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ON THE ZEROS OF THE RIEMANN ξ-FUNCTION

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Abstract. Necessary and sufficient conditions for the validity of the Riemann Hypothesis are established in terms of the sum of the reciprocal squares of the zeros of the Riemann ξ-function. Numerical work, in conjunction with density estimates, involving the first 100,000 zeros of the Riemann ξ-function are presented.

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1. Introduction. This paper is motivated by a recent result of P. C. Hu dealing with the Riemann Hypothesis. After a brief review of the standard terminology and notation adopted in the theory of the Riemann zeta function (Section 2), we proffer a new proof of Hu's theorem (Theorem 2.2). This theorem provides a necessary and sufficient condition for the validity of the Riemann Hypothesis in terms of the sum of the reciprocal squares of the zeros of the Riemann ξ-function. We also prove an analogous theorem (Theorem 2.3) using the Fourier integral representation of the Riemann ξ-function. In Section 3, we elucidate the aforementioned theoretical results by means of concrete examples involving the first 100,000 zeros the Riemann ξ-function. With the aid of a refinement of the Riemann–von Mangoldt formula (see Backlund's density theorem, Theorem 3.1), we are able to estimate the tail end of the series of the sum of the reciprocal squares of the zeros (Lemma 3.2 and Proposition 3.3). The paper ends with an open problem pertaini! ng to the distribution of zeros of a a real entire function related to the Riemann ξ-function.

2. The zeros of the Riemann ξ-function. The function, $\zeta(s)$, is defined for $\sigma = \operatorname{Re} s > 1$ by the simplest of Dirichlet series, the series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (s = \sigma + it). \tag{2.1}$$

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This Dirichlet series is convergent for $\operatorname{Re} s = \sigma > 1$ and is uniformly convergent for $\sigma \geq \sigma_0 > 1$. The function $\zeta(s)$ admits analytic (meromorphic) continuation to the entire complex plane where its only singularity is a simple pole at $s = 1$ with residue 1 ([15, p. 13]) For general information about $\zeta(s)$ we refer the reader to the excellent monographs [10], [11], [12], [14] and [15]. The functional equation (for the zeta function) can be expressed in a number of different ways (cf. [15, p. 13, p. 16] or [11, p. 8]):

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \qquad (s \in \mathbb{C}) \tag{2.2}$$

or, for example, in the symmetric form

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{2.3}$$

The Riemann ξ-function defined by the formula

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{2.4}$$

satisfies the relation $\xi(s) = \xi(1-s)$ which is equivalent to (2.3). Now it is known that $\xi(s)$ is an entire function of order 1 and of maximal type ([15, p. 29] or [12, p. 20].

The non-real (non-trivial) zeros of $\zeta(s)$, $\{\rho_k\}_{k=1}^{\infty}$, are the zeros of $\xi(s)$ and these zeros all lie in the *critical strip* $0 \leq \sigma \leq 1$. (The "trivial zeros" of $\zeta(s)$, $-2, -4, -6, \cdots$, are not zeros of $\xi(s)$ since they are cancelled by the poles of $\Gamma(s/2)$.) Now, $\zeta(s)$ (and *a fortiori* $\xi(s)$) has an infinite number of zeros in the critical strip ([15, p. 30] and the Riemann Hypothesis is the assertion that the non-trivial zeros of $\zeta(s)$ (that is, the zeros of $\xi(s)$) all lie on the*critical line* $\operatorname{Re} s = \sigma = 1/2$. Since $\zeta(s)$ is real on the real axis, $\overline{\xi(s)} = \xi(\overline{s})$, and whence $\overline{\rho}$ is a zero of $\xi(s)$ whenever ρ is. Also, by the functional equation $\xi(s) = \xi(1-s)$, if ρ is a zero of $\xi(s)$, then $1-\rho$ and $1-\overline{\rho}$ are also zeros of $\xi(s)$. Thus, the zeros of $\xi(s)$ are situated symmetrically with respect to the lines $\operatorname{Im} s = 0$ and $\operatorname{Re} s = \frac{1}{2}$.

Since $\xi(s)$ is an entire function of order 1, and $\xi(0) = -\zeta(0) = \frac{1}{2}$, the Hadamard factorization theorem gives (cf. [15, p. 30]) for all $s \in \mathbb{C}$

$$\xi(s) = \frac{1}{2}e^{B_0 s}\prod_{\rho}\left(1 - \frac{s}{\rho}\right)e^{s/\rho}, \qquad \left(\sum_{\rho}\frac{1}{|\rho|^2} < \infty\right), \tag{2.5}$$

where the product extends over all the zeros of $\xi(s)$ and B_0 is the constant

$$B_0 := \frac{1}{2}\log 4\pi - 1 - \frac{1}{2}\gamma, \quad \text{where} \quad \gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) \tag{2.6}$$

and where γ is Euler's constant ($\gamma = -\Gamma'(1) = 0.577\ 215\ 664\cdots$).

Remark 2.1. *In 1896, Hadamard and de la Vallée-Poussin proved independently that $\zeta(s)$ has no zeros on the line* $\operatorname{Re} s = \sigma = 1$ *(*[15, p. 45] *or* [12, p. 25]*). Moreover, Hardy proved in 1914 that $\zeta(s)$ has an infinite number of zeros on the critical line* $\operatorname{Re} s = \sigma = \frac{1}{2}$*. Thus, the zeros of $\xi(s)$ all lie in the open critical strip* $0 < \sigma < 1$ *and $\xi(s)$ has an infinite number of zeros on the critical line $\sigma = \frac{1}{2}$.*

Theorem 2.2 (Hu [9, Theorem 2]). *Let $\rho_k = \frac{1}{2} + i\gamma_k$, $(k = 1, 2, 3 \dots)$, $0 < \gamma_k \leq \gamma_{k+1} \leq \cdots$, denote the zeros of $\xi(s)$ (that is, the nontrivial zeros of $\zeta(s)$ on the critical line in the upper half-plane $\operatorname{Im} s > 0$). Then the Riemann Hypothesis is true if and only if*

$$\sum_{k=1}^{\infty} \frac{1}{\frac{1}{4} + \gamma_k^2} = 1 + \frac{\gamma}{2} - \frac{1}{2}\log(4\pi), \tag{2.7}$$

where γ (without a subscript) denotes Euler's constant.

Proof. In view of the Hadamard factorization of $\xi(s)$ ((2.5)), logarithmic differentiation yields

$$\frac{\xi'(s)}{\xi(s)} = B_0 + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) \quad \text{and} \quad -\frac{\xi'(1-s)}{\xi(1-s)} = -B_0 - \sum_{\rho}\left(\frac{1}{1-s-\rho} + \frac{1}{\rho}\right), \tag{2.8}$$

where B_0 is defined by (2.6), ρ extends over all the zeros of $\xi(s)$, and the series converge uniformly on compact subsets of the plane punctured at the zeros of $\xi(s)$. Since $\xi(s) = \xi(1-s)$, $\dfrac{\xi'(s)}{\xi(s)} = -\dfrac{\xi'(1-s)}{\xi(1-s)}$ and so by (2.8) we have

$$\sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + \sum_{\rho}\left(\frac{1}{1-s-\rho} + \frac{1}{\rho}\right) = -2B_0. \tag{2.9}$$

Setting $s = 0$ in (2.9), yields

$$\sum_{\rho}\left(\frac{1}{1-\rho} + \frac{1}{\rho}\right) = -2B_0. \tag{2.10}$$

