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# FLUTTER PANEL EQUATION IN NON CYLINDRICAL DOMAIN

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Abstract. The motion for small-amplitude of a thin elastic panel with both linear-weak and linear-strong damping are investigated in one-dimensional domain with moving boundary. Existence and uniqueness of global solutions are shown. Asymptotic behavior of energies are also established.

**A.M.S. (MOS) Subject Classification Codes.** 35B40, 35G30, 35K30.

**Key Words and Phrases.** Flutter panel equation, non cylindrical domain, existence of solutions, uniqueness, asymptotic behavior.

## 1. Introduction

The one-dimensional equation of motion of a thin panel fixed at both ends and undergoing cylindrical bending can be written as

(1.1)
$$\begin{aligned}&u_{tt}(x,t)-\Big(\zeta_0+\zeta_1\int_\Omega|u_x(t)|^2dx+\sigma\int_\Omega u_x(t)u_{xt}(t)dx\Big)u_{xx}(x,t)+\\&\sqrt{\rho}\delta u_t(x,t)+\rho u_x(x,t)+u_{xxxx}(x,t)+\nu u_{xxxxt}(x,t)=0\ \text{ in }\ Q,\end{aligned}$$

where $u$ is the plate transverse displacement, $x$ is the spatial coordinate in the direction of the fluid flow, $t$ is the time, the viscoelastic structural damping terms are denotes by $\nu$ and $\sigma$, $\sqrt{\rho}\delta$ represents an aerodynamic damping, $\zeta_1$ is the nonlinear stiffness of membrane, $\rho$ is the aerodynamic pressure, $\zeta_0$ is an in-plane tensile load, and $(x,t)$ belongs to $Q=\Omega\times[0,\infty[$ with $\Omega=(0,1)$. All quantities are physically non-dimensionalized and $\nu,\ \sigma,\ \delta,\ \zeta_1$ are fixed positive and $\zeta_0,\ \rho$ are fixed non-negative.

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Equation (1.1) arise in a wind tunnel experiment for a panel at supersonic speeds, for a derivation of this model see, for instance, Dowell [9].

Equation (1.1) include in its mathematical formulation some special situations in elasticity. Omitting the dissipative terms and the aerodynamic pressure $\rho$ we have the beam equation

$$u_{tt}(x,t) - \Big(\zeta_0 + \zeta_1 \int_\Omega |u_x(t)|^2 dx\Big) u_{xx}(x,t) + u_{xxxx}(x,t) = 0 \text{ in } Q. \tag{1.2}$$

The beam equation (1.2) has been studied by several authors, among them, Ball [2], Biler [3], Brito [4], Medeiros [15] and Pereira [18].

By omission of the term $u_{xxxx}$ in (1.2) we have the well known Kirchhoff equation

$$u_{tt}(x,t) - \Big(\zeta_0 + \zeta_1 \int_\Omega |u_x(t)|^2 dx\Big) u_{xx}(x,t) = 0 \text{ in } Q. \tag{1.3}$$

Equation (1.3) has also been extensively studied by several authors in both $\{1, 2, \cdots, n\}$-dimensional cases and general mathematical model over a Hilbert space $H$. Local solutions and global solutions has been shown to exist in several physical-mathematical contexts. Among them, Arosio-Spagnolo [1], Clark [5, 6], Dickey [7], Dix [8], Kirchhoff [13], Ferrel-Medeiros [10], