)x_6\,, \quad x_3' = x_5\,, \quad x_5' = u_1\,,\\ x_2' &= x_1x_5 + l_2\cos(x_3+x_4)x_6\,, \quad x_4' = x_6\,, \quad x_6' = u_2\,. \end{aligned}\right. \tag{4}$$

Before we actually start looking for a Lyapunov function for system (4), we may want to look for any difficulty inherent in the basic geometric structure of the robot. As shown by Meyer, there is indeed a problem: there exists at least one direction in which the end-effector cannot be moved no matter how we choose the joint velocities θ_1' and θ_2'. This arises when $\theta_2 = 0, \pi$. In our scheme, we want to avoid these *singular configurations*. We may also want to take into account other constraints in the system. These include, for example, the allowable space or area to work in, or the allowable velocity or acceleration of the end-effector. Hence, our *acceleration control scheme* via the Lyapunov method can be stated as follows:

Step 1. Let the final destination of the end-effector be $(x, y) = (p_1, p_2)$, achieved at the angular positions $x_3 = p_3$ and $x_4 = p_4$ and let us call $(p_1, p_2, p_3, p_4, 0, 0)$ the *target*.

Step 2. Identify system constraints (these include singular configurations) and appropriately construct mathematical functions to model them. Call the constraints *antitargets* or *obstacles* to be avoided.

Step 3. Construct a Lyapunov function such that u_1 and u_2 render system (4) stable, meaning that $(p_1, p_2, p_3, p_4, 0, 0)$ is an equilibrium point of system (4) and that the trajectory $(x_1, x_2, x_3, x_4, x_5, x_6)$ starts and remains, for all time $t \geq 0$, near the target $(p_1, p_2, p_3, p_4, 0, 0)$ while it avoids stationary and/or mobile obstacles in the workspace.

In Step 1, intuitively, we want to have a kind of a yardstick that measures, at time t, the position of the end-effector from the point $(x, y) = (p_1, p_2)$ and the rate at which it approaches or moves away from (p_1, p_2). The following choice of probable functions accomplishes this (on suppressing t),

$$V_0(x_1, \dots, x_6) = \frac{1}{2}\left[(x_1-p_1)^2 + (x_2-p_2)^2) + x_5^2 + x_6^2\right]\,,$$

noting that

$$V_0(p_1, p_2, p_3, p_4, 0, 0) = 0\,,$$

and

$$V_0 > 0 \text{ for } (x_1, x_2, x_3, x_4, x_5, x_6) \neq (p_1, p_2, p_3, p_4, 0, 0)\,.$$

In Step 3, if a trajectory ever converges to the target, then it remains there for all time t since $V_0(p_1, p_2, p_3, p_4, 0, 0) = 0$. The use of V_0, together with appropriate collision avoidance schemes, should enable us to construct Lyapunov functions that ensure that trajectories remain near $(p_1, p_2, p_3, p_4, 0, 0)$ while avoiding obstacles.

In the next section, we consider several possible avoidance scenarios, in which we use the vector notations $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{x}^* = (p_1, p_2, p_3, p_4, 0, 0)$.

5. Possible Avoidance Scenarios

5.1 Scenario 1: Avoidance of Singular Configurations.

The singular configurations occur when $\theta_2 = x_4 = 0$, $\theta_2 = x_4 = \pi$ in the anticlockwise direction or $\theta_2 = x_4 = -\pi$ in the clockwise direction. For the configurations, consider the functions

$$W_1(\mathbf{x}) = |x_4| \quad \text{and} \quad W_2(\mathbf{x}) = \pi - |x_4|\,,$$

for $x_4 \in (-\pi, 0) \cup (0, \pi)$. Now, let us, for the moment, consider the effect of the ratios

$$\frac{\beta_1}{W_1} \quad \text{and} \quad \frac{\beta_2}{W_2}\,,$$

for some constants β_1 and β_2. If the robot arm approaches any one of the singular configurations, it is clear that one of the ratios will increase. Hence, if the ratios form parts of a Lyapunov function for system (4), intuitively the ratios will act as avoidance functions that repel the robot arm from the singular configurations. Indeed, by the mere fact that we will have a Lyapunov function, all trajectories will converge to a neighbourhood of the target, implying therefore that we cannot have the situation where $W_1 = W_2 = 0$. This is in turn means that we cannot have, at any time, the angular position of Link 2 as $p_4 = 0$, $p_4 = \pi$ or $p_4 = -\pi$. In this sense, we can think about singular configurations as mobile obstacles.

5.1.1 A Lyapunov Function.

Let us finally suggest the following function as a tentative Lyapunov function for system (4):

$$V(\mathbf{x}) = V_0(\mathbf{x}) + F(\mathbf{x}) \sum_{i=1}^{2} \frac{\beta_i}{W_i(\mathbf{x})}\,,$$

where β_1 and β_2 are positive constants and

$$F(\mathbf{x}) = \frac{1}{2}\left[(x_1 - p_1)^2 + (x_2 - p_2)^2\right]\,.$$

Clearly, V is continuous and positive on the domain

$$\begin{aligned}\mathbb{D}(V) &= \{\mathbf{x} \in \mathbb{R}^6 : W_1 > 0, W_2 > 0\}\\ &= \{\mathbf{x} \in \mathbb{R}^6 : -\pi < x_4 < 0 \text{ or } 0 < x_4 < \pi\}\,,\end{aligned}$$

and $V(\mathbf{x}^*) = 0$, $\mathbf{x}^* \in \mathbb{D}(V)$, so that we may take $\mathbb{D}(V)$ as the neighborhood $\mathbb{D}$ of the equilibrium point $\mathbf{x}^*$ in Lyapunov's Theorem 1.

Along a particular trajectory of system (4), we have (on suppressing t),

$$
\begin{aligned}
\frac{d}{dt}[V]_{(4)} = & \left\{[(x_2 - p_2)x_1 - (x_1 - p_1)x_2]\left(1 + \sum_{i=1}^{2}\frac{\beta_i}{W_i}\right) + u_1\right\}x_5 \\
& + \left\{l_2\left[(x_2 - p_2)\cos(x_3 + x_4) - (x_1 - p_1)\sin(x_3 + x_4)\right]\left(1 + \sum_{i=1}^{2}\frac{\beta_i}{W_i}\right)\right. \\
& \left. -F\frac{x_4}{|x_4|}\left(\frac{\beta_1}{W_1^2} - \frac{\beta_2}{W_2^2}\right) + u_2\right\}x_6 \\
\stackrel{\text{def}}{=} & \{G_1(\mathbf{x}) + u_1\}x_5 + \{G_2(\mathbf{x}) + u_2\}x_6 \,.
\end{aligned}
$$

For some numbers $\alpha_1 > 0$ and $\alpha_2 > 0$, define

$$-\alpha_1 x_5 = G_1 + u_1 \quad \text{and} \quad -\alpha_2 x_6 = G_2 + u_1 \,.$$

Then

$$V_{(4)}{}' = -\alpha_1 x_5^2 - \alpha_2 x_6^2 \,,$$

provided we have the *nonlinear controllers* as $u_1 = -\alpha_1 x_5 - G_1$ and $u_2 = -\alpha_2 x_6 - G_2$. Since $V_{(4)}{}'$ nonpositive and continuous for all $\mathbf{x} \in \mathbb{D}(V)$, by Theorem 1, V is a Lyapunov function for system (4) on $\mathbb{D}(V)$ establishing the stability of $\mathbf{x}^*$. Hence, the stable system that governs the motion of the arm is

$$
(5) \qquad \begin{cases} x_1' = -x_2 x_5 - l_2 \sin(x_3 + x_4)x_6 \,, & x_3' = x_5 \,, \quad x_5' = -\alpha_1 x_5 - G_1 \,, \\ x_2' = x_1 x_5 + l_2 \cos(x_3 + x_4)x_6 \,, & x_4' = x_6 \,, \quad x_6' = -\alpha_2 x_6 - G_2 \,. \end{cases}
$$

We see that an equilibrium point is $(p_1, p_2, p_3, p_4, 0, 0)$. Our Lyapunov function guarantees only stability of this equilibrium point. Clearly, a Lyapunov function which ensures asymptotic stability is more desirable. This is the most challenging part of the Lyapunov method, given the difficulty in constructing such a function for the pathpath problem. A new development in this direction is to guarantee stability and also prove that certain sets of initial conditions ensure asymptotic stability. This looks promising and at present is applicable to a point-mass system with one stationary obstacle and one stationary target. The interested reader can refer to the paper by Ha and Shim [8] for more details.

In this article, to simplify discussions, it is enough to consider a stable equilibrium point and then use a computer to help us decide which system parameters to use for a desirable run. These parameters are β_1 and β_2, which we shall call *control parameters*, and α_1 and α_2, which we shall call *convergence parameters*. It can be shown that the larger a control parameter is, the greater is the repulsion from the associated obstacle, and the larger a convergence parameter is, the slower is the trajectory [5].

5.1.2 A Computer Simulation.

The computer is used to numerically integrate system (5) to obtain the solution $(x_1, \ldots, x_6)$ and plot the points $(x_1(t), x_2(t))$ at time t in the x_1x_2-plane until the points converge to a neigborhood of (p_1, p_2) and stay there as $t \to \infty$. For our example, Table I gives the parameters, and Figure 2 gives the trajectory of the arm. Notice the slowing down of the arm as it approaches its final configuration. This could be explained in terms of the potential energy "cup" of the Liapunov function [5]. The arm reaches its final configuration in about 15 units of time. In Figure 2, the links are drawn every 1 unit of time.

Table I: A Scenario 1 example.

Lengths of Links	$l_1 = l_2 = 3$
Initial Conditions	$\mathbf{x} = \left(-3\sqrt{3}/2, 3/2, \pi/2, 2\pi/3, \pi/90, \pi/90\right)$
Planar Target	$(p_1, p_2) = (4, 4)$
Control Parameters	$\beta_1 = \beta_2 = 0.01$
Convergence Parameters	$\alpha_1 = 110$, $\alpha_2 = 20$

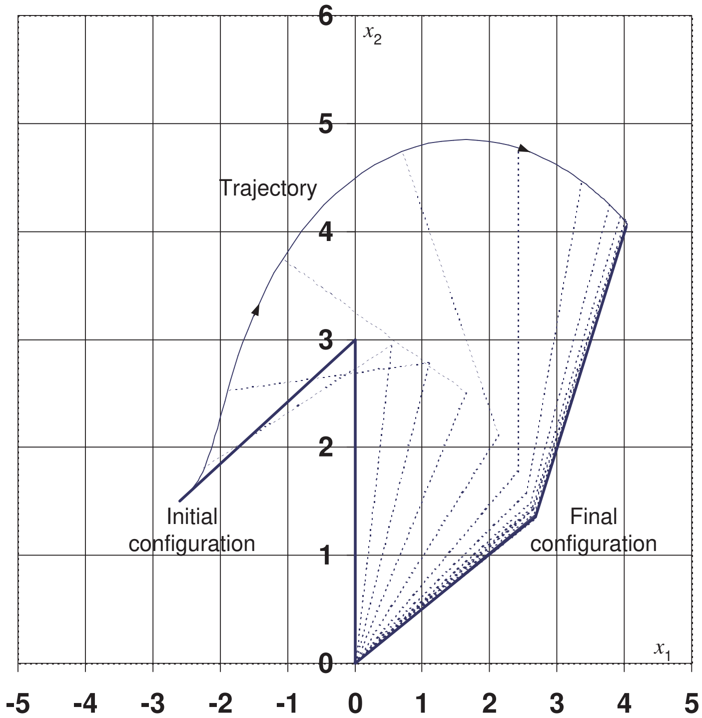

Figure 2: The motion of the arm from an initial configuration to a final configuration as determined by system (5).

5.2 Scenario 2: Movement in A Rectangle.

An interesting scene involves restricting the motion of the arm within a rectangular region in the first quadrant of the xy-plane. We place the bottom left corner of the rectangle at $(0,0)$ where the base of the arm is located as shown in Figure 3, in which we call the constraints or obstacles the *left wall, right wall, floor* and *roof.* These are fixed obstacles. The mobile obstacles are the singular configurations.

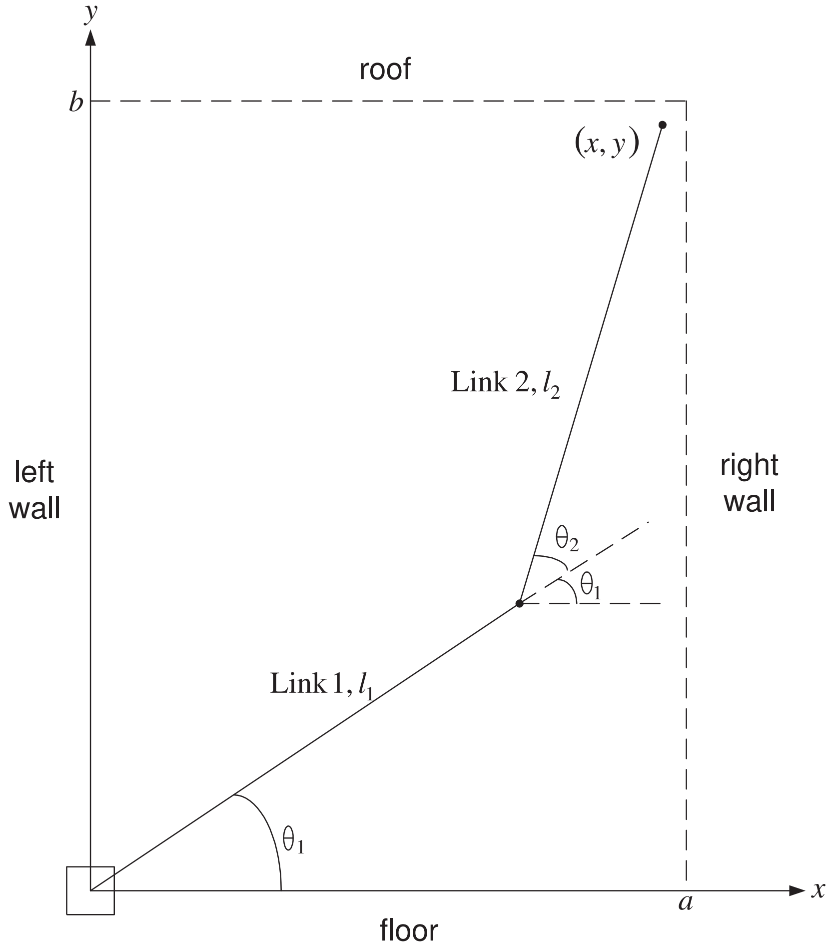

Figure 3: The planar robot arm in a constrained environment.

For obvious geometrical reasons, we construct the following avoidance functions that ensure that the end-effector (and, hence, Link 2), Link 1 and the joint do not collide with the fixed obstacles.

5.2.1 Avoidance Functions for the End-effector.

Table II gives the functions that will ensure that the end-effector does not collide with fixed obtacles.

Table II: Avoidance functions for the end-effector.

Obstacle	Avoidance Function	Sign
Left wall, $x_1 = 0$	$W_3 = x_1$	$W_3 > 0$ for all $x_1 \in (0, a)$
Right wall, $x_1 = a$	$W_4 = a - x_1$	$W_4 > 0$ for all $x_1 \in (0, a)$
Floor, $x_2 = 0$	$W_5 = x_2$	$W_5 > 0$ for all $x_2 \in (0, b)$
Roof, $x_2 = b$	$W_6 = b - x_2$	$W_6 > 0$ for all $x_2 \in (0, b)$

5.2.2 Avoidance Functions for Link 1.

Table III gives the functions that will ensure that Link 1 does not collide with fixed obtacles.

Table III: Avoidance functions for Link 1.

Obstacle	Avoidance Function	Sign
Floor, $x_3 = 0$	$W_7 = x_3$	$W_7 > 0$ for all $x_3 \in (0, \pi/2)$
Left wall, $x_3 = \pi/2$	$W_8 = \pi/2 - x_3$	$W_8 > 0$ for all $x_3 \in (0, \pi/2)$

5.2.3 Avoidance Functions for Elbow.

Table IV gives the functions that will ensure that the joint or elbow between Link 1 and Link 2 does not collide with fixed obtacles.

Table IV: Avoidance functions for elbow.

Obstacle	Avoidance Function	Sign
Roof, $x_2 = b$	$W_9 = b - l_1 \sin x_3$	$W_9 > 0$ in rectangle
Right wall, $x_1 = a$	$W_{10} = a - l_1 \cos x_3$	$W_{10} > 0$ in rectangle

5.2.4 A Lyapunov Function.

Here, we simply provide a Lyapunov function and the controllers, the derivation of which is similar to that in Scenario 1.

For constants $\beta_i > 0$, $i = 1, \ldots, 10$, we consider

$$V = V_0 + F \sum_{i=1}^{10} \frac{\beta_i}{W_i},$$

on the domain $\mathbb{D}(V) = \{\mathbf{x} \in \mathbb{R}^6 : W_i > 0, i = 1, \ldots, 10\}$, where V_0, W_1, W_2 and F are as those in Scenario 1. Then along a trajectory of system (4) in $\mathbb{D}(V)$, we have, for $\alpha_1, \alpha_2 > 0$, the time-derivative

$$V_{(4)}{}' = -\alpha_1 x_5^2 - \alpha_2 x_6^2 \,,$$

if we define the controllers as

$$u_1 = -\alpha_1 x_5 - G_3 \quad \text{and} \quad u_2 = -\alpha_2 x_6 - G_4 \,,$$

where

$$\begin{aligned} G_3 = G_3(\mathbf{x}) = {} & [(x_2 - p_2)x_1 - (x_1 - p_1)x_2] \left(1 + \sum_{i=1}^{10} \frac{\beta_i}{W_i}\right) \\ & - F\left[\left(-\frac{\beta_3}{W_3^2} + \frac{\beta_4}{W_4^2}\right) x_2 + \left(\frac{\beta_5}{W_5^2} - \frac{\beta_6}{W_6^2}\right) x_1 \right. \\ & \left. + \frac{\beta_7}{W_7^2} - \frac{\beta_8}{W_8^2} - \frac{\beta_9}{W_9^2} l_1 \cos x_3 + \frac{\beta_{10}}{W_{10}^2} l_1 \sin x_3 \right] \,, \end{aligned}$$

and

$$\begin{aligned} G_4 = G_4(\mathbf{x}) = {} & l_2 \left[(x_2 - p_2)\cos(x_3 + x_4) - (x_1 - p_1)\sin(x_3 + x_4)\right] \left(1 + \sum_{i=1}^{10} \frac{\beta_i}{W_i}\right) \\ & - F\left[\left(\frac{\beta_1}{W_1^2} - \frac{\beta_2}{W_2^2}\right) \frac{x_4}{|x_4|} + \left(-\frac{\beta_3}{W_3^2} + \frac{\beta_4}{W_4^2}\right) l_2 \sin(x_3 + x_4) \right. \\ & \left. + \left(\frac{\beta_5}{W_5^2} - \frac{\beta_6}{W_6^2}\right) l_2 \cos(x_3 + x_4) \right] \,. \end{aligned}$$

These render the system

$$(6) \qquad \begin{cases} x_1' = -x_2 x_5 - l_2 \sin(x_3 + x_4) x_6 \,, & x_3' = x_5 \,, \quad x_5' = -\alpha_1 x_5 - G_3 \,, \\ x_2' = x_1 x_5 + l_2 \cos(x_3 + x_4) x_6 \,, & x_4' = x_6 \,, \quad x_6' = -\alpha_2 x_6 - G_4 \,, \end{cases}$$

stable at the equilibrium point $\mathbf{x}^* = (p_1, p_2, p_3, p_4, 0, 0)$.

5.2.5 A Computer Simulation.

Table V shows the parameters used for a case in Scenario 2, and Figure 4 gives the motion of the arm. The arm reaches its final configuration in about 10 units of time.

Table V: A Scenario 2 example.

Lengths of Links	$l_1 = l_2 = 3$
Initial Conditions	$\mathbf{x} = (2.95, 0.52, 7\pi/18, -2\pi/3, \pi/90, \pi/90)$
Planar Target	$(p_1, p_2) = (2.2, 5.2)$
Control Parameters	$\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_6 = \beta_8 = \beta_9 = \beta_{10} = 1\,,$ $\beta_5 = 10, \beta_7 = 5$
Convergence Parameters	$\alpha_1 = \alpha_2 = 100$
Right wall, Roof	$a = 3.2,\, b = 5.5$

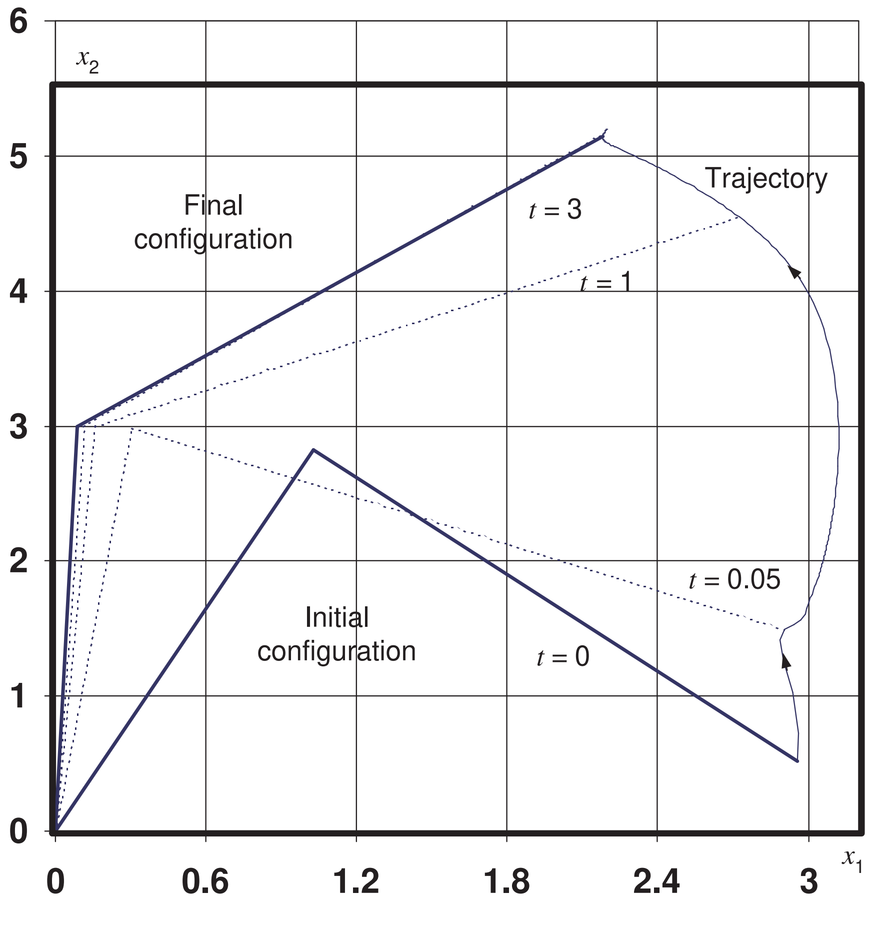

Figure 4: The motion of the robot arm from an initial configuration to a final configuration as determined by system (6).

6. Conclusion

A known drawback of the Lyapuov method, which is classified in the robotic literature as a *potential field* method, is the possibility of having collision-free paths leading not to the intended target but to "traps" outside of the target. These traps, like the target, are in fact local minima or points of zero kinetic and potential energy. As mentioned earlier, an encouraging development to deal with this problem was recently reported by Ha and Shim [8]. Another likely drawback of the Lyapunov method might well turn out to be the large amount of computation required for a multirobot environment. However, this may eventually cease to be a problem given the ever improving power of the digital computer. Hence, as far as the application of the Lyapunov method to the findpath problem is concerned, there are, at the present stage of development, some encouraging signs that could promote the wider use of the method.

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A UNIFYING THEOREM ON NEWTON'S METHOD IN A BANACH SPACE WITH A CONVERGENCE STRUCTURE UNDER WEAK ASSUMPTIONS

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Abstract. We provide a semilocal analysis for Newton's method on a Banach space with a convergence structure. Our main result contains as special cases earlier results on semilocal theorems of Newton–Kantorovich-type as well as global results based on monotonicity. It turns out that under the same hypotheses our sufficient convergence conditions are weaker and the error bounds on the distances involved finer than before. Some numerical examples are provided to show that our results apply where others fail.

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Key Words and Phrases. Newton's method, Banach space with a convergence structure, order convex operator, Newton–Kantorovich hypothesis, monotone convergence, multilinear/bounded/symmetric operator.

1. Introduction

In this study we are concerned with the problem of approximating a zero x^* of an operator G on a Banach space X. Our results will be in terms of the operator

$$F(x) := AG(x_0 + x), \tag{1}$$

where x_0 is the initial guess for Newton's method

$$x_0 = 0, \quad x_{n+1} = x_n + F'(x_n)^*(-F(x_n)) \quad (n \geq 0), \tag{2}$$

A is an approximation to $G'(x_0)^{-1}$ and $F'(x_n)^*$ is an operator to be precised later.

Semilocal convergence theorems were given in [1], [2], [4], [5] under various assumptions on a Banach space with a convergence structure. Here we introduce finer "majorizing" sequences and show that under the same hypotheses as in [5] we can obtain semilocal results under weaker assumptions and with finer error bounds.

We complete our study with some examples to justify the claims made above.

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2. Semilocal Convergence Analysis

To make the study as self-contained as possible we need to state some definitions and lemmas that can also be found in [5] (see also [1], [2]).

Definition 1. A Banach space with a convergence structure is a triplet (X, V, E) such that:

(3) $(X, \|\cdot\|)$ is a real Banach space;

(4) $(V, C, \|\cdot\|_V)$ is a real Banach space which is partially ordered by the closed convex cone C with the norm $\|\cdot\|_V$ assumed to be monotone on C;

E is a closed convex cone in $X \times V$ such that

$$\{0\} \times C \subseteq E \subseteq X \times C; \tag{5}$$

the operator $|\cdot|: D \to C$ is well defined:

$$|x| := \inf\{r \in C \mid (x, r) \in E\}$$

where

(6) $x \in D = \{x \in X \mid \exists r \in C : (x, r) \in E\}$;

(7) $\|X\| \leq \|\,|x|\|_V$ for all $x \in D$.

The set

$$U(p) = \{x \in X \mid (x, p) \in E\} \tag{8}$$

is a generalized neighborhood of zero.

The convergence analysis is based on monotonicity considerations in the space $X \times V$.

We have

$$(x_n, r_n) \leq (x_{n+m}, r_{n+m}) \Rightarrow 0 \leq (x_{n+m} - x_n, r_{n+m} - r_n) \tag{9}$$

for all increasing sequences $(x_n, r_n) \in E^\pi$. If

$$r_n \to r \tag{10}$$

we get

$$0 \leq (x_{n+m} - x_n, r - r_n) \tag{11}$$

and by (5) of Definition 1

$$\|x_{n+m} - x_n\| \leq \|r - r_n\|_V \to 0 \quad \text{as } n \to \infty. \tag{12}$$

Hence $\{x_n\}$ $(n \geq 0)$ is a Cauchy sequence in a Banach space and as such it converges to some point $x^* \in X$. Let $\{v_n\} \in C^\pi$ be an increasing sequence and set $r_n = v_0 - v_n$ to obtain error estimates:

$$0 \leq (x_{n+m} - x_n, v_n - v_{n+m}) \leq (x_{n+m} - x_n, v_n). \tag{13}$$

Note that if $x_n \to x^*$ as $n \to \infty$ this implies

$$|x^* - x_n| \leq v_n \quad (n \geq 0). \tag{14}$$

Definition 2. An operator $L \in C^1(V_0 \to V)$ on an open subset V_0 of an ordered Banach space V is called order convex on $[a, b] \subseteq V_0$ if

$$c, d \in [a, b],\ c \leq d \Rightarrow L'(d) - L'(c) \in L_+(V), \tag{15}$$

where,

$$L_+(V^n) = \{L \in L(V^n) \mid 0 \leq x_i \Rightarrow 0 \leq L(x_1, \dots, x_n)\}. \tag{16}$$

Definition 3. The set

$$\begin{aligned} B(A) = \big\{ L \in L_+(V^n) \mid (x_i, r_i) \in E \\ \Rightarrow [A(x_1, \dots, x_n), L(r_1, \dots, r_n)] \in E \big\} \end{aligned} \tag{17}$$

is the set of bounds for $A \in L(X^n)$.

Lemma 1. *Let $A\colon [0,1] \to L(X^n)$ and $L\colon [0,1] \to L_+(V^n)$ be continuous operators. Then the following implication holds*:

$$\forall t \in [0,1]\colon L(t) \in B(A(t)) \Rightarrow \int_0^1 L(t)dt \in B\Big(\int_0^1 A(t)dt\Big). \tag{18}$$

Definition 4. Let $Q\colon Y \to Y$ be an operator on a subset Y of a normed space. By $Q^n(x)$ we mean $Q(Q^{n-1}(x))$ and in case of convergence we write

$$Q^\infty(x) = \lim_{n\to\infty} Q^n(x). \tag{19}$$

Definition 5. Let $A \in L(X)$ and $y \in X$ be given. Then we have

$$\begin{aligned} A^* y = z \Leftrightarrow z \in Q^\infty(0),\ Q(x) = (I - A)(x) + y \\ \Leftrightarrow z = \sum_{i=0}^{\infty} (I - A)^i (y). \end{aligned} \tag{20}$$

Lemma 2. *Let $L \in L_+(V)$ and $p, r \in C$ be given such that*

$$L(r) + p \leq r \quad \text{and} \quad L^n(r) \to 0 \quad \text{as } n \to \infty. \tag{21}$$

Then the operator

$$(I - L)^*\colon [0, p] \to [0, p]$$

is well defined and continuous.

Lemma 3. *Let $A \in L(X)$, $L \in B(A)$, $y \in D$ and $r \in C$ be given such that*:

$$L(r) + |y| \leq r \quad \text{and} \quad L^n(r) \to 0 \quad \text{as } n \to \infty. \tag{22}$$

Then the following hold:

$$x = (I - A)^*(y) \tag{23}$$

is well defined, $x \in D$, and

$$|x| \leq (I - L)^* |y| \leq r. \tag{24}$$

We can now provide the main semilocal convergence theorem for Newton's method (2).

Theorem 1. *Assume: given a Banach space X with a convergence structure (X, V, E) with $V = (V, C, \|\cdot\|_V)$, an operator $F \in C^1(X_0 \subseteq X \to X)$, operators M, L_0, $L \in L^1(V_0 \subseteq V \to V)$, and a point $p \in C$ such that the following conditions hold*:

$$U(p) \subseteq X_0, \quad [0,p] \subseteq V_0; \tag{25}$$

M, L_0, L are order convex on $[0,p]$, and such that for $x, y, z \in U(a)$ with $|x| \le p$, $|y| + |z| \le p$

$$L_0'(|x|) - L_0'(0) \in B(F'(0) - F'(x)), \tag{26}$$

$$L'(|y| + |z|) - L'(|y|) \in B(F'(y) - F'(y+z)), \tag{27}$$

$$M(p) \le p, \tag{28}$$

$$L_0(p_0) \le M(p_0) \quad \text{for all } p_0 \in [0,p], \tag{29}$$

$$L_0'(p_0) \le M'(p_0) \quad \text{for all } p_0 \in [0,p], \tag{30}$$

$$L_0'(0) \in B(I - F'(0)), \quad (-F(0), L_0(0)) \in E, \tag{31}$$

$$M'(p)^n(p) \to 0 \quad \text{as } n \to \infty, \tag{32}$$

$$M'(d_n)(b - d_n) + L(d_n) \le M(b) \quad \text{for all } n \ge 0, \tag{33}$$

where,

$$d_0 = 0,\ d_{n+1} = L(d_n) + L_0'((x_n))(c_n),\ c_n = |x_{n+1} - x_n| \quad (n \ge 0) \tag{34}$$

and

$$b = M^{\infty}(0). \tag{35}$$

Then sequence $\{x_n\}$ $(n \ge 0)$ generated by Newton's method (2) is well defined, remains in E^{π}, is monotone and converges to a unique zero x^* in $U(b)$, where b is the smallest fixed point of M in $[0,p]$. Moreover the following bounds hold for all $n \ge 0$

$$d_n \le b, \tag{36}$$

$$|x^* - x_n| \le b - d_n, \tag{37}$$

$$|x^* - x_n| \le M^n(p) - M^n(0), \tag{38}$$

$$c_n + d_n \le d_{n+1}, \tag{39}$$

and

$$c_n = |x_{n+1} - x_n|. \tag{40}$$

Proof. We first show that (x_n, d_n) is well defined and monotone for all $n \ge 0$ using induction on n.

By (5), (22) and the order convexity of L_0 we get

$$L_0'(0)(b) + |-F(0)| \le L_0(b - 0) + L_0(0) = L_0(b) \le M(b) = b. \tag{41}$$

Hence equation

$$r = (I - F'(0))(r) - F(0) \tag{42}$$

has a solution x_1, which is well defined, and $(x_1, b) \in E$. We also have

$$x_1 = (I - F'(0))(x_1) + (-F(0)), \tag{43}$$
$$|x_1| \leq L_0'(0)|x_1| + L(0) = d_1, \tag{44}$$

and

$$d_1 = L_0'(0)|x_1| + L(0) \leq M'(0)(b) + L(0) \leq M(b) = b. \tag{45}$$

Assume (x_k, d_k) is well defined and monotone with

$$0 \leq (x_{n-1}, d_{k-1}) \leq (x_k, d_k), \quad k = 1, 2, \ldots, n \tag{46}$$

and

$$d_k \leq b. \tag{47}$$

We must solve

$$r = (I - F'(x_n))(r) + (-F(x_n)). \tag{48}$$

Using (26) and (31) we can have

$$\begin{aligned} |I - F'(x_n)| &= |I - F'(x_n) - F'(0) + F'(0)| \\ &= |(I - F'(0)) + (F'(0) - F'(x_n))| \\ &\leq L_0'(0) + L_0'(|0| + |x_n|) - L_0'(|0|) = L_0'(|x_n|). \end{aligned} \tag{49}$$

That is

$$L_0'(|x_n|) \in B(I - F'(x_n)). \tag{50}$$

By the approximation

$$-F(x_n) = -F(x_n) + F(x_{n-1}) + F'(x_{n-1})(x_1 - x_{n-1}) \tag{51}$$

we can have in turn using (27), (33) and (34)

$$\begin{aligned} |-F(x_n)| &\leq \int_0^1 [L'(|x_{n-1}| + tc_{n-1}) - L'(|x_{n-1}|)]c_{n-1}dt \\ &= L(|x_{n-1}| + c_{n-1}) - L(|x_{n-1}|) - L'(|x_{n-1}|)c_{n-1} \\ &\leq L(d_{n-1} + d_n - d_{n-1}) - L(d_{n-1}) - L_0'(|x_{n-1}|)c_{n-1} \\ &= L(d_n) - d_n, \end{aligned} \tag{52}$$

and

$$L_0'(|x_n|)(b - d_n) + |-F(x_n)| + d_n \leq M'(d_n)(b - d_n) + L(d_n) \leq M(b) = b. \tag{53}$$

Hence we showed

$$M'(|x_n|)(b - d_n) + |-F(x_n)| \le b - d_n. \tag{54}$$

By Lemma 3, x_{n+1} is defined and

$$c_n \le b - d_n. \tag{55}$$

Therefore d_{n+1} is well defined too, and we obtain:

$$d_{n+1} \le L(d_n) + L_0'(d_n)(b - d_n) \le L(d_n) + M'(d_n)(b - d_n) \le M(b) = b. \tag{56}$$

Moreover we can have:

$$c_n + d_n \le L_0'(|x_n|(c_n) + |-F(x_n)| + d_n \le L_0'(|x_n|)(c_n) + L(d_n) = d_{n+1}, \tag{57}$$

which shows the monotonicity $(x_n, d_n) \le (x_{n+1}, d_{n+1})$ $(n \ge 0)$.

Furthermore we get

$$M^n(0) \le d_n \le b, \tag{58}$$

and since

$$M^n(0) \to b \quad \text{as } n \to \infty \tag{59}$$

we get

$$d_n \to b \quad \text{as } n \to \infty. \tag{60}$$

That is, $x_n \to x^* \in U(b)$, and $F(x^*) = 0$. Clearly $(y_n, M^n(0))$ is monotone in $X \times V$, and if $x^{**} \in U(p)$ is a solution of F, then as above we get

$$|x^{**} - y_n| \le M^n(p) - M^n(0) \le M'(p)^n(p) \to 0 \quad \text{as } n \to \infty \tag{61}$$

where,

$$y_{n+1} = y_n - F(y_n). \tag{62}$$

That is $\lim_{n\to\infty} y_n = x^{**}$, which shows the uniqueness of x^* in $U(p)$.

That completes the proof of Theorem 1. ☐

Remark 1. Under the hypotheses of Theorem 1 define

$$R_n(r) = (I - L_0'(|x_n|))^* S_n(r) + c_n, \tag{63}$$

where,

$$S_n(r) = L(|x_n| + r) - L(|x_n|) - L_0'(|x_n|)(r). \tag{64}$$

As in [1], [5] assume there exists $r_n \in C$ such that $|x_n| + r_n \le p$, and

$$S_n(r_n) + L_0'(|x_n|)(r_n - c_n) \le r_n - c_n \tag{65}$$

then $R_n \colon [0, r_n] \to [0, r_n]$ is well defined and monotone. Since

(66)
$$\begin{aligned} d_n + c_n \le d_{n+1} \Rightarrow\ & L(p) - L(d_n) - L_0'(|x_n|)c_n \\ & \le p - d_n - c_n, \end{aligned}$$

$$\Rightarrow s_n(p - d_n) + L_0'(|x_n|)(p - d_n - c_n) \le p - d_n - c_n, \tag{67}$$

which implies that $r_n = p - d_n$ is a suitable r_n. That is using solutions of

$$R_n(r) \le r \tag{68}$$

we obtain better a posteriori error estimates than

$$|x^* - x_n| \le b - d_n \le r - d_n \quad (n \ge 0). \tag{69}$$

The proof of the following propositions can be obtained from Lemmas 10, 11, 12 in [5] by simply replacing L' by L_0' in their proofs.

Proposition 1. *If $r \in [0, p - |x_n|]$ satisfies $R_n(r) \leq r$ then the following holds for all $n \geq 0$:*

$$c_n \leq R_n(r) = a \leq r \tag{70}$$

and

$$R_{n+1}(a - c_n) \leq a - c_n. \tag{71}$$

Proposition 2. *Assume hypotheses of Theorem 1 hold and let $r_n \in [0, p - |x_n|]$ be a solution (69). Then the following a posteriori estimates hold for all $n \geq 0$:*

$$|x^* - x_m| \leq q_m, \tag{72}$$

where,

$$q_n = r_n, \quad q_{m+1} = R_m(q_m) - c_m \quad (m \geq n). \tag{73}$$

Proposition 3. *Under the hypotheses of Theorem 1 any solution $r \in [0, p - |x_n|]$ of $R_n(r) \leq r$ yields the a posteriori estimate*

$$|x^* - x_n| \leq R_n^\infty(0) \leq r \quad (n \geq 0). \tag{74}$$

3. Monotone Convergence

Let X be partially ordered and set

$$X = V, \quad D = C^2 \quad \text{and} \quad |\cdot| = I. \tag{75}$$

We can show:

Theorem 2. *Let X be partially ordered, Y a Banach space, $G \in C^1(X_0 \subseteq X \to Y)$, $A \in L(Y \to X)$ and $x, y \in X$ such that:*

$$1. \quad [x, y] \in X_0, \tag{76}$$

$$2. \quad I - AG'(x) \in L_+(X), \tag{77}$$

$$3. \quad z_1 \leq z_2 \Rightarrow AG'(z_1) \leq AG'(z_2) \ \ \textit{for all } z_1, z_2 \in [x, y], \tag{78}$$

$$4. \quad AG(x) \leq 0, \quad AG(y) \geq 0, \tag{79}$$

$$5. \quad (I - AG'(y))^n (y - x) \to 0 \quad \textit{as } n \to \infty. \tag{80}$$

Then sequence $\{y_n\}$ $(n \geq 0)$ generated by

$$y_0 = x, \quad y_{n+1} = y_n + AG'(y_n)^*[-AG(y_n)] \tag{81}$$

is well defined for all $n \geq 0$, monotone and converges to a unique zero y^* of AG in $[x, y]$.

Proof. Define

$$F(u) = AG(x + u), \quad M(r) = r - F(r), \quad p = y - x. \tag{82}$$

The result now follows from Theorem 1.

That completes the proof of Theorem 2. □

Remark 2. Conditions (28) and (32) can be replaced by:

$$\text{there exists } \theta \in (0,1)\colon M(p) \leq \theta p. \tag{83}$$

Then there exists $p' \in [0, \theta p]$ satisfying all hypotheses of Theorem 1. The solution $x^* \in U(p')$ and is unique in $U(p)$.

Remark 3. (a) If $L_0 = L = M$ the results obtained here reduce to the corresponding ones in [5] (see also [1], [2], [4]).

(b) If $L = M$, condition (33) holds automatically. Inequalities (29) and (30) hold in general. However if strict inequality holds in (29) and (30) then our error bounds are finer than the ones in Lemmas 9–12 [5] in this case, and under the same sufficient convergence conditions (see, e.g. (28) and (32)). Let us demonstrate this observation with one of the error bounds. Define as in [5] sequence $\{q_n\}$ $(n \geq 0)$ by

$$q_0 = 0, \quad q_{n+1} = L(q_n) + L'(|x_n|)(c_n). \tag{84}$$

Then we get

$$d_n < q_n \quad (n \geq 1). \tag{85}$$

(c) Estimate

$$M'(|x_n|)(\overline{d}_n) + |-F(x_n)| \leq \overline{d}_n \quad \text{for some } 0 \leq \overline{d}_n\colon \overline{d}_n + d_n \leq b \tag{33'}$$

or condition

$$x_n \text{ given by (2) is well defined for all } n \geq 1 \tag{33''}$$

can replace (33) in Theorem 1. Hypotheses $(33)''$ is the weakest of the three.

(d) (M, M') can be replaced by a sequence of in practice more precise pairs (M_n, M_n') $(n \geq 0)$ in the hypotheses of Theorem 1 (see (e) that follows).

(e) If $L \neq M$ one would expect: (28) and (32) (or (83)) will be weaker than the corresponding ones in [5] given by

$$L(p) \leq p, \tag{86}$$

$$L'(p)^n(p) \to 0 \quad \text{as } n \to \infty \tag{87}$$

(or $L(p) \leq \theta p$) (because of (85)); and that (48) will be solvable in which case (33) can be dropped.

As an example of such a case we consider a Banach space X with a real norm $\|\cdot\|$. Assume $F'(0) = I$ and

$$\|F'(x) - F'(y)\| \leq \ell \|x - y\|, \tag{88}$$

$$\|F'(x) - F'(0)\| \leq \ell_0 \|x\| \tag{89}$$

for all $x, y \in U(r_0)$, and some $\ell_0, \ell \neq 0$ where

$$r_0 = \frac{1 - \sqrt{1 - 2\ell\eta}}{\ell} \tag{90}$$

and

$$\eta \geq \|F(0)\|. \tag{91}$$

Define operators L_0, L by

$$L_0(r) = \|F(0)\| + \frac{1}{2}\ell_0 r^2, \tag{92}$$

$$L(r) = \|F(0)\| + \frac{1}{2}\ell r^2, \tag{93}$$

sequences α_n, β_n, γ_n by

$$\alpha_0 = \beta_0 = \gamma_0 = 0, \quad \alpha_1 = \beta_1 = \alpha_1 = \eta, \tag{94}$$

$$\alpha_{n+1} = \alpha_n + \frac{\ell_0 \alpha_n^2}{2(1-\ell_0\alpha_n)}, \tag{95}$$

$$\beta_{n+1} = \beta_n + \frac{\ell \beta_n^2}{2(1-\ell_0\beta_n)}, \tag{96}$$

$$\gamma_{n+1} = \gamma_n + \frac{\ell \gamma_n^2}{2(1-\ell\gamma_n)}, \tag{97}$$

and points α^*, β^*, γ^* by

$$\alpha^* = \frac{1-\sqrt{1-2\ell_0\eta}}{\ell_0}, \tag{98}$$

$$\beta^* = \lim_{n\to\infty} \beta_n, \tag{99}$$

$$\gamma^* = \frac{1-\sqrt{1-2\ell\eta}}{\ell} \tag{100}$$

provided that

$$h = 2\ell\eta \leq 1. \tag{101}$$

Note that (101) is the famous Newton–Kantorovich hypothesis for the convergence of Newton's method [4], [5].

As it can easily be seen by (2), (34), (49), (52), (62), and (84) iterations (95), (96), (97) are the majorizing sequences (denoted by d_n in Theorem 1) corresponding to methods (62), (2) (via Theorem 1) and (2) (via Lemma 9 in [5] and (84)) respectively.

Condition $(33)''$ needed to solve (48) can be dropped since iterates say β_n (i.e. d_n) can be found for all $n > 0$.

The pair (M_n, M_n') is defined "discretely" for each $n \geq 0$ by sequence β_n which is monotonically convergence under (107). Therefore the verification of (28) and (32) is not needed since it is replaced by (107), which also makes the explicit construction of (M_n, M_n') unnecessary.

Then it can easily be seen

$$\alpha^* = L_0^\infty(0), \tag{102}$$

$$\gamma^* = L^\infty(0), \tag{103}$$

$$\alpha_n \le \beta_n \le \gamma_n \quad (n \ge 2), \tag{104}$$

and

$$\alpha^* \le \beta^* \le \gamma^*. \tag{105}$$

Note that in general

$$\ell_0 \le \ell. \tag{106}$$

Moreover if strict inequality holds in (106) then so does in (105). Furthermore it can easily be seen that (96) monotonically converges to β^* provided that

$$h_0 = (\ell + \ell_0)\eta \le 1, \tag{107}$$

which is a weaker hypothesis than (0). That is

$$h \le 1 \Rightarrow h_0 \le 1 \tag{108}$$

but not vice versa unless if $\ell = \ell_0$. That is in the Banach space case with real norm $\|\ \|$, our Theorem 1 guarantees convergence of Newton's method under (107), whereas Lemma 9 in [5] can only guarantee convergence under the stronger condition (101).

We complete this study with an example to show $\frac{\ell}{\ell_0}$ can be arbitrarily large, which implies that (106) holds a strict inequality and that (107) may hold but not (101).

Example 1. Let $X = Y = \mathbf{R}$, $x_0 = 0$, $A = G'(x_0)^{-1}$, and define function G on $\mathbf{R}$ by

$$G(x) = c_0 x + c_1 + c_2 \sin e^{c_3 x}, \tag{109}$$

where c_i, $i = 0, 1, 2, 3$ are given parameters. Using (109) it can easily be seen that for c_3 large and c_2 sufficiently small, $\frac{\ell}{\alpha_2}$ may be arbitrarily large. That is (107) can be satisfied but not (101).

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DOUBLE BARRIER OPTIONS WITH CURVED BOUNDARIES

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Abstract. We consider a double barrier option between two generic curved boundaries and use a Green's function analysis to reformulate the problem as a pair of coupled integral equations, and solve the integral equations for the special case of flat boundaries.

A.M.S. (MOS) Subject Classification Codes. 44A10, 45K05.

Key Words and Phrases. options, financial derivatives, Green's function, integral equations

1. Introduction

Since the publication of the celebrated Black-Scholes option pricing formula in 1973 [2,10], there has been an explosion in the use of financial derivatives, which are instruments that derive their value from some other underlying instrument. An example of such a derivative is an option, which typically carries the right but not the obligation to either buy or sell the underlying. In addition to the publicly traded puts and call with which many investors are familiar, there are numerous other options, usually termed "exotic", which typically are traded over-the-counter, or OTC, meaning that typically they are custom-made by financial institutions for clients. Once such exotic is the barrier option, which is is either "knocked in" or "knocked out", depending on the flavor of the barrier, when the price of the underlying is equal to the barrier price. One reason for the existence of barrier options is that they can be cheaper than their plain vanilla counterparts because they risk either not being knocked in or being knocked out. Another reason is that they may be a part of a more complicated structured contract. Therefore, barrier options have become increasingly popular in world markets. According to

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[15], barriers have been traded in the USA since at least 1967, some 6 years before the Chicago Board of Options Exchange (CBOE) opened its doors in 1973, with a typical user being a hedge fund. Like all equity options, the value $V(S,t)$ of a barrier option is given by the Black-Scholes-Merton PDE,

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r - D_0)\, S\frac{\partial V}{\partial S} - rV0 \tag{1.1}$$

with the various options differentiated by the boundary conditions [11].

There are single barrier options and double barrier options. Closed-form solutions are available for single-barrier options in the Black-Scholes framework [10,6]. Other closed-form solutions have been found in Black-Scholes setup (e.g., [13,3]). On the other hand, no simple formula currently exists for other options with multiple barriers. Double barrier options are an ideal method for investors in particular to take advantage of markets expected to trade within a range. They are available in most underlying markets, including commodities, interest rate and equity markets and most commonly in foreign exchange. In the present study, we will consider a double barrier option, occasionally referred to as a dual barrier option, where there is both an upper boundary $S_u(t)$ and a lower boundary $S_l(t)$, which are typically specified in the option contract. It is possible to recover an option with a single barrier from this by either taking the limit $S_u(t) \to \infty$, when we recover an option with a lower barrier, or the limit $S_l \to 0$, when we recover an option with an upper barrier. Work on this problem goes back at least as far as 1965, when [4] gave a density function involving Fourier series for two flat boundaries, $S_u(t) = S_{u0}$ and $S_l(t) = S_{l0}$, and also used the method of images to construct a Green's function, and [5] revisited that problem using Laplace transforms, while [8] gave a formula for two exponentially curved boundaries, with $S_u(t) = S_{u0}e^{\delta_u(T-t)}$ and $S_l(t) = S_{l0}e^{\delta_l(T-t)}$. [8] also listed several examples of double barrier options which were traded in Tokyo at the time of their study. Other than the papers mentioned above, other contributions to the pricing of double barrier options include the studies of [12,14], who used residue theory to invert the transforms in [5] those transforms having been inverted numerically in the original paper, and the studies of [7,1], where the connection between the solutions obtained by the different methods mentioned above was discussed, and an early numerical study by [9] who evaluated a LYON (liquid option yield note) which had curved boundaries.

While [8] considered exponentially curved boundaries, in the present work, we will consider two generic curved boundaries $0 \leq S_l(t) < S_u(t)$, and also allow for the possibility of rebates when the barriers are hit, so that the pay-off for a double knock-out option is assumed to be $R_u(t)$ on $S_u(t)$ and $R_l(t)$ on $S_l(t)$, while if the barriers are never hit, and the price of the underlying is always between the two barriers, $S_l(t) < S < S_u(t)$ for $t \leq T$, the pay-off at expiry is $\max(S - E, 0)$, meaning we have a plain vanilla European call with strike price E. Similarly, we can define a double knock-in option, for which the pay-off at expiry is zero if the barriers are never hit, $S_l(t) < S < S_u(t)$ for $t \leq T$, while if we hit either of the barriers, we receive a rebate as above together with a plain vanilla European call. Since the value of a European call on the barriers, $V_e(S_u(t), t)$ or $V_e(S_l(t), t)$, can be readily calculated using the Black-Scholes option pricing formula, we can assume for the double knock-in option that the value of the European call is included within the rebate, so that the pay-off for a double knock-in option is assumed to be $R_u(t)$ on $S_u(t)$ and $R_l(t)$ on $S_l(t)$.

To price such options, we will use a Green's function approach. The key to this, which we will present in the next section, is that we can write down a solution to the Black-Scholes-Merton equation of the form

$$V(S,t) = \int_{-\infty}^{\infty} v(z,0)G(S,z,T-t)dz, \tag{1.2}$$

where $v(z,0)$ is related to the pay-off at expiry, and the variables in this formula are defined in the next section. If we are pricing options which have no early exercise features, such as vanilla European options, then the formula (1.2) will give us a solution. However, for problems with boundaries, such as American options where early exercise is allowed or the double barrier options considered here, the problem is that the pay-off at expiry is only defined for certain values of the underlying price. Because of this, the function $v(z,0)$ is only known for certain values of z, while in the formula (1.2) above, we need to know $v(z,0)$ for all z, $-\infty < z < \infty$. There are at least two possible ways to overcome this. One approach is, rather than use the solution (1.2) presented above, to instead the Green's function for the Black-Scholes-Merton equation with boundaries present, so that the solution can be cast as

$$V(S,t) = \int_{z_l}^{z_u} v(z,0)G(S,z,T-t)dz, \tag{1.3}$$

where the Green's function $G(S,z,T-t)$ in (1.3) differs from that in (1.2); for flat boundaries, it is possible to do this using the method of images, which was the approach used in [4]. An alternative approach, which we consider here, is to use the formula (1.2) and assign some form to the function $v(z,0)$ outside the barriers: in essence, we are supposing that we have a European option with some nonlinear pay-off at expiry whose value on the barriers is equal to the (known) value of the double barrier option. essentially, This second approach means that we have to find $v(z,0)$ outside the barriers for this new option, and then the value of our double barrier option is given by (1.2) together with the pay-off.

The remainder of the paper is as follows. In Section 2, we will introduce the Green's function and perform our analysis, then in Section 3, we will make some concluding remarks.

2. Analysis

The value $V(S,t)$ of an option is given by the Black-Scholes-Merton PDE,

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r - D_0)\, S\frac{\partial V}{\partial S} - rV = 0\ , \tag{2.1}$$

where S is the price of the underlying, and D_0, r and σ, all of which are assumed constant, are the dividend yield, risk-free interest rate, and volatility respectively. This equation was initially presented by [2] for options without dividends and extended by [10] to cover options with a continuous dividend yield. For a vanilla European option, this equation is valid for all S, $0 \le S < \infty$, and for all times up to expiry $t < T$, together with the pay-off at expiry, $V(S,T)$.

For the double barrier options considered here, the equation (2.1) is only valid while the price of the underlying remains between the two barriers, $S_l(t) < S < S_u(t)$ for $t \le T$, while if the underlying price hits one of the barriers, we assume that a rebate is paid, which we take to be $R_u(t)$ on $S_u(t)$ and $R_l(t)$ on $S_l(t)$. In our analysis, we require R_u and R_l both tend to 0 as $t \to -\infty$, so that the option price will also tend to 0 in that limit. Although we have the value of the option on the boundary, we do not know the delta of the option, $(\partial V/\partial S)$, on the boundary; this can be contrasted with plain vanilla American options, where the delta is known but the location of the boundary is unknown. The pay-off at expiry between the barriers is $V(S,T) = \max(S-E,0)$ for a double knock-out option and $V(S,T) = 0$ for a double knock-in option.

This problem can be recast as a more conventional diffusion problem,

$$\frac{\partial v}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2}, \tag{2.2}$$

by using the transformation

$$V(S,t) = S^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} e^{-r_2\tau} v(x,\tau), \tag{2.3}$$

where $x = \log S$ and $\tau = T-t$ and $r_2 = \frac{(r-D_0)^2}{2\sigma^2} + \frac{D_0+r}{2} + \frac{\sigma^2}{8}$. This has a solution using a Green's function

$$v(x,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} v(z,0) e^{-\frac{(x-z)^2}{2\sigma^2\tau}}\, dz, \tag{2.4}$$

so that in the original variables we have

$$V(S,t) = S^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{e^{-r_2(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} v(z,0) e^{-\frac{(\log S - z)^2}{2\sigma^2(T-t)}}\, dz. \tag{2.5}$$

As noted in the introduction, we only know $v(z,0)$ between the barriers. We require this to be equal to the rebate on the barriers, so we have a pair of twin integral equations for $v(z,0)$, which is unknown except between $S_l(T) \le S \le S_u(T)$ or equivalently $\log S_l(T) \le z \le \log S_u(T)$,

$$\begin{aligned} R_u(t) &= (S_u(t))^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{e^{-r_2(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} v(z,0) e^{-\frac{(\log S_u(t)-z)^2}{2\sigma^2(T-t)}}\, dz \\ R_l(t) &= (S_l(t))^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{e^{-r_2(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} v(z,0) e^{-\frac{(\log S_l(t)-z)^2}{2\sigma^2(T-t)}}\, dz. \end{aligned} \tag{2.7}$$

We can rewrite this so that the exponential term is the same in both integrals

$$\begin{aligned} R_u(t) &= (S_u(t))^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{e^{-r_2(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} v(z+\log S_u(t),0) e^{-\frac{z^2}{2\sigma^2(T-t)}}\, dz \\ R_l(t) &= (S_l(t))^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{e^{-r_2(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} v(z+\log S_l(t),0) e^{-\frac{z^2}{2\sigma^2(T-t)}}\, dz, \end{aligned} \tag{2.8}$$

and in (2.8) we have a pair of coupled integral equations for $v(z,0)$; once $v(z,0)$ has been determined, the price of the option can be recovered using (2.5). We can write down an explicit solution to the equations (2.8) in some special cases. When the boundaries are flat, $S_u(t) = S_{u0}$ and $S_l(t) = S_{l0}$, we have

$$\begin{aligned} R_u(t)e^{-r_2(T-t)} &= (S_{u0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{1}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} v(z+\log S_{u0},0) e^{-\frac{z^2}{2\sigma^2(T-t)}} dz \\ R_l(t)e^{-r_2(T-t)} &= (S_{l0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{1}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} v(z+\log S_{l0},0) e^{-\frac{z^2}{2\sigma^2(T-t)}} dz, \end{aligned} \tag{2.9}$$

We can rewrite these two equations as

$$\begin{aligned} \tilde{R}_u(\tau)e^{-r_2\tau} &= (S_{u0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{1}{\sigma\sqrt{2\pi\tau}} \int_0^{\infty} \left[v_1(z)+v_2(z)\right] e^{-\frac{z^2}{2\sigma^2\tau}} dz \\ \tilde{R}_l(\tau)e^{-r_2\tau} &= (S_{l0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{1}{\sigma\sqrt{2\pi\tau}} \int_0^{\infty} \left[v_3(z)+v_4(z)\right] e^{-\frac{z^2}{2\sigma^2\tau}} dz, \end{aligned} \tag{2.10}$$

where $v_1(z) = v(z+\log S_{u0}, 0)$, $v_2(z) = v(-z+\log S_{u0}, 0)$, $v_3(z) = v(z+\log S_{l0}, 0)$ and $v_4(z) = v(-z+\log S_{l0}, 0)$, $\tau = T - t$ is the time remaining until expiry and $\tilde{R}_u(\tau) = R_u(T-\tau)$, with a similar definition for $\tilde{R}_l$. If we apply a Laplace transform in τ to these equations, we obtain

$$\begin{aligned} \sigma\sqrt{2p}L\left[\tilde{R}_u\right](p+r_2) &= (S_{u0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \left[L\left[v_1\right](\frac{\sqrt{2p}}{\sigma}) + L\left[v_2\right](\frac{\sqrt{2p}}{\sigma})\right] \\ \sigma\sqrt{2p}L\left[\tilde{R}_l\right](p+r_2) &= (S_{l0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \left[L\left[v_3\right](\frac{\sqrt{2p}}{\sigma}) + L\left[v_4\right](\frac{\sqrt{2p}}{\sigma})\right], \end{aligned} \tag{2.11}$$

which we can write as

$$\begin{aligned} L\left[v_1\right](p) + L\left[v_2\right](p) &= (S_{u0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \sigma^2 p L\left[\tilde{R}_u\right]\left(\frac{\sigma^2p^2}{2}+r_2\right) \\ L\left[v_3\right](p) + L\left[v_4\right](p) &= (S_{l0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \sigma^2 p L\left[\tilde{R}_l\right]\left(\frac{\sigma^2p^2}{2}+r_2\right), \end{aligned} \tag{2.12}$$

or inverting, we get

$$\begin{aligned} v(z+\log S_{u0},0) + v(-z+\log S_{u0},0) &= (S_{u0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \sigma^2 \\ &\qquad \times \frac{d}{dz}L^{-1}\left(L\left[\tilde{R}_u\right]\left(\frac{\sigma^2p^2}{2}+r_2\right)\right)[z] \\ v(z+\log S_{l0},0) + v(-z+\log S_{l0},0) &= (S_{l0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \sigma^2 \\ &\qquad \times \frac{d}{dz}L^{-1}\left(L\left[\tilde{R}_l\right]\left(\frac{\sigma^2p^2}{2}+r_2\right)\right)[z]. \end{aligned} \tag{2.13}$$

Since we already know $V(S,T)$ for $S_{l0} \le S \le S_{u0}$, or equivalently $v(z,0)$ for $\log S_{l0} \le z \le \log S_{u0}$, for specific functions R_l and R_u, it is possible to deduce the function $v(z,0)$ for all z, $-\infty < z < \infty$ and then we can recover the value of the option from (2.5).

When we have exponentially curved boundaries $S_u(t) = S_{u0}e^{\delta_u(T-t)}$ and $S_l(t) = S_{l0}e^{\delta_l(T-t)}$, the integral equations (2.8) become

$$\begin{aligned}
\tilde{R}_u(\tau) = &(S_{u0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left[\left(-r_2 + \delta_u\left(\frac{D_0-r}{\sigma^2}+\frac{1}{2}\right)\right)\tau\right] \\
&\times \int_{-\infty}^{\infty} v(z + \log S_{u0} + \delta_u\tau, 0)e^{-\frac{z^2}{2\sigma^2\tau}}\,dz, \\
\tilde{R}_l(\tau) = &(S_{l0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left[\left(-r_2 + \delta_l\left(\frac{D_0-r}{\sigma^2}+\frac{1}{2}\right)\right)\tau\right] \\
&\times \int_{-\infty}^{\infty} v(z + \log S_{l0} + \delta_l\tau, 0)e^{-\frac{z^2}{2\sigma^2\tau}}\,dz,
\end{aligned} \tag{2.14}$$

which in principal could be solved, and for the case $R_l(t) = R_u(t) = 0$, the formula in [8] recovered.

3. Discussion

In the previous section, we used a Green's function to consider the valuation of a double barrier option with two curved boundaries, $0 \le S_l(t) < S_u(t)$, with rebates of $R_l(t)$ and $R_u(t)$ when these boundaries are hit. By doing so, we were able to arrive at our main result, which was the pair of coupled integral equations, (2.8); once these equations are solved, the price of the option can be recovered using (2.5). We were able to solve (2.8) using Laplace transforms for the special case of flat boundaries, arriving at the expression (2.13). When there are no rebates, (2.13) is equivalent to the well-known expression [4] using the method of images, but one contribution of our present study is to offer a framework for pricing double barrier options with general rebates, with the other main contribution of this work being to offer a framework for pricing double barrier options with generic curved boundaries, as opposed to the exponentially curved boundaries considered by [8].

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INTERNATIONAL FISHING AS PARTIALLY COOPERATIVE GAME

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Abstract. A game theoretical model of international fishing is introduced under partial cooperation of the countries involved. The interior equilibrium is determined and it is shown that both increasing cooperation levels and smaller available fish stock have decreasing effects on the total fish harvest. A dynamic system model is developed for the dynamic behavior of the fish stock.

A.M.S. (MOS) Subject Classification Codes. 91-99.

Key Words and Phrases. Partial cooperation, equilibrium, international fishing, dynamic system.

1. Introduction

International fishing became the focus of many researchers of game theory during the last decade. Okuguchi [1] has introduced a fishing duopoly in which the fishing countries behave as Cournot oligopolists at each time period, so the fish harvest is determined by the Cournot -Nash equilibrium. The fishing activity changes the biological reproduction law of the fish stock, so its long term properties depend on the fishing activity. The asymptotic behavior of the fish stock was examined in this paper. This model and the asymptotic results were extended to the general n-country case in Szidarovszky and Okuguchi [2]. If the fishing countries form a coalition, then the fish harvest is always lower than that in the non-cooperative case resulting in a more stable fish-stock (Szidarovszky and Okuguchi, [3]). Entry of a new country, merging, and their effects on the fish stock and profitability were investigated in Szidarovszky et al. [4], and the effect of information delay on asymptotic behavior of the fish stock was examined by Engel et al. [5].

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In this paper partial cooperation of the fishing countries will be considered, when the payoff of each country contains her profit and some portions of the profits of the others. After the mathematical model will be introduced, the dependence of the fish harvest on the cooperation levels of the countries as well as on the available fish stock will be investigated. And finally, dynamic models describing the long-term behavior of the fish stock will be developed.

2. The Mathematical Model

Assume that n countries or firms harvest fish of a single species in a common property fishing ground and sell the fish in m markets. Let x_{ri} denote the amount of fish harvest of country r and sold in market i. The total harvest of country r is therefore

$$X_r = x_{r1} + x_{r2} + x_{r3} + ... + x_{rm} \ , \tag{1}$$

and the total fish supply in market i is the following:

$$Y_i = x_{1i} + x_{2i} + ... + x_{ni}. \tag{2}$$

The price of the fish in market i is assumed to be linear:

$$P_i(Y_i) = a_i - b_i Y_i, \tag{3}$$

and the cost function of country r is given as

$$C_r(X_r) = c_r + \gamma_r \frac{X_r^2}{X}, \tag{4}$$

here X is the available fish stock. With these price and cost functions we follow the models of the earlier studies mentioned before. The profit of country r is then

$$\Psi_r = \sum_{i=1}^{m} (a_i - b_i Y_i) x_{ri} - (c_r + \gamma_r \frac{X_r^2}{X}), \tag{5}$$

wich is the difference of the revenue and cost of the country. Let $\alpha_{rl} \in [0,1]$ denote the cooperation level of country r towards country l, then the payoff function of country r is

$$\pi_r = \Psi_r + \sum_{l \neq r} \alpha_{rl} \Psi_l. \tag{6}$$

This payoff can be interpreted as country r takes α_{rl} percents of the interest of country l into account in addition to his own full interest in his payoff function. Using equation (5),

$$\pi_r = \sum_{i=1}^{m} (a_i - b_i Y_i) x_{ri} - (c_r + \gamma_r \frac{X_r^2}{X}) + \sum_{l \neq r} \alpha_{rl} [\sum_{i=1}^{m} (a_i - b_i Y_i) x_{li} - (c_l + \gamma_l \frac{X_l^2}{X})]. \tag{7}$$

For all values $X > 0$ of the fish stock we have now defined an n-person non-cooperative game with strategy sets

$$S_r = \{(x_{r1}, x_{r2}, ..., x_{rm)} \mid X_r = \sum_{i=1}^{m} x_{ri} \in [0, X]\}, \tag{8}$$

and payoff functions π_r with the additional constraint:

$$\sum_{r=1}^{n} X_r \in [0, X]. \tag{9}$$

In the next section the interior Cournot-Nash equilibrium of this game will be determined.

The equilibrium of the game

At a Cournot-Nash eqilibrium each π_r is maximal in $x_{r1}, x_{r2}......, x_{rm}$ with fixed strategies of all other countries. Assuming interior equilibrium, the first order conditions can be written as

$$\frac{\partial \pi_r}{\partial x_{ri}} = a_i - b_i Y_i - b_i x_{ri} - \frac{2\gamma_r X_r}{X} + \sum_{l \neq r} \alpha_{rl}(-b_i x_{li}) = 0, \tag{10}$$

implying that for all r and i,

$$x_{ri} + \sum_{l \neq r} \alpha_{rl} x_{li} = \frac{a_i}{b_i} - Y_i - \frac{2\gamma_r X_r}{X b_i}. \tag{11}$$

Since π_r is concave in $(x_{r1,}......., x_{rm})$, the second order conditions are satisfied. For simplicity, introduce the following notations:

$$\underline{H} = \begin{pmatrix} 1 & \alpha_{12} & ...... & \alpha_{1n} \\ \alpha_{21} & 1 & ...... & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & ...... & 1 \end{pmatrix}, \underline{Z}_i = \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix}, \underline{x} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$$

$$\underline{1} = (1 \quad 1 \quad ... \quad .. \quad 1)^T, and\, \underline{C} = diag(\gamma_1, \gamma_{2,....,}\gamma_n).$$

Then (11) can be rewritten as

$$\underline{HZ}_i = (\frac{a_i}{b_i} - Y_i)\underline{1} - \frac{2}{b_i X}\underline{Cx}, \tag{12}$$

that is ,

$$\underline{Z}_i = (\frac{a_i}{b_i} - Y_i)\underline{H}^{-1}\underline{1} - \frac{2}{b_i X}\underline{H}^{-1}\underline{Cx}\ . \tag{13}$$

By adding these equations for all values of i we have

$$\underline{x} = (A - S)\underline{H}^{-1}\underline{1} - \frac{2B}{X}\underline{H}^{-1}\underline{Cx}, \tag{14}$$

where $A = \sum_{i=1}^{m} \frac{a_i}{b_i}$ and $B = \sum_{i=1}^{m} \frac{1}{b_i}$, furthermore $S = \sum_{i=1}^{m} Y_i$ is the total fish harvest of all countries. From (14) we see that

$$\underline{x} = (A - S)(\underline{I} + \frac{2B}{X}\underline{H}^{-1}\underline{C})^{-1}\underline{H}^{-1}\underline{1} = (A - S)(\underline{H} + \frac{2B}{X}\underline{C})^{-1}\underline{1}. \tag{15}$$

Since $\underline{1}^T\underline{x} = X_1 + X_2 + ... + X_n = S$, we obtain a single equation for S:

$$S = (A - S)\underline{1}^T(\underline{H} + \frac{2B}{X}\underline{C})^{-1}\underline{1} \tag{16}$$

implying that

$$S = \frac{Af(X)}{1 + f(X)} \tag{17}$$

with

$$f(X) = \underline{1}^T(\underline{H} + \frac{2B}{X}\underline{C})^{-1}\underline{1}. \tag{18}$$

Consider the special case of noncooperating countries. Then $\alpha_{rl} = 0$ for all $l \neq r$, so $\underline{H} + \frac{2B}{X}\underline{C}$ is a diagonal matrix showing that

$$f(X) = \sum_{r=1}^{n} \frac{1}{1 + \frac{2B\gamma_r}{X}}. \tag{19}$$

Notice that this result coincides with those obtained earlier in Szidarovszky and Okuguchi [2]. The completely cooperative case occurs when the firms maximize the sum of their profits, so in this case $\alpha_{rl} = 1$ for all $l \neq r$. Next we compute the equilibrium in a more general case, when $\alpha_{rl} = \alpha$ for all $l \neq r$. The case of $\alpha = 1$ gives the completely cooperative case. Now we have

$$\underline{H} + \frac{2B}{X}\underline{C} = \begin{pmatrix} 1 & \alpha & \cdots & \alpha \\ \alpha & 1 & \cdots & \alpha \\ \vdots & \vdots & \vdots & \vdots \\ \alpha & \alpha & \cdots & 1 \end{pmatrix} + \begin{pmatrix} \frac{2B\gamma_1}{X} & 0 & 0 & 0 \\ 0 & \frac{2B\gamma_2}{X} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \frac{2B\gamma_n}{X} \end{pmatrix}$$

$$= \underline{D} + \alpha\underline{1}\underline{1}^T, \tag{20}$$

where

$$\underline{D} = diag\left(1 + \frac{2B\gamma_1}{X} - \alpha, ..., 1 + \frac{2B\gamma_n}{X} - \alpha\right).$$

Therefore

$$\left(\underline{H} + \frac{2B}{X}\underline{C}\right)^{-1} = \underline{D}^{-1} - \frac{\alpha\underline{D}^{-1}\underline{1}\underline{1}^T\underline{D}^{-1}}{1 + \alpha\underline{1}^T\underline{D}^{-1}\underline{1}}, \tag{21}$$

so

$$f(X) = \underline{1}^T \underline{D}^{-1} \underline{1} - \frac{1}{1 + \alpha \underline{1}^T \underline{D}^{-1} \underline{1}} \alpha \left(\underline{1}^T \underline{D} \underline{1} \right)^2$$

$$= \frac{\underline{1}^T \underline{D}^{-1} \underline{1}}{1 + \alpha \underline{1}^T \underline{D}^{-1} \underline{1}} = \frac{\sum_{r=1}^{n} \dfrac{1}{1 + \dfrac{2B\gamma_r}{X} - \alpha}}{1 + \sum_{r=1}^{n} \dfrac{\alpha}{1 + \dfrac{2B\gamma_r}{X} - \alpha}}. \tag{22}$$

In the special case of $\alpha = 1$,

$$f(X) = \frac{\dfrac{X}{2B} C}{1 + \dfrac{X}{2B} C} = \frac{XC}{2B + CX}, \tag{23}$$

with $C = \sum_{r=1}^{n} \dfrac{1}{\gamma_r}$. Notice that this result coincides with that presented earlier in Szidarovszky and Okuguchi [3].

4. Dependence on α_{rl}

In this section we will investigate how the cooperation levels α_{rl} affect the total fish harvest. For this purpose we need to find the derivative of S with respect to α_{rl}. Let

$$\underline{U} = \left(\underline{H} + \frac{2B}{X} \underline{C} \right)^{-1}, \tag{24}$$

then

$$\underline{U} \left(\underline{H} + \frac{2B}{X} \underline{C} \right) = \underline{I}. \tag{25}$$

Differentiate both sides with respect to α_{rl} to have

$$\left(\frac{\partial}{\partial \alpha_{rl}} \underline{U} \right) \left(\underline{H} + \frac{2B}{X} \underline{C} \right) + \underline{U} \underline{e}_r \underline{e}_l^T = \underline{0}, \tag{26}$$

where $\underline{e}_r$ and $\underline{e}_l$ are the r th and l th basis vectors.

So

$$\frac{\partial}{\partial \alpha_{rl}} \underline{U} = -\underline{U} \underline{e}_r \underline{e}_l^T \underline{U} = -\underline{c}_r \underline{r}_l^T, \tag{27}$$

where $\underline{c}_r$ is the rth column and $\underline{r}_l^T$ is the l th row of matrix $\underline{U}$, and therefore

$$\frac{\partial}{\partial \alpha_{rl}} f(X) = -\left(\underline{1}^T \underline{c}_r\right)\left(\underline{r}_l^T \underline{1}\right). \tag{28}$$

It is reasonable to assume that $\alpha_{rl} = \alpha_{lr}$ for all $r \neq l$ which means that any two countries cooperate with each other on equal levels. Then $\underline{H}$ is symmetric, therefore $\underline{U}$ is also symmetric showing that $\underline{c}_l = \underline{r}_l$. Then relation (28) and the fact that

$$\frac{\partial S}{\partial \alpha_{rl}} = \frac{\partial}{\partial \alpha_{rl}}\left(\frac{Af(X)}{1+f(X)}\right) = \frac{A}{\left(1+f(X)\right)^2}\frac{\partial f(X)}{\partial \alpha_{rl}}, \tag{29}$$

imply that

$$\frac{\partial S}{\partial \alpha_{rl}} = -\frac{A}{(1+f(X))^2}\left(\underline{1}^T \underline{c}_r\right)\left(\underline{1}_l^T \underline{c}_l\right). \tag{30}$$

Assume that the sum of the elements in each column has the same sign. Then increasing cooperation levels decrease the total fish harvest. A simple consequence of this observation is the fact that total cooperation among the countries results is the smallest total harvest. If $\alpha_{rl} = 0$ for all $r \neq l$, then $\underline{U}$ is diagonal with positive diagonal, so $\underline{1}^T \underline{c}_r > 0$ for all r, so by selecting sufficiently small α_{rl} values, S still decreases in α_{rl}.

5. Dependence on X

Similarly to relation (26) we now have

$$\left(\frac{\partial}{\partial X}\underline{U}\right)\left(\underline{H} + \frac{2B}{X}\underline{C}\right) + \underline{U}\left(-\frac{2B}{X^2}\underline{C}\right) = 0 \tag{31}$$

implying that

$$\frac{\partial}{\partial X}\underline{U} = \frac{2B}{X^2}\underline{UCU}. \tag{32}$$

Therefore

$$f'(X) = \frac{2B}{X^2}\underline{1}^T \underline{UCU1}. \tag{33}$$

Introduce matrix $\underline{K} = diag\left(\sqrt{\gamma_1}, ..., \sqrt{\gamma_n}\right)$ to see that in the case of

$$\alpha_{rl} = \alpha_{lr} \quad (\ r \neq l),$$

$$f'(X) = \frac{2B}{X^2}\underline{1}^T \underline{UKKU1} = \frac{2B}{X^2} \parallel \underline{KU1} \parallel_2^2. \tag{34}$$

Since both $\underline{K}$ and $\underline{U}$ are invertible, $\underline{KU1} \neq \underline{0}$ showing that $f'(X) > 0$, and from (29) we see that S is strictly increasing in X. Therefore on any cooperation level more available fish stock results in larger total harvest.

6. The Dynamic Model

Assume first that without commercial fishing the fish stock changes according to the well-known logistic equation

$$\dot{X} = X(\alpha - \beta X), \tag{35}$$

(see for example, Clark, [6]). In the presence of the commercial fishing this rule becomes

$$\dot{X} = X(\alpha - \beta X - g(X)), \tag{36}$$

with

$$g(X) = \frac{S}{X} = \frac{Af(X))}{X(1+f(X))}. \tag{37}$$

Notice that from (18) we have

$$f(X) = \underline{1}^T(\underline{H}X + 2B\underline{C})^{-1}\underline{1}X, \tag{38}$$

so

$$g(X) = \frac{A\underline{1}^T(\underline{H}X + 2B\underline{C})^{-1}\underline{1}}{1 + \underline{1}^T(\underline{H}X + 2B\underline{C})^{-1}\underline{1}X}. \tag{39}$$

Therefore

$$g(0) = A\underline{1}^T(2B\underline{C})^{-1}\underline{1} = \frac{A}{2B}\sum_{r=1}^{n}\frac{1}{\gamma_r} = \frac{AC}{2B}, \tag{40}$$

where we use the same notation as in equation (23). Notice furthermore that $f(X) \longrightarrow \underline{1}^T\underline{H}^{-1}\underline{1}$ as $X \longrightarrow \infty$ implying that $g(X) \longrightarrow 0$. Any nonzero equilibrium of the dynamic system is the solution of equation

$$\alpha - \beta X = g(X), \tag{41}$$

and $\overline{X} = 0$ is always an equilibrium.

In model (35) a linearly decreasing growth rate was considered. Clark [6] has suggested more advanced population rules. A straightforward generalization of the logistic equation is

$$\dot{X} = X^r(\alpha - \beta X) \tag{42}$$

with some constant $r > 0$. Then in the presence of commercial fishing this rule becomes

$$\dot{X} = X^r(\alpha - \beta X - g(X)) \tag{43}$$

where now

$$g(X) = \frac{s}{X^r} = \frac{Af(X)}{X^r(1 + f(X))}. \tag{44}$$

In the literature instead of equations (35) and (42) the following is used often:

$$\dot{X} = rX \ln \frac{K}{X}, \tag{45}$$

which modifies as

$$\dot{X} = X(r \ln K - r \ln X - g(X)) \tag{46}$$

in the presence of commercial fishing, where $g(X)$ is the same as given earlier in equation (37).
More complicated model is obtained by rule

$$\dot{X} = rX(\frac{X}{K_0} - 1)(1 - \frac{X}{K}) \tag{47}$$

which modifies in the presence of commercial fishing as

$$\dot{X} = X\left[r(\frac{X}{K_0} - 1)(1 - \frac{X}{K}) - g(X)\right] \tag{48}$$

where the $g(X)$ term is the same as given in equation (37).

In the case of model (43), the eqilibrium equation is again (41) giving the intercept of the linear function $\alpha - \beta X$ and the nonlinear curve $g(X)$. In the case of models (46) and (48) the equilibria are the solutions of the more complicated equations

$$r \ln K - r \ln X - g(X) = 0 \tag{49}$$

and

$$r\left(\frac{X}{K_0} - 1\right)\left(1 - \frac{X}{K}\right) - g(X) = 0 \tag{50}$$

respectively. The dynamic behavior of the equilibria depends on the analytic properties of the functions involved. A detailed analysis will be the subject of a future paper.

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A LOCAL CONVERGENCE ANALYSIS AND APPLICATIONS OF NEWTON'S METHOD UNDER WEAK ASSUMPTIONS

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Abstract. We provide a local convergence theorem for Newton's method in a Banach space setting. We use Lipschitz, center Lipschitz conditions and small majorizing sequences. It turns out that this way our convergence radius (ball) is larger and the error bounds finer than before. This observation is very important in computational mathematics, and finds several applications such as the construction of efficient mesh independence refinement strategies.

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1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1}$$

where, F is defined on the closed ball $\overline{U}(x^*, R) = \{x \in X \mid \|x - x^*\| \leq R\}$ $(R \geq 0)$ which is a subset of a Banach space X with values in a Banach space Y.

The famous Newton–Kantorovich method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad (n \geq 0) \tag{2}$$

has been used extensively to generate a sequence converging to x^* under various assumptions. For a survey of such results we refer the reader to [3], [7], [8] and the references there. The assumptions used involve Lipschitz conditions on the Fréchet derivative $F'(x)$ of $F(x)$, Lipschitz constants [7], [8] and or small majorizing functions [2], [3].

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Here we use Lipschitz as well as center Lipschitz conditions. The center Lipschitz conditions are used for the computation of the upper bounds on $\|F'(x)^{-1}F'(x^*)\|$.

This way we manage to obtain more precise error bounds and a larger convergence radius than before [8]. This observation is very important in computational mathematics, and finds many applications such as the construction of efficient mesh independence refinement strategies [1], [3], [5], [6].

Finally, we complete this study with a numerical example to show that our convergence radius (ball) and error bounds are finer than before.

2. Local Convergence Analysis of Newton's Method

We present the following local result for the convergence of Newton's method:

Theorem 1. *Let $F\colon \overline{U}(x^*, R) \subseteq X \to Y$ be a Fréchet-differentiable operator for some $R \geq 0$, x^* such that $F(x^*) = 0$, and $F'(x^*)^{-1} \in L(Y, X)$. Moreover, assume:*

$$\|F'(x^*)^{-1}[F'(x) - F'(y)]\| \leq w(\|x - y\|), \tag{3}$$

$$\|F'(x^*)^{-1}[F'(x) - F'(x^*)]\| \leq w_0(\|x - x^*\|) \tag{4}$$

for all $x, y \in \overline{U}(x^*, R)$, and

$$\overline{U}(x^*, r_0) \subseteq \overline{U}(x^*, R), \tag{5}$$

where, functions $w, w_0\colon [0, R] \to [0, \infty)$ are monotonically increasing with

$$\lim_{r\to 0} w(r) = 0, \tag{6}$$

$$0 \leq w_0(r) \leq w(r) \quad (0 \leq r \leq R), \tag{7}$$

and r_0 is the positive zero of function f given by

$$f(r) = \int_0^1 w(tr)dt + w_0(r) - 1. \tag{8}$$

Then, Newton's method $\{x_n\}$ $(n \geq 0)$ generated by (2) is well defined, remains in $\overline{U}(x^*, r_0)$ for all $n \geq 0$, and converges to x^* provided that $x_0 \in U(x^*, r_0)$. Moreover the following error bounds hold for all $n \geq 0$:

$$\|x_{n+1} - x^*\| \leq \frac{\int_0^1 w(t\|x_n - x^*\|)dt}{1 - w_0(\|x_n - x^*\|)}\|x_n - x^*\|. \tag{9}$$

Proof. Let $x \in U(x^*, r_0)$. It follows from (4) that

$$\begin{aligned}\|F'(x^*)^{-1}[F'(x) - F'(x^*)]\| &\leq w_0(\|x - x^*\|) \\ &< w_0(r_0) \leq 1 \quad \text{(by (4) and (8))}.\end{aligned} \tag{10}$$

It follows from (10) and the Banach Lemma on invertible operators [7] $F'(x)^{-1}$ $F'(x^*)$ exists and

$$\|F'(x)^{-1}F'(x^*)\| \leq [1 - w_0(\|x - x^*\|)]^{-1}. \tag{11}$$

In particular we have: $x_0 \in U(x^*, r_0)$. Assume $x_k \in U(x^*, r_0)$ for all $k \le n$. Using (2), (3), (8) and (11) we get in turn:

(12)
$$\begin{aligned} \|x_{k+1} - x^*\| &= \Bigg\| [F'(x_k)^{-1}F'(x^*)] \\ &\quad \cdot \left[F'(x^*)^{-1}\int_0^1 (F'(x_k + t(x^* - x_k)) - F'(x_k))(x^* - x_k)dt\right]\Bigg\| \\ &\le \frac{\int_0^1 w(\|x_k - x^*\| + t\|x_k - x^*\| - \|x_k - x^*\|)dt}{1 - w_0(\|x_k - x^*\|)}\|x_k - x^*\| \\ &< \frac{\int_0^1 w(tr_0)}{1 - w_0(r_0)}\|x_k - x^*\| = \|x_k - x^*\|, \end{aligned}$$

which shows (9), $x_{k+1} \in U(x^*, r_0)$, and $\lim\limits_{k\to\infty} x_k = x^*$.

That completes the proof of Theorem 1. □

Remark 1. Condition (4) can be replaced by

$$\|F'(x)^{-1}F'(x^*)\| \le [1 - w_0(\|x - x^*\|)]^{-1}, \tag{13}$$

for all $x \in \overline{U}(x^*, R)$ [2], [3]. Similarly we can show a local result using function $\overline{w}$ instead of w defined by

$$\overline{w}(r) = \sup\{w(v_1) + w_0(v_2) \colon v_1 + v_2 = r\}. \tag{14}$$

Theorem 2. *Under the hypotheses of Theorem 1, denote by r_1 the positive zero of function g, given by*

$$g(r) = \int_0^1 [w((1+t)r) - w_0(r)]dt + w_0(r) - 1, \tag{15}$$

and further assume:

$$\overline{U}(x^*, r_1) \subseteq \overline{U}(x^*, R). \tag{16}$$

Then Newton's method $\{x_n\}$ $(n \ge 0)$ generated by (2) is well defined, remains in $\overline{U}(x^*, r_1)$ for all $n \ge 0$, and converges to x^* provided that $x_0 \in U(x^*, r_1)$. Moreover the following hold:

(17)
$$\begin{aligned} &\|x_{n+1} - x^*\| \le \\ &\frac{\int_0^1 [\overline{w}(\|x_n - x^*\| + t\|x_n - x^*\|) - w_0(\|x_n - x^*\|)]dt}{1 - w_0(\|x_n - x^*\|)}\|x_n - x^*\| \ (n \ge 0) \end{aligned}$$

and

$$r_0 \le r_1. \tag{18}$$

Proof. The proof is exactly the same with the corresponding one in Theorem 1 but to arrive at (12) we use

$$
\begin{aligned}
(19)\qquad & \|x_{k+1} - x^*\| \\
& \le \frac{\int_0^1 w(\|x_k - x^*\| + t\|x_k - x^*\| - \|x_k - x^*\|)dt}{1 - w_0(\|x_k - x^*\|)} \|x_k - x^*\| \\
& \le \frac{\int_0^1 [\overline{w}(\|x_k - x^*\| + t\|x_k - x^*\|) - w_0(\|x_k - x^*\|)]dt}{1 - w_0(\|x_k - x^*\|)} \|x_k - x^*\| \\
(20)\qquad & < \|x_k - x^*\|.
\end{aligned}
$$

Finally estimate (18) follows from the observation that

$$
0 = f(r_0) \le g(r_0). \tag{21}
$$

That completes the proof of Theorem 2. □

3. Application

In order to compare Theorems 1 and 2 with earlier results we state the following theorem. The proof of the following local result can be found in [8].

Theorem 3. *Let $F\colon D \subseteq X \to Y$ be a Fréchet-differentiable operator. Assume:*
there exist a simple zero x^* of equation $F(x) = 0$, a parameter $p \ge 0$ such that

$$
\|F'(x^*)^{-1}[F'(x) - F'(y)]\| \le p\|x - y\| \quad \textit{for all } x, y \in D \tag{22}
$$

and

$$
\overline{U}(x^*, q) \subseteq D, \tag{23}
$$

where,

$$
q = \frac{2}{3p}\,. \tag{24}
$$

Then, Newton's method $\{x_n\}$ $(n \ge 0)$ generated by (2) is well defined, remains in $\overline{U}(x^*, q)$ for all $n \ge 0$ and converges to x^*, provided that $x_0 \in \overline{U}(x_0, q)$. Moreover the following error bounds holds for all $n \ge 0$

$$
\|x_{n+1} - x^*\| \le \frac{p\|x^* - x_n\|^2}{2[1 - p\|x^* - x_n\|]} = \alpha_n. \tag{25}
$$

Remark 2. As noted in [1], [3], [5], [6], [7] the local results obtained here can be used for projection methods such as Arnoldi's, the generalized minimum residual method (GMRES), the generalized conjugate residual method (GCR), for combined Newton/finite-difference projection methods and in connection with the mesh independence principle in order to develop the cheapest mesh refinement strategies.

Remark 3. The local results obtained here can also be used to solve equations of the form $F(x) = 0$, where F' satisfies the autonomous differential equation [3], [5], [7]:

$$F'(x) = T(F(x)), \tag{26}$$

where, $T: Y \to X$ is a known continuous operator. Since $F'(x^*) = T(F(x^*)) = T(0)$, we can apply the results obtained here without actually knowing the solution x^* of equation (1).

We complete this section with a numerical example.

Example 1. Let $X = Y = \mathbf{R}$, $D = U(0,1)$, $R = 1$, and define function F on D by

$$F(x) = e^x - 1. \tag{27}$$

Set $w(r) = pr$, $w_0(r) = p_1 r$, $r \in [0,1]$. Then it can easily be seen that we can set $T(x) = x + 1$ in (26). Using (3), (4), (22), we get $p = e$. Moreover since $x^* = 0$ we obtain in turn

$$\begin{aligned} F'(x) - F'(x^*) &= e^x - 1 = x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \\ &= \left(1 + \frac{x}{2!} + \cdots + \frac{x^{n-1}}{n!} + \cdots\right)(x - x^*) \end{aligned}$$

and for $x \in U(0,1)$, $\|F'(x) - F'(x^*)\| \le (e-1)\|x - x^*\|$. That is, $p_1 = e - 1$. Using (8) and (24) we obtain respectively:

$$q = .245252961 \tag{28}$$

and

$$r_0 = .254028662. \tag{29}$$

That is, our convergence radius r_0 is larger than the corresponding one q due to Rheinboldt [8]. This observation is very important in computational mathematics (see also Remark 2). Moreover the corresponding error bounds (9) (see also (25)) are given by

$$\|x_n - x^*\| \le \beta_n = \frac{p\|x_n - x^*\|^2}{2[1 - p_1\|x_n - x^*\|]} \quad (n \ge 0), \tag{30}$$

and

$$\beta_n < \alpha_n \quad (n \ge 0), \tag{31}$$

since $p_1 < p$. Hence our claims made at the introduction are justified.

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SEMILOCAL CONVERGENCE FOR NEWTON'S METHOD ON A BANACH SPACE WITH A CONVERGENCE STRUCTURE AND TWICE FRÉCHET DIFFERENTIABLE OPERATORS

Ioannis K. Argyros

Abstract. We provide a semilocal convergence theorem for Newton's method on a Banach space with a convergence structure. The operators involved are twice Fréchet differentiable. Our result contains earlier ones as special cases, if the Banach space is equipped with a real norm. Moreover it compares favorably with earlier results involving only once Fréchet-differentiable operators, since our sufficient convergence conditions are shown that they can be weaker.

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Key Words and Phrases. Newton's method, Banach space with a convergence structure, majorizing sequence, Fréchet-differentiable operator, Newton–Kantorovich hypothesis.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of the equation

$$F(x) = 0, \tag{1}$$

where,

$$F(x) = AG(x_0 + x). \tag{2}$$

G is a twice Fréchet differentiable operator on a Banach space X with a convergence structure and values in X, A is meant to be an approximation to $G'(x_0)^{-1}$ and x_0 is the initial guess for Newton's method

$$x_0 = 0, \;\; x_{n+1} = x_n + F'(x_n)^*(-F(x_n)) \quad (n \geq 0). \tag{3}$$

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By a Banach space X with convergence structure we mean a triple (X, V, E) in the sense of Definition 1 in [3] (see also [1], [2], [6]). Operator $F'(x)^*$ has also been defined in [3].

We provide a semilocal convergence analysis for Newton's method (2) in this general setting. In the case of a Banach space with (real) norm $\|\cdot\|$ our results reduce to the ones in [2], [4], [7].

Moreover our results compare favorably with earlier ones involving only once Fréchet-differentiable operators [2], [6], since our sufficient convergence conditions are shown that they can be weaker.

Finally some numerical examples are provided to show that our results can show convergence of Newton's method to x^*, when earlier ones fail.

2. Semilocal Convergence Analysis for Newton's Method

We provide the main semilocal convergence result for Newton's method (2) on a Banach space with a convergence structure and twice Fréchet-differentiable operators.

Theorem 1. *Let there be given a Banach space X with convergence structure (X, V, E) where $V = (V, C, \|\cdot\|_V)$, and operator $F \in C''(X_0 \to X)$ with $X_0 \subseteq X$, and operator $M \in C'(V_0 \to V)$ with $V_0 \subseteq V$, and a point $p \in C$ satisfying:*

$$U(p) \subseteq X_0, \quad [0,p] \subseteq V_0; \tag{4}$$

M is order-convex on $[0,p]$ and for all $x, y \in U(p)$ with $|x| + |y| \leq p$

$$M'(|x+y|) - M'(x) \in B([F''(x) - F''(x+y)](y)); \tag{5}$$

$$M'(0) \in B(I - F'(0)), \quad (-F(0), M(0)) \in E; \tag{6}$$

$$M(p) \leq p; \tag{7}$$

and

$$M'(p)^n p \to 0 \quad \text{as } n \to \infty. \tag{8}$$

Then, sequence $(x_n, t_n) \in (X \times V)^\pi$, where $\{x_n\}$ is generated by Newton's method (3) and $\{t_n\}$ $(n \geq 0)$ is given by

$$t_0 = 0, \quad t_{n+1} = M(t_n) + M'(|x_n|)(a_n), \quad a_n = |x_{n+1} - x_n| \tag{9}$$

is well defined for all $n \geq 0$, belongs in E^π, and is monotone.

