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INTERNATIONAL FISHING AS PARTIALLY COOPERATIVE GAME

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Abstract. A game theoretical model of international fishing is introduced under partial cooperation of the countries involved. The interior equilibrium is determined and it is shown that both increasing cooperation levels and smaller available fish stock have decreasing effects on the total fish harvest. A dynamic system model is developed for the dynamic behavior of the fish stock.

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1. Introduction

International fishing became the focus of many researchers of game theory during the last decade. Okuguchi [1] has introduced a fishing duopoly in which the fishing countries behave as Cournot oligopolists at each time period, so the fish harvest is determined by the Cournot -Nash equilibrium. The fishing activity changes the biological reproduction law of the fish stock, so its long term properties depend on the fishing activity. The asymptotic behavior of the fish stock was examined in this paper. This model and the asymptotic results were extended to the general n-country case in Szidarovszky and Okuguchi [2]. If the fishing countries form a coalition, then the fish harvest is always lower than that in the non-cooperative case resulting in a more stable fish-stock (Szidarovszky and Okuguchi, [3]). Entry of a new country, merging, and their effects on the fish stock and profitability were investigated in Szidarovszky et al. [4], and the effect of information delay on asymptotic behavior of the fish stock was examined by Engel et al. [5].

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In this paper partial cooperation of the fishing countries will be considered, when the payoff of each country contains her profit and some portions of the profits of the others. After the mathematical model will be introduced, the dependence of the fish harvest on the cooperation levels of the countries as well as on the available fish stock will be investigated. And finally, dynamic models describing the long-term behavior of the fish stock will be developed.

2. The Mathematical Model

Assume that n countries or firms harvest fish of a single species in a common property fishing ground and sell the fish in m markets. Let x_{ri} denote the amount of fish harvest of country r and sold in market i. The total harvest of country r is therefore

$$X_r = x_{r1} + x_{r2} + x_{r3} + ... + x_{rm} \ , \tag{1}$$

and the total fish supply in market i is the following:

$$Y_i = x_{1i} + x_{2i} + ... + x_{ni}. \tag{2}$$

The price of the fish in market i is assumed to be linear:

$$P_i(Y_i) = a_i - b_i Y_i, \tag{3}$$

and the cost function of country r is given as

$$C_r(X_r) = c_r + \gamma_r \frac{X_r^2}{X}, \tag{4}$$

here X is the available fish stock. With these price and cost functions we follow the models of the earlier studies mentioned before. The profit of country r is then

$$\Psi_r = \sum_{i=1}^{m} (a_i - b_i Y_i) x_{ri} - (c_r + \gamma_r \frac{X_r^2}{X}), \tag{5}$$

wich is the difference of the revenue and cost of the country. Let $\alpha_{rl} \in [0, 1]$ denote the cooperation level of country r towards country l, then the payoff function of country r is

$$\pi_r = \Psi_r + \sum_{l \neq r} \alpha_{rl} \Psi_l. \tag{6}$$

This payoff can be interpreted as country r takes α_{rl} percents of the interest of country l into account in addition to his own full interest in his payoff function. Using equation (5),

$$\pi_r = \sum_{i=1}^{m} (a_i - b_i Y_i) x_{ri} - (c_r + \gamma_r \frac{X_r^2}{X}) + \sum_{l \neq r} \alpha_{rl} [\sum_{i=1}^{m} (a_i - b_i Y_i) x_{li} - (c_l + \gamma_l \frac{X_l^2}{X})]. \tag{7}$$

For all values $X > 0$ of the fish stock we have now defined an n-person non-cooperative game with strategy sets

$$S_r = \{(x_{r1}, x_{r2}, ..., x_{rm)} \mid X_r = \sum_{i=1}^{m} x_{ri} \in [0, X]\}, \tag{8}$$

and payoff functions π_r with the additional constraint:

$$\sum_{r=1}^{n} X_r \in [0, X]. \tag{9}$$

In the next section the interior Cournot-Nash equilibrium of this game will be determined.