Since the zeros of $\xi(s)$ are situated symmetrically with respect to the lines $\operatorname{Im} s = 0$ and $\operatorname{Re} s = 1/2$, we can express (2.10) in the form

$$\begin{aligned}
-2B_0 &= \sum_{\operatorname{Im}\rho > 0}\left(\frac{1}{1-\rho} + \frac{1}{\rho} + \frac{1}{1-\overline{\rho}} + \frac{1}{\overline{\rho}}\right) = 2\sum_{\operatorname{Im}\rho>0}\left(\frac{1-\operatorname{Re}\rho}{|1-\rho|^2} + \frac{\operatorname{Re}\rho}{|\rho|^2}\right) \\
&= 2\sum_{\operatorname{Re}\rho = 1/2,\, \operatorname{Im}\rho>0}\left(\frac{1-\operatorname{Re}\rho}{|1-\rho|^2} + \frac{\operatorname{Re}\rho}{|\rho|^2}\right) + 2\sum_{\operatorname{Re}\rho \neq 1/2,\, \operatorname{Im}\rho>0}\left(\frac{1-\operatorname{Re}\rho}{|1-\rho|^2} + \frac{\operatorname{Re}\rho}{|\rho|^2}\right) \\
&= 2\sum_{k=1}^{\infty}\frac{1}{\frac{1}{4}+\gamma_k^2} + 2\sum_{\operatorname{Re}\rho \neq 1/2,\, \operatorname{Im}\rho>0}\left(\frac{1-\operatorname{Re}\rho}{|1-\rho|^2} + \frac{\operatorname{Re}\rho}{|\rho|^2}\right).
\end{aligned}$$

Thus, if we set $S := \{\rho \,|\, \xi(\rho) = 0, \operatorname{Re}\rho \neq \frac{1}{2}, \operatorname{Im}\rho > 0\}$, we have (using (2.6)),

$$\sum_{k=1}^{\infty}\frac{1}{\frac{1}{4}+\gamma_k^2} + \sum_{\rho\in S}\left(\frac{1-\operatorname{Re}\rho}{|1-\rho|^2} + \frac{\operatorname{Re}\rho}{|\rho|^2}\right) = 1 + \frac{\gamma}{2} - \frac{1}{2}\log(4\pi), \tag{2.11}$$

where we adopt the usual convention that any sum over the empty set is zero. Now, if the Riemann Hypothesis is valid, then $S = \emptyset$ in (2.11) and so (2.7) holds.

Conversely, suppose that (2.7) holds. If the Riemann Hypothesis is false, then $S \neq \emptyset$. But then, since $\operatorname{Re}\rho > 0$ and $1 - \operatorname{Re}\rho > 0$ for $\rho \in S$ (cf. Remark 2.1), the second summand on the left-hand side of (2.11) is positive. This contradicts (2.7) and thus, if (2.7) holds, then the Riemann Hypothesis is true. □

In the foregoing proof, it was convenient to evaluate the sum in (2.9) at $s = 0$. We remark that any other fixed value of s could have been used to carry out the proof.

We next establish another necessary and sufficient condition, analogous to Theorem 2.2, for the validity of the Riemann Hypothesis. To this end, we recall that the Riemann Ξ-function admits the Fourier integral representation (see, for example, [15, p. 255] or [3])

$$H(x) := \frac{1}{8}\Xi\left(\frac{x}{2}\right) = \int_0^\infty \Phi(t)\cos(xt)\,dt \tag{2.12}$$

where $\Xi(x) := \xi(\frac{1}{2} + ix)$ ([15, p. 16]) and

$$\Phi(t) := \sum_{n=1}^{\infty} \left(2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t}\right)\exp\left(-\pi n^2 e^{4t}\right) \quad (t \in \mathbb{R}). \tag{2.13}$$