Moreover the following hold for all $n \geq 0$

$$t_n \leq b, \tag{10}$$

where,

$$b = M^\infty(0), \tag{11}$$

is the smallest fixed point of M in $[0,p]$.

Proof. We first show that $(x_1, b) \in E$. By (6) and (7) we get (for $p = b$)

$$\begin{aligned} M'(0)(b) + |-F(0)| &\leq M'(0)(b-0) + M(0) \\ &\leq M(b) - M(0) + M(0) = M(b) \leq b, \end{aligned} \tag{12}$$

which shows that equation

$$q = (I - F'(0))(q) - F(0) \tag{13}$$

has a solution $q = x_1$, and $(x_1, b) \in E$.

Hence by (9) and the order convexity of M we get

$$|x_1| \leq M'(0)|x_1| + M(0) = t_1 \tag{14}$$

and

$$t_1 = M'(0)|x_1| + M(0) \leq M'(0)b + M(0) \leq M(b) \leq b. \tag{15}$$

Let us assume sequence (x_n, t_n) is well defined and monotone so that

$$0 \leq (x_{k-1}, t_{k-1}) \leq (x_k, t_k) \tag{16}$$

and

$$t_k \leq b \tag{17}$$

for all $k = 1, 2, \ldots, n$.

Using (5), (6) and the approximation

$$\begin{aligned} I - F'(x_n) &= I - F'(x_n) - F'(0) + F'(0) \\ &= [I - F'(0)] + [F'(0) - F'(x_n)] \\ &= [I - F'(0)] - \int_0^1 F''(\theta x_n) d\theta (x_n), \end{aligned} \tag{18}$$

we get

$$\begin{aligned} |I - F'(x_n)| &\leq |I - F'(0)| + \int_0^1 |-F''(\theta x_n) d\theta (x_n)| \\ &\leq M(0) + M'(|x_n|) - M'(0) = M'(|x_n|). \end{aligned} \tag{19}$$

That is

$$M'(|x_n|) \in B(I - F'(x_n)). \tag{20}$$

Moreover using (3), (5), (9) and the approximation

$$\begin{aligned} |-F(x_n)| &= |-F(x_n) + F(x_{n-1}) + F'(x_{n-1})(x_n - x_{n-1})| \\ &= \left| -\int_0^1 F''[x_n + \theta(x_{n+1} - x_n)](1-\theta) d\theta (x_{n+1} - x_n)^2 \right| \\ &\leq \int_0^1 [M'(|x_{n-1}| + \theta a_{n-1}) - M'(|x_{n-1}|)] a_{n-1} d\theta \\ &= M(|x_{n-1}| + a_{n-1}) - M(|x_{n-1}|) - M'(|x_{n-1}|) a_{n-1} \\ &\leq M(t_{n-1} + t_n - t_{n-1}) - M(t_{n-1}) - M'(|x_{n-1}|) a_{n-1} \end{aligned} \tag{21}$$

$$= M(t_n) - t_n. \tag{22}$$

Let

$$q = b - t_n. \tag{23}$$

Using (20), (22) and (23) we obtain in turn

$$\begin{aligned} M'(|x_n|)(q) + |-F(x_n)| &\leq M'(t_n)(b - t_n) + M(t_n) - t_n \\ &\leq M(b) - M(t_n) + M(t_n) - t_n \leq b - t_n = q. \end{aligned} \tag{24}$$

Hence equation

$$q = (I - F'(x_n))(q) + (-F(x_n)) \tag{25}$$

has a solution x_{n+1} and $a_n \leq b - t_n$. We also get

$$t_{n+1} \leq M(t_n) + M'(t_n)(b - t_n) \leq M(b) \leq b, \tag{26}$$

and

$$\begin{aligned} a_n + t_n &\leq M'(|x_n|)(a_n) + |-F(x_n)| + t_n \\ &\leq M'(|x_n|)(a_n) + M(t_n) = t_{n+1}. \end{aligned} \tag{27}$$

By (9) we obtain using induction on n

$$M^n(0) \leq t_n \leq b. \tag{28}$$

That is $t_n \to b$ since $M^n(0) \to b$. Hence sequence $\{x_n\}$ converges to $x^* \in U(b)$ which is a solution of F. To show uniqueness, define the modified Newton iteration

$$y_{n+1} = y_n - F(y_n), \quad y_0 = 0. \tag{29}$$

This sequence also converges to x^*, and $(y_n, M^n(0))$ is monotone in $X \times V$. Let $y^* \in U(p)$ be a solution of F. Then it can easily be seen

$$|y^* - y_n| \leq M^n(p) - M^n(0) \to 0 \quad \text{as } n \to \infty. \tag{30}$$

Hence we get $x^* = y^*$.

That completes the proof of Theorem 1. □

Remark. In order for us to obtain a posteriori estimates we define as in [3], [6]

$$R_n(q) = (I - M'(|x_n|)^* S_n(q) + a_n, \tag{31}$$

$$S_n(q) = M(|x_n| + q) - M(|x_n|) - M'(|x_n|)(q) \tag{32}$$

on interval

$$I_n = [0, p - |x_n|]. \tag{33}$$

As in [3], [6] we can show the following Propositions but for twice Fréchet-differentiable operators F.

Proposition 1. *If $q \in I_n$ satisfies*

$$R_n(q) \leq q, \tag{34}$$

then the following error bounds hold for all $n \geq 0$:

$$a_n \leq R_n(q) = d \leq q \tag{35}$$

and

$$R_{n+1}(d - a_n) \leq d - a_n. \tag{36}$$

Proposition 2. *Assume hypotheses of Theorem 1 hold and there exists $q_n \in I_n$ satisfying (34). Then the following hold:*

$$|x^* - x_m| \leq \overline{t}_m, \tag{37}$$

where sequence $\{\overline{t}_n\}$ $(n \geq 0)$ is defined by

$$\overline{t}_n = q_n, \quad \overline{t}_{m+1} = R_m(\overline{t}_m) - a_m \quad (m \geq n). \tag{38}$$

Proposition 3. *Let $q \in I_n$ be any solution of (34), and assume hypotheses of Theorem 1 hold. Then the following error bounds hold for all $n \geq 0$:*

$$|x^* - x_n| \leq R_n^{\infty}(0) \leq q. \tag{39}$$

Remark 2. In order for us to consider monotone convergence for method (3) as in [3], [6] we assume X is as partially ordered. Set

$$X = V, \quad D = C^2 \quad \text{and} \quad |\cdot| = I. \tag{40}$$

Then we obtain the result:

Proposition 4. *Let Y be a Banach space, $G \in C''(V_0 \to Y)$, $A \in L(Y \to X)$, and $x, y \in V$ be such that*

$$[x, y] \in V_0; \tag{41}$$

$$I - AG'(x) \in L_+(V); \tag{42}$$

for all

$$z_1, z_2 \in [x, y], \quad z_1 \leq z_2 \Rightarrow AG'(z_1) \geq AG'(z_2); \tag{43}$$

$$AG(x) \leq 0, \quad AG(y) \geq 0 \tag{44}$$

and

$$(I - AG'(y))^n (y - x) \to 0 \quad \text{as } n \to \infty. \tag{45}$$

Then Newton's method $\{z_n\}$ $(n \geq 0)$ given by

$$z_0 = x, \quad z_{n+1} = z_n + [AG'(z_n)]^* [-AG(z_n)] \tag{46}$$

is well defined for all $n \geq 0$, and monotone. Moreover it converges to the unique solution x^* of AG in $[x, y]$.

Proof. Define

$$F(z) = AG(x + z), \quad M(q) = q - F(q), \quad p = y - x. \tag{47}$$

Then the results follow as special cases of Theorem 1.

That completes the proof of Proposition 4. □

Remark 3. It turns out that the error bounds obtained here can be improved under the same hypotheses of Theorem 1.

Indeed let us assume that there exists $M_0 \in C'(V_0 \to V)$ which is order convex on $[0, p]$ and for all $x \in U(p)$ with $|x| \leq p$

$$M_0'(|x|) - M_0'(0) \in B([F''(0) - F''(x)](x)), \tag{48}$$

$$M_0'(0) \in B(I - F'(0)), \quad (-F(0), M_0(0)) \in E, \tag{49}$$

and define sequence $\{r_n\}$ by

$$r_0 = 0, \quad r_{n+1} = M(r_n) + M_0'(|x_n|)(a_n). \tag{50}$$

Furthermore assume:

$$M_0(p_0) \leq M(p_0) \tag{51}$$

and

$$M_0'(p_0) \leq M'(p_0) \tag{52}$$

for all $p_0 \in [0, p]$.

It can then easily be seen from the proof of Theorem 1 that sequence $(x_n, r_n) \in (X \times V)^{\pi}$ is well defined for all $n \geq 0$, monotone and satisfies

$$r_n \leq t_n, \tag{53}$$

$$r^* = \lim_{n\to\infty} r_n \leq b. \tag{54}$$

Note also that M' must be replaced by M_0' in the definition of operator R_n and S_n. Conditions (51) and (52) hold in general. Moreover if strict inequality holds in (51) or (52) then so does in (53) say. That is under the same hypotheses as in Theorem 1 and Propositions 1–3 we can improve on the error bounds. Moreover one hopes that sequence $\{r_n\}$ must converge under conditions weaker than (7), (8) corresponding to $\{t_n\}$. If we can show the convergence of $\{r_n\}$ without using (7) and (8), then one hopes that the new sufficient convergent conditions must be indeed weaker than (7) and (8) because of (51) and (52).

In the next section we justify these claims.

3. Applications

As an application let us discuss the case of a Banach space with a (real) norm $\|\cdot\|$. To check the conditions of Theorem 1 and also compare it with earlier ones assume there exist parameters ℓ, ℓ_0, ℓ_1 and ℓ_2 such that for $F'(0) = I$

$$\|F''(x) - F''(y)\| \leq \ell\|x - y\|, \tag{55}$$

$$\|F''(x) - F''(x_0)\| \leq \ell_0\|x - x_0\|, \tag{56}$$

$$\|F''(x_0)\| \leq \ell_1, \tag{57}$$

$$\|F(x_0)\| \leq \ell_2, \tag{58}$$

and

$$\|F(x) - F'(y)\| \leq \ell_3 \|x - y\|. \tag{59}$$

Define sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $(n \geq 0)$ by

$$\alpha_0 = 0, \quad \alpha_1 = \ell_2, \tag{60}$$

$$\alpha_{n+2} = \alpha_{n+1} + \frac{\frac{\ell}{2}[\alpha_n^2 + \frac{1}{3}(\alpha_{n+1} - \alpha_n)] + \ell_1}{1 - \ell_1 \alpha_{n+1} - \frac{1}{2}\ell t_{n+1}^2}(\alpha_{n+1} - \alpha_n)^2$$

$$\beta_0 = 0, \quad \beta_1 = \ell_2, \tag{61}$$

$$\beta_{n+2} = \beta_{n+1} + \frac{\frac{\ell}{2}[\beta_n^2 + \frac{1}{3}(\beta_{n+1} - \beta_n)] + \ell_1}{1 - \ell_1 \beta_{n+1} - \frac{1}{2}\ell_0 \beta_{n+1}^2}(\beta_{n+1} - \beta_n)^2$$

and

$$\gamma_0 = 0, \quad \gamma_1 = \ell_2, \quad \gamma_{n+2} = \gamma_{n+1} + \frac{\ell_3(\gamma_{n+1} - \gamma_n)^2}{2(1 - \ell_3 \gamma_{n+1})}. \tag{62}$$

In general we have:

$$\ell_0 \leq \ell. \tag{63}$$

Note that if equality holds in (62) then $\alpha_n = \beta_n$ for all $n \geq 0$.

It was shown in [2], [4], [7] that if

$$3\ell_2\ell^2 + 3\ell_1\ell + \ell_1^3 \leq [\ell_1^2 + 2\ell]^{3/2} \tag{64}$$

then $\{\alpha_n\}$ $(n \geq 0)$ is monotonically convergent to some α^* and majorizes sequence $\{x_n\}$ $(n \geq 0)$. Moreover sequence $\{x_n\}$ $(n \geq 0)$ has a solution x^*. If strict inequality holds in (63) then under (64) sequence $\{\beta_n\}$ $(n \geq 0)$ is also monotonically convergent to some β^* and majorizes sequence $\{x_n\}$, with

$$\beta_n < \alpha_n, \tag{65}$$

$$\beta_{n+1} - \beta_n < \alpha_{n+1} - \alpha_n \tag{66}$$

and

$$\beta^* \leq \alpha^* \tag{67}$$

for all $n > 1$.

That is the error bounds are finer and the information on the location of the solution at least as precise. It can also easily be seen that in the case when strict inequality holds in (63) then sequence $\{\beta_n\}$ must hold, be monotonically convergent under sufficient conditions at least as weak as (64).

Furthermore sequences $\{\alpha_n\}$, $\{\beta_n\}$ correspond to $\{t_n\}$ and $\{r_n\}$ respectively in the (real) norm case $\|\cdot\|$. All the above justify the claims made in Remark 3.

In order for us to compare our Theorem 1 with earlier ones [2], [6] we note that $\{\gamma_n\}$ is monotonically convergent to some γ^* and majorizes sequence $\{x_n\}$ if the famous Newton–Kantorovich hypothesis

$$2\ell_3\ell_2 \leq 1 \tag{68}$$

holds [2], [5]. However condition (64) can be weaker than (68), which implies that our Theorem 1 can guarantee convergence of sequence x_n to x^* but the corresponding Theorem 5 in [6] cannot (note that γ_n corresponds to d_n there).

We complete this study by providing such a case considered also in [2], [4], [7]:

Example 1. Let $X = [-1, 1]$, $x_0 = 0$ and define G on X by

$$G(x) = \frac{1}{6}x^3 + \frac{1}{6}x^2 - \frac{5}{6}x + \frac{1}{3}. \tag{69}$$

Choose $A = G'(x_0)^{-1}$. Then using (55)–(59) and (60) we get

$$\ell = \ell_0 = \frac{6}{5}, \quad \ell_1 = \frac{2}{5}, \quad \ell_2 = \frac{2}{5} \quad \text{and} \quad \ell_3 = \frac{8}{5}. \tag{70}$$

Then we get

$$2\ell_2\ell_3 = \frac{32}{25} > 1. \tag{71}$$

Therefore, the Newton–Kantorovich hypothesis fails. Hence Theorem 5 in [6] cannot guarantee the convergence of Newton's method (3) starting at $x_0 = 0$. However condition (64) becomes

$$\frac{404}{125} < \frac{512}{125}\,.$$

That is (64) holds in this case, which guarantees the convergence of sequence x_n to x^*.

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ALGEBRAS OF FUNCTIONS ON GROUPOID OF SOME SPECIAL FOLIATIONS

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Abstract. Here we study foliated manifold M, F. We assume that foliation F is given by locally free action of the commutative Lie group H. We also assume that M possess Ehresmann connection (i.e. a full connection transversal to the leaves of the foliation F).

A.M.S. (MOS) Subject Classification Codes. 55,57,86,20,22.

Key Words and Phrases. Foliated manifold, holonomy groupoid, Lie Groupoid, Seifert manifold, Ehresman connection

Introduction.

We obtain here certain invariants of functional algebras describing general properties of leaves and their mutual location on the manifold.

As soon as action of the commutative group H on M induces action of the universal covering $\widetilde{H}$ of H, say $\tilde{g}p = \rho(\tilde{g})p$, here $\tilde{g} \in \widetilde{H}$, $p \in M$, $\rho : \widetilde{H} \to H$ being the universal cover, we can let H be simply-connected, even R^n.

Let us point out some crucial definitions and facts.

We can associate groupoid of homotopy classes of paths to any leaf $L \subset M$ with fixed range and source. Let us denote its total space by $G(L)$. Projections on the beginning and the end are denoted respectfully s and $r : G(L) \to L$. There are certain spaces $G^x = \{\gamma \in G | r(\gamma) = x\}$, $G_y = \{\gamma \in G | s(\gamma) = y\}$, $G^x_y = \{\gamma \in G | r(\gamma) = x, s(\gamma) = y\}$.

Now we can associate holonomy groupoid to the foliated manifold M as in the Section 2 of [7].

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1. [1], [7] Groupoid G with base X is called Lie groupoid in case the following conditions are met:

1) G is a manifold,
2)s and $r : G \to X$ are surjective submersions,
3)multiplication of paths and embedding $\iota : X \to G$ are smooth functions.

Groupoids considered in this article are Lie groupoids due to the results from [8]. It is proved there that groupoid of the foliation generated by group action is in fact fibre bundle.

2. [1], [7] Left Haar system on locally compact groupoid G is the family of measures $(\lambda^x)_{x\in L}$ on G, such that

1) Support of λ^x is precisely G^x,
2) System of measures is invariant with respect to the action $L_\gamma : G^{s(\gamma)} \to G^{r(\gamma)}$ for any $\gamma \in G_x = \{\gamma \in G(L)|r(\gamma) = x\}$, peculiarly G^x_x
3) For all $\phi \in C_c(G(L))$ ($C_c(G(L))$ being variety of complex-valued functions with compact support) map

$$x \mapsto \lambda(\phi)(x) = \int \phi d\lambda^x$$

defines continuous function on L.

The equivalence relation on groupoid G^x given by: $\gamma_1 \equiv \gamma_2 \Leftrightarrow \exists\gamma \in G^x_x \gamma_1 = \gamma_2\gamma$ is measurable [5] since groupoids under consideration are Lie groupoids, moreover its factor space $G^x/\equiv$ is precisely space of units of $G(L)$ being L itself. Hence we can define measure $\lambda^{x'}$ (called tangential measure) on the set of equivalence classes as factor-measure of the λ^x .

Thus we can associate algebra $F(L)$ of measurable functions for any leaf L of the foliation F.

We also assume the existence of so called invariant transverse measure ν [7] defined on the set of leaves of the foliated manifold M. One then can form measure μ on M with the help of the transversal and tangential measures given above.

Let us first take into consideration the following set of functions on groupoid G:

$$C(G) = \{f : G \to C| \int\limits_M \int\limits_{G^x} f(\gamma)\overline{f}(\gamma)d\lambda^{x'} d\mu < \infty\}$$

3. Let us consider variety of continuous, infinitely differentiable functions with compact on the total space of the groupoid G $C^\infty_c(G)$.

Let us define multiplication and involution on this set as follows:

$$f * g = \int\limits_{G^{s(\gamma)}} f(\gamma_1\gamma)g(\gamma_1^{-1})d\lambda^{s(\gamma)}(\gamma_1);$$

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

Let us define $\|f\|_x = \|f\|_{L_2(G^{s(\gamma)}}$ and $\|f\| = \sup_{x\in M} \|f\|_x$

Closure of $C^\infty_c(G)$ in this norm is C^*-algebra which we will denote $C^*_r(G)$.

One can show that C^* algebra $C^*_r(G)$ can be represented as subalgebra of the algebra of bounded operators acting on H $B(H)$

Common algebra.

Let us take into consideration a manifold M with foliation F, generated by action of the commutative Lie group H. We suppose that there is also an integrable Ehresmann connection on (M, F).

Let L_1, L_2 be two (maybe the same) leaves of the foliation F and γ be a horizontal curve such that $x = \gamma(0) \in L_1$, $y = \gamma(1) \in L_2$.Then there is an isomorphism $A : G^x \leftrightarrow G^y$, which is given by Ehresmann connection. Namely, let δ be a representative of $\alpha \in G^x$. then by definition of the connection γ and δ give rise to rectangle $\Pi : [0,1] \times [0,1] \to M$, $\Pi(t,0) = \gamma(t)$,$\Pi(0,s) = \delta(t)$. Let us define $A([\delta]) = [\Pi(1,s) : [0,1] \to L_2]$. This definition is correct due to the properties of the Ehresmann connection.

Now we can define multivalent correspondence between two groupoids in the following way. Group action on any fixed leaf L_1 can be extended to the action of H on $G(L_1)$, $h([\beta]) = [h\beta]$. Let us denote by H_1 isotropy subgroup of the element $\alpha \in G^x$ (from the commutativity of H we infer that H_1 is isotropy subgroup for any element of $G(L_1)$). Thus we put in correspondence for any $\alpha \in G^x$ a subset $T(\alpha) = \{hA(\alpha)|h \in H_1\}$. Similarly for any $\beta \in G^y$ we take $T'(\beta) = \{hA^{-1}(\beta)|h \in H_2\}$, here H_2 is an isotropy subgroup for $\beta \in G^x$.

Let us introduce certain equivalence relation $\sim$ on the groupoid $G(L_1)$, generated by the relation $\equiv$, here $\alpha_1 \equiv \alpha_2$ in case $\alpha_1, \alpha_2 \in T'(\beta)$ for some β. The same relation maybe introduced on the $G(L_2)$.

We consider map $B : G(L_1)/\!\sim\, \to G(L_2)/\!\sim$. For $\alpha \in G(L_1)$, $r(\alpha) = x_1$ we take $B[\alpha] = [A(h^{-1}\alpha)]$, with $hx = x_1$.

Statement1. The map $B : G(L_1)/\!\sim\, \to G(L_2)/\!\sim$ is an isomorphism.

• It is enough to show injectivity of B. Let $[\alpha] \neq [\alpha']$, but $\beta_0 = A([\alpha]) = B([\alpha'])$. Then there $\exists \beta_i$, $i = \overline{1,k}$, $\beta_0 = B([\alpha])$, $\beta_k = B([\alpha'])$, $\beta_0 \equiv \ldots \equiv \beta_k$, $\beta_i, \beta_{i-1} \in T(\alpha_i)$. But in this case $\alpha \equiv \alpha_1 \equiv \ldots \equiv \alpha_k \sim \alpha'$ hence $\alpha \sim \alpha'$, what contradicts to the assumption. □

There is a subalgebra of A_1, let us denote it A_{12} consisting of functions constant on classes of the introduced equivalence relation, the same subalgebra may be considered in $A_{21} \subset A_2$. Clearly $A_{12} \cong A_{21}$.

Let L_1 be a fixed leaf. For any leaf L we can find subalgebra $A(L) \subset A_1$ in the given above way. Let $A_E = \cap A(L)$ (we take intersection over all leaves L).

A. Let foliation F be foliated bundle and horizontal curve γ be contractible. In this case algebra $A_E = A(F)$, F being standard leaf.

B. Foliation of the parallels on torus $T^2 = R^2/Z \times Z$ with the Ehresmann connection given by the vector fields tangent to the constant vector field $V_{(x,y)} = (a,b)$ $a/b \in R \setminus Q$. Factor of any leaf by equivalence relation from 1) won't be Hausdorff variety hence equivalence relation is not measurable.

C. For Seifert manifold M with standard leaf T^k [9] and Ehresmann connection compatible with group action. Let F be standard foliation on M. Then the following holds true:

1) Let foliation F have codimension 1 and singular leaves have Z_2 as isotropy subgroup. Then M is not oriented and $A_E \cong A_t/Z_2$, here A_t is algebra of measurable functions on the standart leaf T^k.

2) Let group T^k act locally free on M. Then in case $A_E = A_t/Z_3$ we have codimension of foliation greater than 1.

It is important to note that in case F being of codimension greater than 1 it is impossible to state that $A_E \cong A/Z_p$ for some p in general case [6].

3) $R \times C$. We can identify $(t, z) \equiv (t + m, e^{\frac{2\pi i m}{q}} z)$. Horizontal distribution is given by the condition $dt = 0$ with t coordinate on $[0, 1]$. Group $S^1 = R/Z$ acts on this manifold in natural way. $\alpha(t, z) = (t + \alpha, ze^{\frac{2\pi i \alpha}{q}})$.

We get maps $B_1 : S^1 \times R \to S^1 \times R$, $\phi(\alpha, x) = (q\alpha, qx)$ from groupoid of the typical leaf on the groupoid of the leaf which is q times covered by typical ones and $B_2 : S^1 \times R \to S^1 \times R$ vice versa. We have for $\phi \in A(F)$, with F being typical leaf, $\phi(\alpha, x) = (\frac{\alpha}{q} + \frac{2i\pi}{q}, \frac{x}{q} + \frac{2i\pi}{q})$, $i = \overline{0, q-1}$ from the groupoid of typical leaf onto groupoid of the singular one and back.

Thus in this case common subalgebra $A = A(S^1)$ coincides with algebra of the singular leaf and is subalgebra of $\frac{2\pi}{q}$-periodic functions on typical leaves.

D. Foliation on $R \times S^1$ given by integral curves of the vector field $(x, 1)$. Let us denote algebra on the leaves laying on the "left" side of the circle S^1 (let us denote them S_e) and on the "right" side by (S_i). We introduce Ehresmann connection E on $R^2 \setminus \{0\}$, in polar coordinates (ρ, φ) by equation $d\varphi = 0$. On $R^2 \setminus \{0\}$ acts S^1, $\phi(\rho, \alpha) = (\rho(\alpha + \phi), \alpha + \phi)$. Let us note that algebras of measured functions for two spirals inside S_i and outside S_e of the circle are isomorphic. Here $A^E(F) = A(S^1)$ (cm. Th. 1 [2]).

*Note*1. 1) We can extend equivalence relation to the abstract groupoids. At first we construct map from G^x to G^y then we slide along paths corresponding one to another at the first stage.

2) *(Partial case of given before equivalence relation in case Ehresmann connection is compatible with group action.) Let us build translation of one point* $x \to y$, *then we map path* $\gamma : [0, 1] \to G\pi_{L_1} \longrightarrow L_1$ *to* $\gamma' : [0, 1] \to G\pi_{L_2} \longrightarrow L_2$, *here first arrows coincide.*

Statement2. Let (M, F) be foliated manifold with Ehresmann connection, and L_1, L_2 be some leaves of F. In case $A_{12} = A_1$, $A_{21} = A_2$ we have L_1 diffeomorphic to L_2, moreover this diffeomorphism is compatible with group action. Inverse statement is clearly true as well the direct one.

• Let us construct map $f : L_1 \to L_2$. Let $x_0 \in L_1$, $y_0 \in L_2$ be points and γ be horizontal path, connecting x_0 and y_0, by which we construct algebra A_{12}.

For arbitrary $x = hx_0$ we define $f(x) = hy_0$. Let us show that f is diffeomorphism. By assumption $A_{12} = A_1$, $A_{21} = A_2$ and some properties of measurable functions we have in case $\alpha \sim \beta$ $\alpha = \beta$. Thus multivalent map T is here univalent moreover T' is also univalent and is inverse to T. Hence $T : G(L_1) \to G(L_2)$ is bijection of groupoids (T is isomorphism of groupoids only if Ehresmann connection is invariant with respect to the group action).

Then if $\gamma : [0, 1] \to L$, $\gamma(0) = x_0$, $\gamma(1) = hx_0$, there is $T(hx_0) = s(T(\gamma)) = hT(x_0) = hy_0$.

Differentiability of the map, induced by connection is due to the properties of the given connection. ▷

Conditions of this statement are met for example in partial case of foliations with Lie groupoids of leaves with system of smooth Haar measures.

Now let $A_1 \neq A_{12}$ or $A_2 \neq A_{21}$. For measures on each of the leaves L_1, L_2 induced by measures on groupoids $G(L_1)$ and $G(L_2)$ the following is true. Let us consider variety of equivalence classes on the groupoid $G(L_i)$. We can construct an equivalence relation on the leaf L_i itself in the way parallel to the way given on groupoids. Now let equivalence class of any point $[x]$, $x \in L_i$, $i = 1, 2$ is at most countable. We can define Borel maps $T : L_1 \to L_2/\sim$, $T' : L_2 \to L_1/\sim$. It is possible to take families of varieties K_1, K_2 consisting of points on leaves which belong to all different equivalence classes with measures and topologies, inherited from factors of leaves in case the latter have good structure and to consider maps of leaves to this varieties.

Corollary. There exists Borel measurable bijection (not a homeomorphism in general) of the elements from families $k_1 \in K_1$, $k_2 \in K_2$, $T : k_1 \leftrightarrow k_2$.

• By Theorem 4.5 from [7] there exists such a division of every leaf L_i, $i = 1, 2$ on Borel measurable subsets U_i^j that induced maps T and T' restricted to U_i^j are bijections. □

Statement3. Invariant subalgebra with respect to Ehresmann connection A_{12} does not depend on Ehresmann connection. This means that for any Ehresmann connection invariant subalgebras coincide up to isomorphism.

• Let there exist two points on the second leaf $y_1, y_2 \in L_2$. Then it is enough to take into consideration that $G^x \approx G^{y_1}$ $G^x \approx G^{y_2}$, hence $G^{y_1} \approx G^{y_2}$. The rest part of proof is true because of dependence of the last part of the construction only on the group action. □

Let us take into consideration partial case of manifold M with foliation induced by locally free action of the commutative group H and Ehresmann connection compatible with the group action.

Let $L_1, L_2 \in M$ be two leaves of the foliation. We define *covering subgroup of* L_2 *by* L_1 $H_{12} \equiv H_1/H_1 \bigcap H_2$, with H_i being isotropy subgroup of the leaf L_i, $i = 1, 2$. The same construction gives us H_{21}.

Statement4. Covering subgroup H_{12} is factor-group of the holonomy group J for the Ehresmann connection.

• Let us take the horizontal curve $\gamma : [0, 1] \to M$, $\gamma(0) = x \in L_1$, $\gamma(1) = y \in L_2$. It is enough to build surjective homomorphism $\rho : J \to H_{12}$.

Let $\alpha \in J$, $\exists a \in \pi_1(x, L_1)$, $[a] = \alpha$. There is a covering $\psi : H \to L_1$, $\psi(e) = x$. We will consider lift $\tilde{f} : [0, 1] \to H$, $\tilde{f}(0) = e$ of the path a. Let us define $g = \tilde{f}(1)$. Due to $a(0) = a(1)$ we have $g \in H_1$ and let $\phi \in H_{12}$ be its' equivalence class in $H_1/H_1 \bigcap H_2$. Now we define $\rho(\alpha) = \phi$.

If $h(t)$, $h(0) = e$ is some path in H, then thanks to invariance property of the Ehresmann connection we have a rectangle $h(t)\delta(s)$ for any horizontal curve $\delta(s)$. Thus parallel translation of the path γ along loop a is $\tilde{f}(t)\gamma$. So, if the loop a translate into itself every horizontal curve beginning in x and consequently γ then $\tilde{f}(1)\gamma(0) = \gamma(0)$ and $\tilde{f}(1)\gamma(1) = \gamma(1)$ or $\tilde{f}(1) \in H_1 \bigcap H_2$. That's because ρ is correctly defined

Surjectivity of the map is an immediate consequence of its construction. □

Corollary. For A_{12} isomorphic to A_1 but not isomorphic (by means of the connection) to A_2 leaf L_2 covers L_1 with covering group H_{21} and "standard fiber" from **Corollary 2.1**.

Structure of $C_c^\infty(G)$

Lemma1. Let M be a manifold with foliation given by action of the commutative group H. Let P be variety of points which can be connected with the given one point $x_0 \in M$ by the horizontal curve. [3]

Let P satisfy the following properties:

1) P is a manifold.

2) $\dim P = n - p$ with $n - p$ being codimension of the foliation.

3) There exists leaf S which has only one common point with P.

Then $S = H/span(\bigcup_{y\in M} H_y)$, where H_y is isotropy subgroup of the leaf L_y passing through y. By *span* we mean that generators of this group are nothing more but union of generators for H_y. Use of *span* is correct due to the fact that $H_y \subset R^n$ and group, generated by elements from the groups H_y can be considered as $span_Z H_y$

Let S' be connected subset of H such that universal cover $p : H \to S$, with $p(e) = x_0$ maps S' bijectively onto S being at the same time homeomorphism when restricted to the inner part of S'. Then there exists almost everywhere continuous bijection $\phi : M \to P \times S'$, $\phi(x) = (\pi(x), \pi'(x))$

• Let us show that for $h \in H_y$, $h \in H_{x_0}$. Due to $S \cap P = \{x_0\}$ and the property of h to map horizontal curve to horizontal one we can say that h maps horizontal curve connecting x_0 with y to horizontal curve connecting y with hx_0 but then hx_0 belongs to P. Thus $hx_0 = x_0$ and h is an element of the isotropy subgroup of the leaf S.

Without any loss in generality we can assume that $x_0 \in S$. Let us define projections $\pi_P : M \to P$, $\pi_S : M \to S$ as follows.

Let $x \in M$. Let us consider some curve connecting x_0 with x. It defines a rectangle lower side of which is a curve on P with the end point y and left side of which is a curve on S defining some point s on the universal covering space H.

Let S' be a connected subset of H such that universal cover $p : H \to S,\ p(e) = x_0$ maps S' bijectively onto S being at the same time homeomorphism restricted to the internal part of S'. Then there is in S' the only element s' with the property $s = s'h_0$ where $h_0 \in H_{x_0}$.

We define $\pi_P(x) = yh_0^{-1}$, $\pi_S(x) = s'$.

Let us take two curves $\gamma_1, \gamma_2 : [0,1] \to M$, $\gamma_1(0) = \gamma_2(0) = x_0$, $\gamma_1(1) = \gamma_2(1) = x$. They define two rectangles with lower sides $g_1^P(t)$ $g_2^P(t)$ and left sides $g_1^S(t)x_0$ and $g_2^S(t)x_0$ where $g_1^S, g_2^S : [0,1] \to H$, such that $g_1^P(0) = g_2^P(0) = x_0$, $g_1^S(0) = g_2^S(0) = e$, where e is unit element of H, $g_1^P(1) = y_1$, $g_2^P(1) = y_2$. Thanks to the uniqueness of the rectangle with given vertical and horizontal sides we have the following upper sides of them: $g_1' = g_1^P g_1^S(1)$ and $g_2' = g_2^P g_2^S(1)$ and we have $g_1'(1) = g_2'(1) = x$. So we get horizontal curve with the beginning in $g_1'(0)$ and end point in $g_2'(0)$ lying in S. After the action of $(g_1^S(1))^{-1}$, we get horizontal curve connecting x_0 with some point $x_1 \in S$. Due to 3) $x_1 = x_0$ henceforth $g_1'(0) = g_2'(0)$.

Thus $g_1^S(1)x_0 = g_1^S(1)g_1^P(0) = g_1'(0) = g_2'(0) = g_2^S(1)g_2^P(0) = g_2^S(1)x_0$ but then $g_1^S(1)^{-1}g_2^S(1)$ is an element of H_{x_0}. That is $y_1 = g_1^P(1) = g_1'(1)g_1^S(1)^{-1} = xg_1^S(1)^{-1} = g_2'(1)g_1^S(1)^{-1} = g_2^P(1)g_2^S(1)g_1^S(1)^{-1} = y_2 g_2^S(1)g_1^S(1)^{-1}$, or $y_1 g_1^S(1) = y_2 g_2^S(1)$.

So now we have a map $\phi : M \to P \times S'$

Continuity of this map almost everywhere is the consequence of bifoliated structure on M defined by P and F.

Really, let us show that inverse image of $U \times V$ $U = \circ U \subset S'$, $V = \circ V \subset P$ is open on the manifold M ($\circ V$ is an interior part of some set V).

It is enough to mention that for any cylinder $B_r(s) \times B_{r'}(p)$, $B_{r'}(p) \subset \circ S'$, $p \in P$, $s \in S'$, where $B_r(x)$ is an open ball of the radius r with the center in x there exists open ball $B_{min\{r,r'\}}(\phi^{-1}(p,s))$ in the inverse image of the cylinder.

There is no continuity only on the variety $\partial S' \subset \overline{S' \times P}$ which is the border of S'. $\square$

Corollary. Let M be a manifold with foliation given by action of the commutative group H. Let P be a variety of points which can be connected to the the given one $x_0 \in L_0 \subset M$ by horizontal curve. Let P satisfy the following conditions:

1) P is a manifold.

2)$\dim P = n - p$ $n - p$ being codimension of foliation.

Then $P \times L_0$ covers M with covering group L_0/S, $S = H/span(\bigcup_{y \in M} H_y)$, here H_y is isotropy group of the leaf L_y passing through the point y.

• It is clear that L_0 covers S and the action of L_0 on S meets $S(L_0/S) = L_0$. There exists variety of points $|L_0/S|$ from $P \times L_0$ defining the same point on M as defined in the beginning of the proof of the Lemma 1 as pairs $(hs, h^{-1}p)$, $s \in S$, $p \in P$, $h \in L_0/S$ represent different points from $P \times L_0$ but define the same curve on M. $\square$

E. Seifert manifold with the group S^1. Any manifold of this type bijectively corresponds to $C \times [0, 1)$.

F. Let us take ordinary torus T^2 with irrational foliation and connection given by parallels. Then we have covering $\phi : S^1 \times R \to T^2$

G. Spirals and circle on $R^2 \setminus 0 = S^1 \times R$ given by integral curves of the distribution $v_{r,\alpha} = (r, 1)$ in polar coordinate system. We can introduce connection E with lines $\alpha = const$ as horizontal curves. With the help of this construction we obtain isomorphism of algebras on external and internal spirals. Here everything depends on the leaf on which we choose point x if it lies on the circle we have a bijection $\phi : R \times [0, 1) \to S^1 \times R$ or in any other case covering $\phi : R \times R \to S^1 \times R$.

Statement5. Let all conditions of the previous Lemma be satisfied. Then algebra $C_c^\infty(G)$ can be embedded into $C(S') \otimes C(P) \otimes L_2(H)$.

• Let us mention that topology of the foliated bundle G over M with fiber H is weaker than the topology of the trivial bundle $P \times S' \times H$. It holds true because as it is shown in Lemma 2 M bijectively corresponds to $P \times S'$ and topology on M is stronger than that on $P \times S'$.

The rest is simple consequence of the results from [R]. $\square$

Let us consider algebra of functions $C^\infty_{r,e}$ from **Definition 2**.

Lemma2. Let all the conditions from Lemma 1 be satisfied.