The equilibrium of the game

At a Cournot-Nash eqilibrium each π_r is maximal in $x_{r1}, x_{r2}......, x_{rm}$ with fixed strategies of all other countries. Assuming interior equilibrium, the first order conditions can be written as

$$\frac{\partial \pi_r}{\partial x_{ri}} = a_i - b_i Y_i - b_i x_{ri} - \frac{2\gamma_r X_r}{X} + \sum_{l \neq r} \alpha_{rl}(-b_i x_{li}) = 0, \tag{10}$$

implying that for all r and i,

$$x_{ri} + \sum_{l \neq r} \alpha_{rl} x_{li} = \frac{a_i}{b_i} - Y_i - \frac{2\gamma_r X_r}{X b_i}. \tag{11}$$

Since π_r is concave in $(x_{r1,}......., x_{rm})$, the second order conditions are satisfied. For simplicity, introduce the following notations:

$$\underline{H} = \begin{pmatrix} 1 & \alpha_{12} & ...... & \alpha_{1n} \\ \alpha_{21} & 1 & ...... & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & ...... & 1 \end{pmatrix}, \underline{Z}_i = \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix}, \underline{x} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$$

$$\underline{1} = (1 \quad 1 \quad ... \quad .. \quad 1)^T, and\, \underline{C} = diag(\gamma_1, \gamma_{2,....,}\gamma_n).$$

Then (11) can be rewritten as

$$\underline{HZ}_i = (\frac{a_i}{b_i} - Y_i)\underline{1} - \frac{2}{b_i X}\underline{Cx}, \tag{12}$$

that is ,

$$\underline{Z}_i = (\frac{a_i}{b_i} - Y_i)\underline{H}^{-1}\underline{1} - \frac{2}{b_i X}\underline{H}^{-1}\underline{Cx}\ . \tag{13}$$

By adding these equations for all values of i we have

$$\underline{x} = (A - S)\underline{H}^{-1}\underline{1} - \frac{2B}{X}\underline{H}^{-1}\underline{Cx}, \tag{14}$$

where $A = \sum\limits_{i=1}^{m} \frac{a_i}{b_i}$ and $B = \sum\limits_{i=1}^{m} \frac{1}{b_i}$, furthermore $S = \sum\limits_{i=1}^{m} Y_i$ is the total fish harvest of all countries. From (14) we see that

$$\underline{x} = (A - S)(\underline{I} + \frac{2B}{X}\underline{H}^{-1}\underline{C})^{-1}\underline{H}^{-1}\underline{1} = (A - S)(\underline{H} + \frac{2B}{X}\underline{C})^{-1}\underline{1}. \tag{15}$$

Since $\underline{1}^T\underline{x} = X_1 + X_2 + ... + X_n = S$, we obtain a single equation for S:

$$S = (A - S)\underline{1}^T(\underline{H} + \frac{2B}{X}\underline{C})^{-1}\underline{1} \tag{16}$$

implying that

$$S = \frac{Af(X)}{1 + f(X)} \tag{17}$$

with

$$f(X) = \underline{1}^T(\underline{H} + \frac{2B}{X}\underline{C})^{-1}\underline{1}. \tag{18}$$

Consider the special case of noncooperating countries. Then $\alpha_{rl} = 0$ for all $l \neq r$, so $\underline{H} + \frac{2B}{X}\underline{C}$ is a diagonal matrix showing that

$$f(X) = \sum_{r=1}^{n} \frac{1}{1 + \frac{2B\gamma_r}{X}}. \tag{19}$$

Notice that this result coincides with those obtained earlier in Szidarovszky and Okuguchi [2]. The completely cooperative case occurs when the firms maximize the sum of their profits, so in this case $\alpha_{rl} = 1$ for all $l \neq r$. Next we compute the equilibrium in a more general case, when $\alpha_{rl} = \alpha$ for all $l \neq r$. The case of $\alpha = 1$ gives the completely cooperative case. Now we have

$$\underline{H} + \frac{2B}{X}\underline{C} = \begin{pmatrix} 1 & \alpha & \cdots & \alpha \\ \alpha & 1 & \cdots & \alpha \\ \vdots & \vdots & \vdots & \vdots \\ \alpha & \alpha & \cdots & 1 \end{pmatrix} + \begin{pmatrix} \frac{2B\gamma_1}{X} & 0 & 0 & 0 \\ 0 & \frac{2B\gamma_2}{X} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \frac{2B\gamma_n}{X} \end{pmatrix}$$

$$= \underline{D} + \alpha\underline{1}\underline{1}^T, \tag{20}$$

where

$$\underline{D} = diag\left(1 + \frac{2B\gamma_1}{X} - \alpha, ..., 1 + \frac{2B\gamma_n}{X} - \alpha\right).$$