The Riemann Hypothesis is the statement that all the zeros of $H(x)$ are real. (We mention here parenthetically that Riemann stated his hypothesis in terms of the reality of the zeros of the Ξ-function, see, for example, [10, Appendix, pp 299–305] for a translation of Riemann's famous memoir *"Ueber die Anzahl der Primzahlen unter einer gegebene Grösse".*) For the various properties of $\Phi(t)$ we refer the reader to [3, Theorem A] and [7, Theorem 2.1]. Now, $H(x)$ is an *even*, *real* entire function of order 1 (and of maximal type) all of whose zeros lie in the horizontal strip $|\operatorname{Im} z| < 1$. Thus, if $z = \alpha + i\beta$ ($|\operatorname{Im}\beta| < 1$) is a zero of $H(x)$, then $-\alpha + i\beta$, $-\alpha - i\beta$ and $\alpha - i\beta$ are also zeros of $H(x)$. Since $H(x) = \frac{1}{8}\Xi\left(\frac{x}{2}\right) = \frac{1}{8}\xi(\frac{1}{2} + i\frac{x}{2})$, we see that the zeros of $\xi(s)$ on the critical line are precisely of the form $\dfrac{1}{2} + i\dfrac{x_j}{2}$, where x_j is a real zero of $H(x)$. Hence, if x_j is a real zero of $H(x)$ in the right half-plane, then $x_j = 2\gamma_j$, where $\rho_j = \frac{1}{2} + i\gamma_j$ is a zero of $\xi(s)$ on the critical line in the upper half-plane, $\operatorname{Im} s > 0$. Finally, we also note that the Taylor series expansion of $H(x)$ about the origin can be written in the form

$$H(x) = \sum_{k=0}^{\infty} \frac{(-1)^k b_k}{(2k)!} x^{2k}, \tag{2.14}$$

where the b_k's are the moments defined by

$$b_k = \int_0^\infty t^{2k}\Phi(t)\,dt, \quad k = 0, 1, 2, \ldots. \tag{2.15}$$

Preliminaries aside, we are now in position to prove the following analog of Theorem 2.2.

Theorem 2.3. *Let* $\{x_j\}_{j=1}^{\infty}$, $0 < x_1 \le x_2 \le \cdots$, *denote the real zeros of* $H(x)$ *in the right half-plane. Then the Riemann Hypothesis is true if and only if*

$$\sum_{j=1}^{\infty} \frac{1}{x_j^2} = \frac{b_1}{2b_0} \qquad \text{or} \qquad \sum_{j=1}^{\infty} \frac{1}{\gamma_j^2} = \frac{2b_1}{b_0} \tag{2.16}$$

where b_k *denotes the* k^{th} *moment of* $\Phi(t)$ *defined in (2.15) and* $x_j = 2\gamma_j$ $(j = 1, 2, 3 \cdots)$.

Proof. By the Hadamard factorization theorem $H(z)$ can be represented in the form

$$H(z) = H(0) \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{z_k^2}\right), \qquad \left(\sum_{k=1}^{\infty} \frac{1}{|z_k|^2} < \infty\right), \tag{2.17}$$

where $H(0) > 0$ (since $\Phi(t) > 0$ for all $t \in \mathbb{R}$) and where the zeros of $H(z)$ are numbered according to increasing moduli, i.e., $0 < |z_1| \le |z_2| \le \cdots$. Now let $S := \{z_k = \alpha_k + i\beta_k \,|\, H(z_k) = 0, \operatorname{Im} \beta_k \neq 0\}$ and let $\{x_j\}_{j=1}^{\infty}$, denote the real zeros of $H(x)$ in the right half-plane. Then (2.17) can be expressed in the form

$$H(z) = H(0) \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{x_j^2}\right) \prod_{z_k \in S} \left(1 - \frac{z^2}{z_k^2}\right), \tag{2.18}$$

where the product over the zeros in S is defined to be 1, if S is the empty set. Now using (2.18), a calculation shows that the derivative of the logarithmic derivative of $H(z)$ is

$$\frac{H(z)H''(z) - (H'(z))^2}{(H(z))^2} = -\sum_{j=1}^{\infty} \left(\frac{1}{(z - x_j)^2} + \frac{1}{(z + x_j)^2}\right) - \sum_{z_k \in S} \left(\frac{1}{(z - z_k)^2} + \frac{1}{(z + z_k)^2}\right). \tag{2.19}$$

Next, using (2.15), (2.19) and the fact that $H(z)$ is an even function (so that $H'(0) = 0$), we find that

$$\left(\frac{H'}{H}\right)'(0) = \frac{H''(0)}{H(0)} = -\frac{b_1}{b_0}. \tag{2.20}$$