Algebra $C(G_P^P)$ can be embedded into $C^\infty_{r,e}$.

• Since all paths from G_P^P are images of some paths on S, we can say that functions on them are elements from $C^\infty_{r,e}$. $\square$

We have $TM = F' \oplus F$, where F' is a transversal distribution. Let us take into consideration set Ω consisting of differential forms on M, then the ordinary differential operator $d = d' + d'' + \partial$. Here the members of the sum are the operators of the type $(1, 0)$, $(0, 1)$, $(2, -1)$, respectively.

4. Let us denote $H_F^{0,i} = \frac{Ker(d'' : \Omega^{0,i} \to \Omega^{0,i+1})}{Im(d'' : \Omega^{0,i-1} \to \Omega^{0,i})}$— leaf cohomologies [10].

Now we can formulate the following

Corollary. Let all the conditions of the Lemma 1 be met.

Group $H_F^{0,1}$ can be embedded into $C^*_{r,e}$ in case of the finite dimensional $H_1(L)$ $\forall L \in F$.

• Since all loops on each leaf are elements from isotropy subgroup of the leaf all homologies are described by loops on S in terms of Lemma 1. This means that all possible functions on homology group of any leaf are embedded into the set of functions on homology group of S.

It is enough to mention now, that for full description of the function from $H_F^{0,1}$, which is an ordinary commutative function on the homotopy class of the loops $f(\gamma_1\gamma_2) = f(\gamma_2\gamma_1)$ it is enough to find its' values on the generators, but for compactness of the latter set we must suppose to be finite. We will define function outside of this set by continuity in the way, which provides compact support. □

We have now as a consequence from Theorem 6.14 of [7] and Lemma 2

Statement6. $C_r^*(G) \subset C(\pi_1(S)) \otimes C(P) \otimes L_2(H)$

Structure of the $C_r^*(G)$ with respect to the local convergence.

General notes.

5. Let $L_1 \subset M$ be some fixed leaf. Let us take restriction $C_r^*(L_1)$ ($\neq C_r^*(G(L_1))$ in general case) of the algebra $C_r^*(G)$ to this leaf. We can construct subalgebra $A(L_2) = A_{12} \subset C_r^*(L_1)$ for any other leaf L_2 by means of the equivalence relation given above. Let $C_{r,e}^*(L_1) = \bigcap\limits_{i \in I} A(L_i)$ (intersection over all leaves of the foliation).

This subalgebra is the first nontrivial invariant, defined for foliated manifold M and Ehresmann connection on the space of functions.

Of course here we do not suppose that the measure on the groupoids of the leaves is Haar one. This is a necessary condition for the non-triviality of the objects defined in this section.

H. Let us take into consideration the most trivial case of the foliations generated by product of manifolds. Then $C_{r,e}^*(M \times N)$ is $C_r^*(G)$. Here G is groupoid of any leaf N.

Let P be a manifold, defined as in the second section of this paper. Let us fix some so-called bifoliated metric g[Zh] on M which will generate one on P. Let us denote distance between points $x, y \in M$ as $d(x, y)$.

Now we can construct nondecreasing sequence of subalgebras of the algebra $C_r^*(G)$. Let us cover P by open balls of ε radii. So we can say that $P = \bigcup x \in PB_\varepsilon(x)$. We define algebra $C_{r,e}^*(B_\varepsilon(x))$ in any ball $B_\varepsilon(x)$ as above $C_{\varepsilon,r}^*$ for the entire P. The equivalence relation for any ball $B_\varepsilon(x)$ we define as equivalence relation on the manifold $sat(B_\varepsilon(x))$ with horizontal curves laying only in $P \bigcup B_\varepsilon(x)$.

6. Let us define algebra of functions $C_\varepsilon^* = \{f \in C_r^*(G) | \forall x \in P, f|_{B_\varepsilon(x)} \in C_{r,e,i}^*\}$.

It is easy to point out that $\lim\limits_{\varepsilon \longrightarrow \infty} C_\varepsilon^* = C_{r,e}^*$. Here lim means direct limit of algebras.

Now let us find necessary and sufficient conditions for the $\lim\limits_{\varepsilon \longrightarrow 0} C_\varepsilon^*$ to be $C_{r,e}^*$. We formulate them in two subsequent statements.

Statement7. Let foliated manifold M satisfy all the conditions from the introduction. $\lim\limits_{\varepsilon \longrightarrow 0} C_\varepsilon^* = C_r^*(G)$ in case the following conditions holds true:

1) $\forall x, y \in P$ $d(hx, hy) < const$ (action of H doesn't change distance between points on P).

2) there are only finitely many singular points on P.

3) All leaves of the foliation are closed.

• We can say that in this case set of functions $f_h : P \to P$ represented by isotropy group of P $H(P) = \{h \in H | hP = P\}$ is uniform continuous.

It holds true that $\forall \varepsilon > \varepsilon'$ $C^*_\varepsilon \subset C^*_{\varepsilon'}$. This is due to $satB_{\varepsilon'} \bigcap P \subset satB_\varepsilon(x) \bigcap P$ for any $x \in P$ which holds true by assumption.

This statement splits into two different cases.

$\exists \varepsilon' > 0$ $\forall \varepsilon < \varepsilon'$ $L(x) \bigcap B_\varepsilon(x) = x$

In this case it is enough to prove statement locally. Let $x \in P$ be some point lying on the singular leaf. Then $\exists \varepsilon' > 0$ $\forall \varepsilon < \varepsilon'$ $L(x) \bigcap B_\varepsilon(x) = x$. Let $B_\varepsilon(x)$ be some open ball which contains x. Now we can construct sequence of functions (f_n) such that $f_n(x) \to f(x)$ $(n \to \infty)$, $f_n \in C^\infty_\varepsilon$. Let us define this sequence as follows:

At first we must introduce two varieties $U_1(\varepsilon) = \{y \in P | \exists z \in B_\varepsilon(x) L(z) = L(y)\}$, $U_2(\varepsilon) = \{y \in P | \exists z \in P \setminus B_\varepsilon(x) L(z) = L(y)\}$. We have $\forall \varepsilon_1$ $\exists \varepsilon_2(\varepsilon_1)$ $U_2(\varepsilon_1) \bigcap U_1(\varepsilon_2) = \emptyset$ by assumption, moreover distance between these varieties is not 0.

So we define sequence $(f_n)_{n \in N}$ as follows:

$$f_n(\gamma) := 0$$

for $\gamma \in G(L(y))$ $\exists z \in P$ $L(y) = L(z)$, $z \in P \setminus B_{1/n}(x)$

$$f_n(\gamma) := f(\gamma')$$

whenever $\gamma \in G(L(y)) \simeq \gamma' \in G(L(x))$ for $y \in B_{\varepsilon_2(1/n)}(x)$.

$$f_n(\gamma) = n\rho f(\gamma')$$

for $\gamma \in G(L(y))$ $\rho = \min\{d(z,x) | L(y) = L(z)\}$.

Then we pass to the partition of unity defined by balls $B_{1/n}(x)$ $x \in P$ on the whole manifold M. $\square$

I. Seifert manifold [6]. Partial case with R^2 as the base and S^1 as leaf, passing through points $ze^{i\pi/n}$ in success. This is the example for the first case of the theorem.

J. Foliation on the cylinder $S^1 \times R$ given by circles up to some moment and spirals from that moment i.e. by vector field which has the following coordinates in natural coordinate chart $v_{(\alpha,x)} = (1,0)$ for $x \leq 0$ $v_{(\alpha,x)} = (1,x)$ $x \geq 0$. This is a situation in which theorem doesn't hold true.

Statement8. Condition from the previous statement is satisfied for $h \in \bigcup_{x \in P} H_x$ in case $\lim_{\varepsilon \longrightarrow 0} C^*_{\varepsilon,x} = C^*_r(G)$

• Let us suppose the contrary, that is the condition of the above statement is violated.

Then there exist $x \in P$ and $\varepsilon > 0$ such that $\forall \delta > 0$ $\exists h(\delta) \in H$ $\exists y(\delta) \in B_\delta(x)$ $d(h(\delta)x, h(\delta), y) \geq \varepsilon$. This means that any function from $C^*_{\varepsilon,x}$ must take same values on the γ $s(\gamma) = y$ and γ' $s(\gamma') = hy$ but since for some relatively close to the unit e of H element $h' \in H$ (it exists due to the fact that action of H is locally free) and map $g : [0,1] \to H$ with $g(0) = e$, $g(1) = h'$ we have different elements of

G given by the maps $g(t)x$, $g(t)y$ and $hg(t)y$ for every y. By general assumption G is Hausdorff so there exists an element from $C_r^*(G)$ corresponding to $g(t)x$ namely δ-function of $g(t)x$. Evidently, all the elements from $C^*_{\varepsilon,x}$ can't represent sequence converging to this characteristic function. $\square$

The last example from above doesn't answer this condition.

7. It is also natural to introduce algebras $C^*_{\varepsilon,x} = \{f \in C_r^*(G) | f|_{B_\varepsilon(x)} \in C^*_{r,e}(B_\varepsilon(x))\}$ Let us call them algebras of ε-neighborhood of the point x.

Of course we can apply aforesaid statements for this algebras $C^*_{\varepsilon,x}$ to converge to $C_r^*(G)$ by considering only local problem. So the statement can be reformulated in the following way:

$$d(hx, hy) = o(1)(\varepsilon \to 0), y \in B_\varepsilon(x)$$

Recent statements then holds true in case $\forall x \in P \lim\limits_{\varepsilon \longrightarrow 0} C^*_{\varepsilon,x} = C_r^*(G)$

It is therefore interesting to find out what happens to foliated manifold in case the behavior of these algebras $C^*_{\varepsilon,x}$ is predictable.

It is easily seen that in case isotropy groups of all the leaves, passing through some ε'-neighborhood of some fixed point x coincide we will have sort of stability $C^*_{\varepsilon,x} = C_r^*(G)$.

The situation in which isotropy group of the leaf $L(x)$ is less than that of all the leaves, passing through the points in some neighborhood of x is impossible, since action of the group is assumed to be continuous.

So the most interesting situation is the situation for which we have sort of discontinuity in the point. This means that in any neighborhood of the point x there exists point y isotropy group of the leaf, passing through it is not equal to the isotropy group of the leaf

Statement9. If for almost all $x \in P$ $\exists \varepsilon > 0$ $C^*_{\varepsilon,x}$ stabilize to C_r^* then variety of points for which this convergence doesn't take place is nowhere dense or, equivalently, closure of this last variety of points in P doesn't contain subsets of dimension greater than $k-1$, $k = \dim P$.

• The question can be reduced to the investigation of the properties of the set T', consisting of points x in any neighborhood of which there exist points lying on the leaves with isotropy group different from one of the leaf, passing through x.

Reductio ad absurdum. Let us assume the contrary, then we take set of points $T' \subset P$ defined above such that $\overline{T'}$ contains at least one connected subset of the topological dimension k. Then by assumption we have in some neighborhood of any point from T' there exist points for which stabilization assumption is true. Then from the closed set $\overline{T'}$ we can extract nonsingular varieties, namely $B_\varepsilon(x)$ for some x. Of course, standard Borel measure of this closed set which clearly contains all points from the T' is less than measure of the $\overline{T'}$ so since this process can be extended to infinity and that it won't stop in case measure of the still decreasing set T' is greater than 0 we conclude that the rest of $\overline{T'}$ must be closed null set, which contradicts the assumption above. $\square$

Now it is useful to describe most suspicious subset of the set of points in which we won't have convergence of algebras $C^*_{\varepsilon,r}$.

Statement10. Set of points in every neighborhood of which we have points laying on leaves with different group of isotropy is at most $k-1$-dimensional (here k is dimension of P) in case $\overline{P} = P$.

• Let us take some subset $U \subset P$ then since action of the group H is free and $\overline{P} = P$ then group, generated by $\bigcup\limits_{x \in U} H_x$ (sort of maximum group) must be discreet, or equivalently, finitely generated. Then there are only finalely many (say n) groups H_i, $i = \overline{1,n}$ of leaves with different isotropy groups. Now we can represent $U = \bigcup\limits_{i=1}^{n} T_i$, here $T_i = \{x \in U | H_x = H_i\}$. Now by Baire theorem we have at least one T_j such that there exists some ball $B_\varepsilon(x) \subset \overline{T_j}$. If some other set T_i is dense in this ball then since it must be invariant under isotropy group H_j then $H_j \subset H_i$ and since inverse must be true then $H_i = H_j$ this contradicts the assumption that these sets are different. So $\forall i \in \overline{1,n} \setminus j$ T_i is nowhere dense in T_i. Thus the set $B_\varepsilon(x) \setminus \bigcup\limits_{i=1, i\neq j}^{n} \overline{T_i}$ satisfies condition of the above statement. □

Let us denote set of points, in which convergence doesn't take place P_s.

Corollary. If H is compact group then set P_s is nowhere dense.

• In this case there exists $K \in R^+$ such that $\forall h \in H$ $\forall x, y \in P$ $d(hx, hy) \leq Kd(x, y)$. Now we can apply statement 4.1 to some neighborhood of almost any point of P. □

Construction of functions describing behavior of C^*_ε.

In this section we construct and consider the behavior of certain functions, which correspond to algebras C^*_ε and $C^*_{\varepsilon,x}$ in some cases. This can help one to estimate "speed" of convergence of algebras $C^*_{\varepsilon,x}$ to C^*_r if there is one. Here we must assume that $\dim P_s \leq k - 1$.

The first variant.

Let us take into consideration space of functions $C(G)$ from the introductory section, Let us recall that norm in this space is given by scalar product as follows:

$$(f, g) = \int\limits_M \int\limits_{G^y} f(\gamma)\overline{g(\gamma)} d\lambda^y d\mu$$

Now we can apply here method used for construction of C^*_ε as in the beginning of the preceding section.

8. As above we can define subspaces of $C(G)$ and $F(G(L)) = \{f : G \to C | f$ is measurable $\}$, $C_{12} \subset F(G(L_1))$, $C_{r,e}(B_\varepsilon(x)) \subset C(G)$, $C_\varepsilon(G) = \{f \in C(G) | \forall x \in P f|_{B_\varepsilon(x)} \in C_{r,e}(B_\varepsilon(x))\} \subset C(G)$, $C_{\varepsilon,x}(G) \subset C(G)$. Let us also consider subspaces of $C(G)$ defined as

Since space $C(G)$ is Hilbert one can construct orthoprojectors onto the spaces defined above. Let us denote them P_ε (projector onto C_ε), $P_{\varepsilon,x}$ (projector onto $C_{\varepsilon,x}$). By corollary 10.1 we have $P_{0,x} = Id$ for almost all $x \in P$.

Here we assume that these projectors commute. Later we will find the condition that is enough to ensure it at least in some cases.

Specter of the algebra generated by projectors from above is at most $P/\simeq$ (note that topology on specter doesn't coincide with one on factor-manifold) due to invariance of the projectors under action of the group H. Here the equivalence relation $\simeq$ is defined as follows:

$$p, q \in P, p \simeq q \Leftrightarrow \exists L (leaf of the foliation) p, q \in L$$

Now we are in situation described by Gelfand-Naimark theorem [4]. We can consider projectors as functions on the specter of this algebra. Let us denote isomorphism built in this theorem by U. Since it can be embedded into P we can use functions defined in Gelfand-Naimark theorem functions with P as domain.

Now we can define functions $f_{\varepsilon,x} : P \to R$, corresponding to projectors $P_{\varepsilon,x}$ as $f_{\varepsilon,x}(y) = U(P())$.

Statement11. $\lim\limits_{\varepsilon \longrightarrow 0} C_\varepsilon = C(G) \Leftrightarrow$ images under Gelfand isomorphism U of the projectors $P_{\varepsilon,x}$ converge to identity (pointwise convergence). $\forall x \in P/\simeq$.

• $\Rightarrow$ Let us assume that on the contrary. $\exists x \in P/\simeq$ and $\exists \varepsilon' > 0$ $\forall \varepsilon < \varepsilon'$ $\exists y \in B_\varepsilon(x)$ $U(P_\varepsilon)(y) = U(P_\varepsilon)(x) = 0$.

Now we must consider subspaces of $C_(G)$ which contain functions, that separate some neighborhood $B_\delta(x)$ of x for $\delta < \varepsilon'$ from any y. Since images of projectors doesn't separate points in which functions from the given spaces may take different values then the projectors which can be represented also as multiplicators by characteristic functions of the varieties $P_{B_\varepsilon(x)/\simeq} \bigcup P \setminus B/\simeq$ won't contain in their image spaces defined in the beginning of this paragraph.

$\Leftarrow$ Is evident. $\square$

Statement12. In case $x \in P$ is a singular point and $C_\infty \neq 0$ then function $U(P_{\infty,x}) = U(P_\infty)$ must be characteristic function of the set of singular points.

• Since in any ε-neighborhood of this point we can find $B_{\varepsilon'(\varepsilon,x)}(x'(\varepsilon,x))$ for which we have $\forall y \in B_{\varepsilon'}(x')$ $P_{0,y} = Id$ then by structure of the projectors this function $U(P_{\infty,x})$ must satisfy $U(P_{\infty,x}) = (U(P_{\infty,x}))^2 = U(P_{\infty,x})U(P_{0,x'}) \neq 1$ then it can be only 0 whenever $x \notin P_s$. Since it can't vanish by assumption then there is only one possibility left. $\square$

K. Fibre bundles. Function f_0 which corresponds to $\lim\limits_{\varepsilon \longrightarrow 0} C_\varepsilon$ is constant and equals 1.

L. Seifert manifold described in example H. Function $f_\infty(z) = \chi_0$, $z \in C$.

M. Spirals and circle given by vector field $(|x|, 1)$ as in Example 10. $f_0(x) = f_\infty(x) = 0$, $x \in R$.

*Note*2. In case projectors, constructed above don't generate commutative algebra we can consider only local problem.

Particularly the following holds true

Statement13. If singular point $x \in P$ is an isolated one and again all leaves of the foliation are proper then we can again formulate and prove previous statements in some neighborhood of x.

• Let ε be such that $P_s \bigcap B_\varepsilon(x) = \{x\}$ then we can take projectors $P_{\varepsilon',y}$ for $y \in B_\varepsilon(x)$ $\varepsilon' \leq \varepsilon - d(x,y)$. $\square$

Second variant. Here we introduce structure of Hilbert space on subset of $C^*_r(G)$.

Let us consider subset R_ε of algebra $C^*_\varepsilon(G)$ consisting of $f \in C^*_\varepsilon(G)$ $f * f \geq 0$. It will consist only of functions, depending only on the range of the γ since $f(\gamma\gamma_1) = \overline{f}(\gamma_1^{-1})$. Now we define scalar product as $(f,g) = \int\limits_M f * g$.

Under these assumptions we can prove all the statements from the first subsection.

Also we obviously can reformulate the first one as

Statement14. $\lim\limits_{\varepsilon \longrightarrow 0} R^*_\varepsilon = R^*_r(G) \Leftrightarrow P_\varepsilon$ is a decomposition of unity for D_0.

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