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DOUBLE BARRIER OPTIONS WITH CURVED BOUNDARIES

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Abstract. We consider a double barrier option between two generic curved boundaries and use a Green's function analysis to reformulate the problem as a pair of coupled integral equations, and solve the integral equations for the special case of flat boundaries.

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1. Introduction

Since the publication of the celebrated Black-Scholes option pricing formula in 1973 [2,10], there has been an explosion in the use of financial derivatives, which are instruments that derive their value from some other underlying instrument. An example of such a derivative is an option, which typically carries the right but not the obligation to either buy or sell the underlying. In addition to the publicly traded puts and call with which many investors are familiar, there are numerous other options, usually termed "exotic", which typically are traded over-the-counter, or OTC, meaning that typically they are custom-made by financial institutions for clients. Once such exotic is the barrier option, which is is either "knocked in" or "knocked out", depending on the flavor of the barrier, when the price of the underlying is equal to the barrier price. One reason for the existence of barrier options is that they can be cheaper than their plain vanilla counterparts because they risk either not being knocked in or being knocked out. Another reason is that they may be a part of a more complicated structured contract. Therefore, barrier options have become increasingly popular in world markets. According to

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[15], barriers have been traded in the USA since at least 1967, some 6 years before the Chicago Board of Options Exchange (CBOE) opened its doors in 1973, with a typical user being a hedge fund. Like all equity options, the value $V(S,t)$ of a barrier option is given by the Black-Scholes-Merton PDE,

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r - D_0)\, S\frac{\partial V}{\partial S} - rV0 \tag{1.1}$$

with the various options differentiated by the boundary conditions [11].

There are single barrier options and double barrier options. Closed-form solutions are available for single-barrier options in the Black-Scholes framework [10,6]. Other closed-form solutions have been found in Black-Scholes setup (e.g., [13,3]). On the other hand, no simple formula currently exists for other options with multiple barriers. Double barrier options are an ideal method for investors in particular to take advantage of markets expected to trade within a range. They are available in most underlying markets, including commodities, interest rate and equity markets and most commonly in foreign exchange. In the present study, we will consider a double barrier option, occasionally referred to as a dual barrier option, where there is both an upper boundary $S_u(t)$ and a lower boundary $S_l(t)$, which are typically specified in the option contract. It is possible to recover an option with a single barrier from this by either taking the limit $S_u(t) \to \infty$, when we recover an option with a lower barrier, or the limit $S_l \to 0$, when we recover an option with an upper barrier. Work on this problem goes back at least as far as 1965, when [4] gave a density function involving Fourier series for two flat boundaries, $S_u(t) = S_{u0}$ and $S_l(t) = S_{l0}$, and also used the method of images to construct a Green's function, and [5] revisited that problem using Laplace transforms, while [8] gave a formula for two exponentially curved boundaries, with $S_u(t) = S_{u0}e^{\delta_u(T-t)}$ and $S_l(t) = S_{l0}e^{\delta_l(T-t)}$. [8] also listed several examples of double barrier options which were traded in Tokyo at the time of their study. Other than the papers mentioned above, other contributions to the pricing of double barrier options include the studies of [12,14], who used residue theory to invert the transforms in [5] those transforms having been inverted numerically in the original paper, and the studies of [7,1], where the connection between the solutions obtained by the different methods mentioned above was discussed, and an early numerical study by [9] who evaluated a LYON (liquid option yield note) which had curved boundaries.

While [8] considered exponentially curved boundaries, in the present work, we will consider two generic curved boundaries $0 \leq S_l(t) < S_u(t)$, and also allow for the possibility of rebates when the barriers are hit, so that the pay-off for a double knock-out option is assumed to be $R_u(t)$ on $S_u(t)$ and $R_l(t)$ on $S_l(t)$, while if the barriers are never hit, and the price of the underlying is always between the two barriers, $S_l(t) < S < S_u(t)$ for $t \leq T$, the pay-off at expiry is $\max(S - E, 0)$, meaning we have a plain vanilla European call with strike price E. Similarly, we can define a double knock-in option, for which the pay-off at expiry is zero if the barriers are never hit, $S_l(t) < S < S_u(t)$ for $t \leq T$, while if we hit either of the barriers, we receive a rebate as above together with a plain vanilla European call. Since the value of a European call on the barriers, $V_e(S_u(t), t)$ or $V_e(S_l(t), t)$, can be readily calculated using the Black-Scholes option pricing formula, we can assume for the double knock-in option that the value of the European call is included within the rebate, so that the pay-off for a double knock-in option is assumed to be $R_u(t)$ on $S_u(t)$ and $R_l(t)$ on $S_l(t)$.

To price such options, we will use a Green's function approach. The key to this, which we will present in the next section, is that we can write down a solution to the Black-Scholes-Merton equation of the form

$$V(S,t) = \int_{-\infty}^{\infty} v(z,0)G(S,z,T-t)dz, \tag{1.2}$$

where $v(z,0)$ is related to the pay-off at expiry, and the variables in this formula are defined in the next section. If we are pricing options which have no early exercise features, such as vanilla European options, then the formula (1.2) will give us a solution. However, for problems with boundaries, such as American options where early exercise is allowed or the double barrier options considered here, the problem is that the pay-off at expiry is only defined for certain values of the underlying price. Because of this, the function $v(z,0)$ is only known for certain values of z, while in the formula (1.2) above, we need to know $v(z,0)$ for all z, $-\infty < z < \infty$. There are at least two possible ways to overcome this. One approach is, rather than use the solution (1.2) presented above, to instead the Green's function for the Black-Scholes-Merton equation with boundaries present, so that the solution can be cast as

$$V(S,t) = \int_{z_l}^{z_u} v(z,0)G(S,z,T-t)dz, \tag{1.3}$$

where the Green's function $G(S,z,T-t)$ in (1.3) differs from that in (1.2); for flat boundaries, it is possible to do this using the method of images, which was the approach used in [4]. An alternative approach, which we consider here, is to use the formula (1.2) and assign some form to the function $v(z,0)$ outside the barriers: in essence, we are supposing that we have a European option with some nonlinear pay-off at expiry whose value on the barriers is equal to the (known) value of the double barrier option. essentially, This second approach means that we have to find $v(z,0)$ outside the barriers for this new option, and then the value of our double barrier option is given by (1.2) together with the pay-off.

The remainder of the paper is as follows. In Section 2, we will introduce the Green's function and perform our analysis, then in Section 3, we will make some concluding remarks.

2. Analysis

The value $V(S,t)$ of an option is given by the Black-Scholes-Merton PDE,

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r - D_0)\, S\frac{\partial V}{\partial S} - rV = 0\ , \tag{2.1}$$

where S is the price of the underlying, and D_0, r and σ, all of which are assumed constant, are the dividend yield, risk-free interest rate, and volatility respectively. This equation was initially presented by [2] for options without dividends and extended by [10] to cover options with a continuous dividend yield. For a vanilla European option, this equation is valid for all S, $0 \leq S < \infty$, and for all times up to expiry $t < T$, together with the pay-off at expiry, $V(S,T)$.

For the double barrier options considered here, the equation (2.1) is only valid while the price of the underlying remains between the two barriers, $S_l(t) < S < S_u(t)$ for $t \leq T$, while if the underlying price hits one of the barriers, we assume that a rebate is paid, which we take to be $R_u(t)$ on $S_u(t)$ and $R_l(t)$ on $S_l(t)$. In our analysis, we require R_u and R_l both tend to 0 as $t \to -\infty$, so that the option price will also tend to 0 in that limit. Although we have the value of the option on the boundary, we do not know the delta of the option, $(\partial V/\partial S)$, on the boundary; this can be contrasted with plain vanilla American options, where the delta is known but the location of the boundary is unknown. The pay-off at expiry between the barriers is $V(S,T) = \max(S-E,0)$ for a double knock-out option and $V(S,T) = 0$ for a double knock-in option.

This problem can be recast as a more conventional diffusion problem,

$$\frac{\partial v}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2}, \tag{2.2}$$

by using the transformation

$$V(S,t) = S^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} e^{-r_2\tau} v(x,\tau), \tag{2.3}$$

where $x = \log S$ and $\tau = T - t$ and $r_2 = \frac{(r-D_0)^2}{2\sigma^2} + \frac{D_0+r}{2} + \frac{\sigma^2}{8}$. This has a solution using a Green's function

$$v(x,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} v(z,0) e^{-\frac{(x-z)^2}{2\sigma^2\tau}} dz, \tag{2.4}$$

so that in the original variables we have

$$V(S,t) = S^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{e^{-r_2(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} v(z,0) e^{-\frac{(\log S - z)^2}{2\sigma^2(T-t)}} dz. \tag{2.5}$$

As noted in the introduction, we only know $v(z,0)$ between the barriers. We require this to be equal to the rebate on the barriers, so we have a pair of twin integral equations for $v(z,0)$, which is unknown except between $S_l(T) \le S \le S_u(T)$ or equivalently $\log S_l(T) \le z \le \log S_u(T)$,

$$\begin{aligned} R_u(t) &= (S_u(t))^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{e^{-r_2(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} v(z,0) e^{-\frac{(\log S_u(t)-z)^2}{2\sigma^2(T-t)}} dz \\ R_l(t) &= (S_l(t))^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{e^{-r_2(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} v(z,0) e^{-\frac{(\log S_l(t)-z)^2}{2\sigma^2(T-t)}} dz. \end{aligned} \tag{2.7}$$

We can rewrite this so that the exponential term is the same in both integrals

$$\begin{aligned} R_u(t) &= (S_u(t))^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{e^{-r_2(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} v(z+\log S_u(t),0) e^{-\frac{z^2}{2\sigma^2(T-t)}} dz \\ R_l(t) &= (S_l(t))^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{e^{-r_2(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} v(z+\log S_l(t),0) e^{-\frac{z^2}{2\sigma^2(T-t)}} dz, \end{aligned} \tag{2.8}$$

and in (2.8) we have a pair of coupled integral equations for $v(z,0)$; once $v(z,0)$ has been determined, the price of the option can be recovered using (2.5). We can write down an explicit solution to the equations (2.8) in some special cases. When the boundaries are flat, $S_u(t) = S_{u0}$ and $S_l(t) = S_{l0}$, we have

$$\begin{aligned} R_u(t)e^{-r_2(T-t)} &= (S_{u0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{1}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} v(z+\log S_{u0},0)e^{-\frac{z^2}{2\sigma^2(T-t)}}\,dz \\ R_l(t)e^{-r_2(T-t)} &= (S_{l0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{1}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} v(z+\log S_{l0},0)e^{-\frac{z^2}{2\sigma^2(T-t)}}\,dz, \end{aligned} \tag{2.9}$$

We can rewrite these two equations as

$$\begin{aligned} \tilde{R}_u(\tau)e^{-r_2\tau} &= (S_{u0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{1}{\sigma\sqrt{2\pi\tau}} \int_0^{\infty} [v_1(z)+v_2(z)]\, e^{-\frac{z^2}{2\sigma^2\tau}}\,dz \\ \tilde{R}_l(\tau)e^{-r_2\tau} &= (S_{l0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \frac{1}{\sigma\sqrt{2\pi\tau}} \int_0^{\infty} [v_3(z)+v_4(z)]\, e^{-\frac{z^2}{2\sigma^2\tau}}\,dz, \end{aligned} \tag{2.10}$$

where $v_1(z) = v(z+\log S_{u0}, 0)$, $v_2(z) = v(-z+\log S_{u0}, 0)$, $v_3(z) = v(z+\log S_{l0}, 0)$ and $v_4(z) = v(-z+\log S_{l0}, 0)$, $\tau = T-t$ is the time remaining until expiry and $\tilde{R}_u(\tau) = R_u(T-\tau)$, with a similar definition for $\tilde{R}_l$. If we apply a Laplace transform in τ to these equations, we obtain

$$\begin{aligned} \sigma\sqrt{2p}L\left[\tilde{R}_u\right](p+r_2) &= (S_{u0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \left[L\left[v_1\right](\frac{\sqrt{2p}}{\sigma}) + L\left[v_2\right](\frac{\sqrt{2p}}{\sigma})\right] \\ \sigma\sqrt{2p}L\left[\tilde{R}_l\right](p+r_2) &= (S_{l0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \left[L\left[v_3\right](\frac{\sqrt{2p}}{\sigma}) + L\left[v_4\right](\frac{\sqrt{2p}}{\sigma})\right], \end{aligned} \tag{2.11}$$

which we can write as

$$\begin{aligned} L\left[v_1\right](p) + L\left[v_2\right](p) &= (S_{u0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \sigma^2 pL\left[\tilde{R}_u\right]\left(\frac{\sigma^2p^2}{2}+r_2\right) \\ L\left[v_3\right](p) + L\left[v_4\right](p) &= (S_{l0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \sigma^2 pL\left[\tilde{R}_l\right]\left(\frac{\sigma^2p^2}{2}+r_2\right), \end{aligned} \tag{2.12}$$

or inverting, we get

$$\begin{aligned} v(z+\log S_{u0},0) + v(-z+\log S_{u0},0) &= (S_{u0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \sigma^2 \\ &\quad \times \frac{d}{dz}L^{-1}\left(L\left[\tilde{R}_u\right]\left(\frac{\sigma^2p^2}{2}+r_2\right)\right)[z] \\ v(z+\log S_{l0},0) + v(-z+\log S_{l0},0) &= (S_{l0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}} \sigma^2 \\ &\quad \times \frac{d}{dz}L^{-1}\left(L\left[\tilde{R}_l\right]\left(\frac{\sigma^2p^2}{2}+r_2\right)\right)[z]. \end{aligned} \tag{2.13}$$

Since we already know $V(S,T)$ for $S_{l0} \le S \le S_{u0}$, or equivalently $v(z,0)$ for $\log S_{l0} \le z \le \log S_{u0}$, for specific functions R_l and R_u, it is possible to deduce the function $v(z,0)$ for all z, $-\infty < z < \infty$ and then we can recover the value of the option from (2.5).

When we have exponentially curved boundaries $S_u(t) = S_{u0}e^{\delta_u(T-t)}$ and $S_l(t) = S_{l0}e^{\delta_l(T-t)}$, the integral equations (2.8) become

$$\begin{aligned}\tilde{R}_u(\tau) &= (S_{u0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}}\,\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left[\left(-r_2+\delta_u\left(\frac{D_0-r}{\sigma^2}+\frac{1}{2}\right)\right)\tau\right]\\ &\quad\times\int_{-\infty}^{\infty} v(z+\log S_{u0}+\delta_u\tau,0)e^{-\frac{z^2}{2\sigma^2\tau}}\,dz,\\ \tilde{R}_l(\tau) &= (S_{l0})^{\frac{D_0-r}{\sigma^2}+\frac{1}{2}}\,\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left[\left(-r_2+\delta_l\left(\frac{D_0-r}{\sigma^2}+\frac{1}{2}\right)\right)\tau\right]\\ &\quad\times\int_{-\infty}^{\infty} v(z+\log S_{l0}+\delta_l\tau,0)e^{-\frac{z^2}{2\sigma^2\tau}}\,dz,\end{aligned}\tag{2.14}$$

which in principal could be solved, and for the case $R_l(t) = R_u(t) = 0$, the formula in [8] recovered.

3. Discussion

In the previous section, we used a Green's function to consider the valuation of a double barrier option with two curved boundaries, $0 \le S_l(t) < S_u(t)$, with rebates of $R_l(t)$ and $R_u(t)$ when these boundaries are hit. By doing so, we were able to arrive at our main result, which was the pair of coupled integral equations, (2.8); once these equations are solved, the price of the option can be recovered using (2.5). We were able to solve (2.8) using Laplace transforms for the special case of flat boundaries, arriving at the expression (2.13). When there are no rebates, (2.13) is equivalent to the well-known expression [4] using the method of images, but one contribution of our present study is to offer a framework for pricing double barrier options with general rebates, with the other main contribution of this work being to offer a framework for pricing double barrier options with generic curved boundaries, as opposed to the exponentially curved boundaries considered by [8].

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