On the other hand, setting $z = 0$ in (2.19), we obtain

$$\left(\frac{H'}{H}\right)'(0) = -2\sum_{j=1}^{\infty} \frac{1}{x_j^2} - 2 \sum_{\substack{\alpha_k + i\beta_k \in S \\ \operatorname{Im} \beta_k > 0}} \left(\frac{\alpha_k^2 - \beta_k^2}{|\alpha_k + i\beta_k|^4}\right)$$

or (using (2.20))

$$\sum_{j=1}^{\infty} \frac{1}{x_j^2} + \sum_{\substack{\alpha_k + i\beta_k \in S \\ \operatorname{Im} \beta_k > 0}} \frac{\alpha_k^2 - \beta_k^2}{|\alpha_k + i\beta_k|^4} = \frac{b_1}{2b_0}. \tag{2.21}$$

But if $S \neq \emptyset$, then the second summand on the left-hand side of (2.21) is positive since all the zeros of $H(z)$ lie in the strip $|Im\, z| < 1$ and $H(z)$ has no zeros ([15, p. 389] or see Section 3 below) in the closed unit square centered at the origin. Thus, we infer from (2.21) that the Riemann Hypothesis is true if and only if (2.16) holds. $\square$

3. Some numerical calculations and density considerations.

With the advent of fast computers there has been a fruitful interplay between mathematical analysis and high-precision computations. In this section our goal is merely to indicate some numerical work pertaining to Theorem 2.2 and Theorem 2.3. All the calculations reported here have been carried out on a Sun Ultra 5 workstation using Mathematica 4.0.1. As before, we let $0 < \gamma_1 \leq \gamma_2 \leq \cdots$ denote the ordinates of the zeros of $\xi(s)$ on the critical line in the upper half-plane $\mathrm{Re}\, s > 0$. Now it is known that the first 1.5×10^9 zeros of $\xi(s)$ lie on the critical line [13]. Thanks to Dr. A. M. Odlyzko, the first 100,000 zeros of the Riemann zeta function, accurate to within 3×10^{-9}, are available on the internet (*http : //www.dtc.umn.edu/ odlyzko/zeta_tables/*). Also, we gratefully acknowledge Dr. H. J. J. te Riele for providing one of the authors with the first 500 moments (see the b_k's defined by (2.15)) accurate to within 200 places. (These moments have also been calculated in the Department of Mathematical sciences at Kent State University, using extended-precision floating-point numbers and 110 digits of precision [3, Table 4.1].)

To begin with, it may be of interest to list here a few values of the γ's: $\gamma_1 = 14.134\ 725 \cdots$, $\gamma_{10,000} = 9877.782\ 654 \cdots$, $\gamma_{20,000} = 18046.464\ 296 \cdots$, $\gamma_{100,000} = 74920.827\ 498 \cdots$. Next, set

$$s_n := \sum_{k=1}^{n} \frac{1}{\frac{1}{4} + \gamma_k^2} \qquad \text{and} \qquad \tau_n := \sum_{j=1}^{n} \frac{1}{\gamma_j^2}.$$

Thus, by Theorem 2.2, if the Riemann Hypothesis is valid, then

$$s_\infty = 1 + \frac{\gamma}{2} - \frac{1}{2}\log(4\pi) = 0.023\ 095\ 708 \cdots . \tag{3.1}$$

Also, by Theorem 2.3, if the Riemann Hypothesis is valid, then

$$\tau_\infty = 4\left(\frac{b_1}{2b_0}\right) = \left(\frac{2b_1}{b_0}\right) = 0.023\ 104\ 993 \cdots , \tag{3.2}$$

where we recall that the x_j's in Theorem 2.3 are related to the γ_j's by $x_j = 2\gamma_j$ and where (cf. [5, Lemma 4.1])

$$b_0 = 0.023\ 073\ 645 \cdots \qquad \text{and} \qquad b_1 = 0.000\ 717\ 873 \cdots .$$