Therefore

$$\left(\underline{H} + \frac{2B}{X}\underline{C}\right)^{-1} = \underline{D}^{-1} - \frac{\alpha\underline{D}^{-1}\underline{1}\underline{1}^T\underline{D}^{-1}}{1 + \alpha\underline{1}^T\underline{D}^{-1}\underline{1}}, \tag{21}$$

so

$$f(X) = \underline{1}^T \underline{D}^{-1} \underline{1} - \frac{1}{1 + \alpha \underline{1}^T \underline{D}^{-1} \underline{1}} \alpha \left(\underline{1}^T \underline{D1} \right)^2$$

$$= \frac{\underline{1}^T \underline{D}^{-1} \underline{1}}{1 + \alpha \underline{1}^T \underline{D}^{-1} \underline{1}} = \frac{\sum_{r=1}^n \frac{1}{1 + \frac{2B\gamma_r}{X} - \alpha}}{1 + \sum_{r=1}^n \frac{\alpha}{1 + \frac{2B\gamma_r}{X} - \alpha}}. \tag{22}$$

In the special case of $\alpha = 1$,

$$f(X) = \frac{\frac{X}{2B} C}{1 + \frac{X}{2B} C} = \frac{XC}{2B + CX}, \tag{23}$$

with $C = \sum_{r=1}^n \frac{1}{\gamma_r}$. Notice that this result coincides with that presented earlier in Szidarovszky and Okuguchi [3].

4. Dependence on α_{rl}

In this section we will investigate how the cooperation levels α_{rl} affect the total fish harvest. For this purpose we need to find the derivative of S with respect to α_{rl}. Let

$$\underline{U} = \left(\underline{H} + \frac{2B}{X} \underline{C} \right)^{-1}, \tag{24}$$

then

$$\underline{U} \left(\underline{H} + \frac{2B}{X} \underline{C} \right) = \underline{I}. \tag{25}$$

Differentiate both sides with respect to α_{rl} to have

$$\left(\frac{\partial}{\partial \alpha_{rl}} \underline{U} \right) \left(\underline{H} + \frac{2B}{X} \underline{C} \right) + \underline{U} \underline{e}_r \underline{e}_l^T = \underline{0}, \tag{26}$$

where $\underline{e}_r$ and $\underline{e}_l$ are the r $\underline{\text{th}}$ and l $\underline{\text{th}}$ basis vectors.

So

$$\frac{\partial}{\partial \alpha_{rl}} \underline{U} = -\underline{U} \underline{e}_r \underline{e}_l^T \underline{U} = -\underline{c}_r \underline{r}_l^T, \tag{27}$$

where $\underline{c}_r$ is the $r\underline{\text{th}}$ column and $\underline{r}_l^T$ is the $l\,\underline{\text{th}}$ row of matrix $\underline{U}$, and therefore

$$\frac{\partial}{\partial \alpha_{rl}} f(X) = -\left(\underline{1}^T \underline{c}_r\right)\left(\underline{r}_l^T \underline{1}\right). \tag{28}$$

It is reasonable to assume that $\alpha_{rl} = \alpha_{lr}$ for all $r \neq l$ which means that any two countries cooperate with each other on equal levels. Then $\underline{H}$ is symmetric, therefore $\underline{U}$ is also symmetric showing that $\underline{c}_l = \underline{r}_l$. Then relation (28) and the fact that

$$\frac{\partial S}{\partial \alpha_{rl}} = \frac{\partial}{\partial \alpha_{rl}} \left(\frac{Af(X)}{1+f(X)} \right) = \frac{A}{\left(1+f(X)\right)^2} \frac{\partial f(X)}{\partial \alpha_{rl}}, \tag{29}$$

imply that

$$\frac{\partial S}{\partial \alpha_{rl}} = -\frac{A}{(1+f(X))^2} \left(\underline{1}^T \underline{c}_r\right)\left(\underline{1}_l^T \underline{c}_l\right). \tag{30}$$

Assume that the sum of the elements in each column has the same sign. Then increasing cooperation levels decrease the total fish harvest. A simple consequence of this observation is the fact that total cooperation among the countries results is the smallest total harvest. If $\alpha_{rl} = 0$ for all $r \neq l$, then $\underline{U}$ is diagonal with positive diagonal, so $\underline{1}^T \underline{c}_r > 0$ for all r, so by selecting sufficiently small α_{rl} values, S still decreases in α_{rl}.