On the other hand, using the values of the γ_k's, $1 \leq k \leq 100,000$, made available by Dr. Odlyzko, we find that

$$s_{100,000} = 0.023\ 073\ 645 \cdots \qquad \text{and} \qquad \tau_{100,000} = 0.023\ 082\ 929 \cdots , \tag{3.3}$$

where we have used several "flags" and procedures (which we will not detail here) in an attempt to control the round-off errors. Thus, with the remainders defined by

$$R_n(s) := s_\infty - s_n \qquad \text{and} \qquad R_n(\tau) := \tau_\infty - \tau_n, \tag{3.4}$$

we find (using (3.1), (3.2) and (3.3)) that

$$R_{100,000}(s) = 2.206\ 306 \cdots \times 10^{-5} \qquad \text{and} \qquad R_{100,000}(\tau) = 3.134\ 776 \cdots \times 10^{-5}. \tag{3.5}$$

It may be of interest to combine the foregoing numerical results with certain theoretical estimates involving the density of the (non-trivial) zeros of Riemann zeta function. Let $N(T)$ $(T > 0)$ denote the number of zeros of the Riemann zeta function $\zeta(s)$, where $s = \sigma + it$, in the rectangle $0 \leq \sigma \leq 1$ and $0 \leq t \leq T$. The distribution of zeros can then be studied with the aid of formulae involving $N(T)$. Perhaps the best known such formula is the Riemann–von Mangoldt formula, that is, $N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log T)$ (as $T \to \infty$). We will use the following refinement of the Riemann–von Mangoldt formula that was proved by Backlund in 1918 [10, p. 134] or [4, Theorem C].

Theorem 3.1. (Backlund's Density Theorem, [1]) $N(T)$ satisfies

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8} + e(T), \tag{3.6}$$

where

$$|e(T)| < U(T) := 0.137 \log T + 0.443 \log\log T + 4.35 \qquad (T \geq 2). \tag{3.7}$$

Lemma 3.2. *Let*

$$q(T) := \frac{1}{2\pi}\left[(T+1)\log\left(1 + \frac{1}{T}\right) - 1 - \log 2\pi\right]. \tag{3.8}$$

Then for all $T \geq T_0 \geq 2$

$$N(T+1) - N(T) \leq \left[\frac{1}{2\pi} + \frac{q(T_0)}{\log T} + \frac{2U(T_0+1)}{\log T_0}\right] \log T, \tag{3.9}$$

where $N(T)$ and $U(T)$ are defined by (3.6) and (3.7) respectively.

Proof. Let

$$f(T) := \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8}.$$

Then algebraic simplifications yield

$$f(T+1) - f(T) = \frac{\log T}{2\pi} + q(T).$$

Since $U(T) > 0$ and $U(T)$ is increasing for all $T \geq T_0 \geq 2$, we obtain the following upper estimate (with $e(T)$ given by (3.6))

$$N(T+1) - N(T) = f(T+1) - f(T) + e(T+1) - e(T) \leq \frac{\log T}{2\pi} + q(T) + 2U(T+1). \tag{3.10}$$

Now elementary, albeit somewhat involved, considerations show that the function $U(T+1)/\log T$ is a decreasing function for $T \geq 2$. Also, since $q(T) < 0$ and $q'(T) < 0$, using (3.10) we obtain the desired inequality (3.9). $\square$

Proposition 3.3. *Let $\rho_k = \frac{1}{2} + i\gamma_k$, $(k = 1, 2, 3 \dots)$, $0 < \gamma_k \le \gamma_{k+1} \le \cdots$, denote the zeros of $\xi(s)$ on the critical line in* Im $s > 0$. *Fix a positive integer m and let $T_0 = [\gamma_m]$, where $[x]$ denotes the greatest integer $\le x$. Then*

$$\sum_{j=m}^{\infty} \frac{1}{\gamma_j^2} \le C_1 \frac{1 + \log(T_0 - 1)}{T_0 - 1} + \frac{C_2}{T_0 - 1}, \tag{3.11}$$

where

$$C_1 := C_1(T_0) = \frac{1}{2\pi} + \frac{2U(T_0 + 1)}{\log T_0} \quad \text{and} \quad C_2 := C_2(T_0) = q(T_0), \tag{3.12}$$

and the functions U and q are defined by (3.7) and (3.8), respectively.

Proof. By Lemma 3.2 and the integral test, we obtain the estimates

$$\begin{aligned}
\sum_{j=m}^{\infty} \frac{1}{\gamma_j^2} \le & \sum_{j=T_0}^{\infty} \sum_{j \le \gamma_k < j+1} \frac{1}{\gamma_k^2} \\
\le & \sum_{j=T_0}^{\infty} \frac{N(j+1) - N(j)}{j^2} \\
\le & \sum_{j=T_0}^{\infty} \frac{\log j}{j^2} \left(C_1 + \frac{C_2}{\log j} \right) \\
\le & C_1 \int_{T_0-1}^{\infty} \frac{\log v}{v^2}\, dv + C_2 \int_{T_0-1}^{\infty} \frac{dv}{v^2} \\
= & C_1 \frac{1 + \log(T_0 - 1)}{T_0 - 1} + \frac{C_2}{T_0 - 1}.
\end{aligned}$$

□

In light of Theorem 2.3, it would be of interest to evaluate the sum $\sum_{j=1}^{m} \frac{1}{\gamma_j^2}$ (3.11) for various "large" values of m, say, $m = 10^6$ or $m = 10^8$. Here, we will report the results for $m = 100,000$. Referring back to the beginning of this section, we know that $\gamma_{100,000} = 74920.827\ 498 \cdots$. Thus, with $T_0 := 74920$, a calculation shows that

$$C_1(T_0) = 1.399\ 142\ 523 \cdots \qquad \text{and} \qquad C_2(T_0) = 0.292\ 506\ 157 \cdots$$

and so in this case the upper bound is

$$C_1 \frac{1 + \log(T_0 - 1)}{T_0 - 1} + \frac{C_2}{T_0 - 1} = 0.000\ 224\ 386 \cdots . \tag{3.13}$$

If $m := 1,048,449,114$, then it is known that the m^{th} zero $\frac{1}{2} + i\gamma_m$ has imaginary part $\gamma_m = 3.888\ 588\ 853 \cdots \times 10^8$ (see [4, (2.8)] or [6]). Therefore, if we set $T_0 := 388,858,885$, then the upper bound in (3.11) becomes

$$C_1 \frac{1 + \log(T_0 - 1)}{T_0 - 1} + \frac{C_2}{T_0 - 1} = 5.304\ 193\ \cdots \times 10^{-8},$$

which, as expected, is a substantial improvement over the upper bound obtained in (3.13).

We conclude this paper with an open problem that bears relevance to the investigation of the distribution of zeros of $H(x)$ defined by (2.12). For $R > 0$, let

$$H_R(x) := \int_0^R \Phi(t)\cos(xt)\,dt, \tag{3.14}$$

where the kernel $\Phi(t)$ is defined by (2.13). Then, $H_R(x)$ is an even, real entire function of order 1 (and of exponential type). Since the zeros of $H(x) := H_\infty(x)$ all lie in the horizontal strip $|\operatorname{Im} z| < 1$, it follows from Hurwitz' theorem that for all R, sufficiently large, the zeros of $H_R(x)$ also lie in this strip. We can establish a result analogous to Theorem 2.3, where now the moments are

$$b_k(R) = \int_0^R t^{2k}\Phi(t)\,dt, \quad k = 0, 1, 2, \ldots. \tag{3.15.}$$

It is known that the moments $b_k(R)$ satisfy the *Turán inequalities* (cf. [2, Corollary 2.9] and [7]); that is,

$$b_k(R)^2 \geq \frac{2k-1}{2k+1} b_{k-1}(R) b_{k+1}(R), \qquad (k \geq 1)$$

These are the simplest necessary conditions for $H_R(x)$ to have only real zeros. If R is “small”, $R \in (0,\, 0.11]$, then it is known that $H_R(x)$ has only real zeros [8, Theorem 3.6]. Does $H_R(x)$ have only real zeros for other values of R?

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