5. Dependence on X

Similarly to relation (26) we now have

$$\left(\frac{\partial}{\partial X} \underline{U}\right)\left(\underline{H} + \frac{2B}{X}\underline{C}\right) + \underline{U}\left(-\frac{2B}{X^2}\underline{C}\right) = 0 \tag{31}$$

implying that

$$\frac{\partial}{\partial X}\underline{U} = \frac{2B}{X^2}\underline{UCU}. \tag{32}$$

Therefore

$$f'(X) = \frac{2B}{X^2}\underline{1}^T \underline{UCU1}\ . \tag{33}$$

Introduce matrix $\underline{K} = diag\left(\sqrt{\gamma_1}, ..., \sqrt{\gamma_n}\right)$ to see that in the case of

$$\alpha_{rl} = \alpha_{lr} \quad (\ r \neq l),$$

$$f'(X) = \frac{2B}{X^2}\underline{1}^T \underline{UKKU1} = \frac{2B}{X^2} \parallel \underline{KU1} \parallel_2^2 . \tag{34}$$

Since both $\underline{K}$ and $\underline{U}$ are invertible, $\underline{KU1} \neq \underline{0}$ showing that $f'(X) > 0$, and from (29) we see that S is strictly increasing in X. Therefore on any cooperation level more available fish stock results in larger total harvest.

6. The Dynamic Model

Assume first that without commercial fishing the fish stock changes according to the well-known logistic equation

$$\dot{X} = X(\alpha - \beta X), \tag{35}$$

(see for example, Clark, [6]). In the presence of the commercial fishing this rule becomes

$$\dot{X} = X(\alpha - \beta X - g(X)), \tag{36}$$

with

$$g(X) = \frac{S}{X} = \frac{Af(X))}{X(1 + f(X))}. \tag{37}$$

Notice that from (18) we have

$$f(X) = \underline{1}^T(\underline{H}X + 2B\underline{C})^{-1}\underline{1}X, \tag{38}$$

so

$$g(X) = \frac{A\underline{1}^T(\underline{H}X + 2B\underline{C})^{-1}\underline{1}}{1 + \underline{1}^T(\underline{H}X + 2B\underline{C})^{-1}\underline{1}X}. \tag{39}$$

Therefore

$$g(0) = A\underline{1}^T(2B\underline{C})^{-1}\underline{1} = \frac{A}{2B}\sum_{r=1}^{n}\frac{1}{\gamma_r} = \frac{AC}{2B}, \tag{40}$$

where we use the same notation as in equation (23). Notice furthermore that $f(X) \longrightarrow \underline{1}^T\underline{H}^{-1}\underline{1}$ as $X \longrightarrow \infty$ implying that $g(X) \longrightarrow 0$. Any nonzero equilibrium of the dynamic system is the solution of equation

$$\alpha - \beta X = g(X), \tag{41}$$

and $\overline{X} = 0$ is always an equilibrium.

In model (35) a linearly decreasing growth rate was considered. Clark [6] has suggested more advanced population rules. A straightforward generalization of the logistic equation is

$$\dot{X} = X^r(\alpha - \beta X) \tag{42}$$

with some constant $r > 0$. Then in the presence of commercial fishing this rule becomes

$$\dot{X} = X^r(\alpha - \beta X - g(X)) \tag{43}$$

where now

$$g(X) = \frac{s}{X^r} = \frac{Af(X)}{X^r(1 + f(X))}. \tag{44}$$

In the literature instead of equations (35) and (42) the following is used often:

$$\dot{X} = rX \ln \frac{K}{X}, \tag{45}$$

which modifies as

$$\dot{X} = X(r \ln K - r \ln X - g(X)) \tag{46}$$

in the presence of commercial fishing, where $g(X)$ is the same as given earlier in equation (37).
More complicated model is obtained by rule

$$\dot{X} = rX(\frac{X}{K_0} - 1)(1 - \frac{X}{K}) \tag{47}$$

which modifies in the presence of commercial fishing as

$$\dot{X} = X\left[r(\frac{X}{K_0} - 1)(1 - \frac{X}{K}) - g(X)\right] \tag{48}$$

where the $g(X)$ term is the same as given in equation (37).

In the case of model (43), the eqilibrium equation is again (41) giving the intercept of the linear function $\alpha - \beta X$ and the nonlinear curve $g(X)$. In the case of models (46) and (48) the equilibria are the solutions of the more complicated equations

$$r \ln K - r \ln X - g(X) = 0 \tag{49}$$

and

$$r\left(\frac{X}{K_0} - 1\right)\left(1 - \frac{X}{K}\right) - g(X) = 0 \tag{50}$$

respectively. The dynamic behavior of the equilibria depends on the analytic properties of the functions involved. A detailed analysis will be the subject of a future paper.

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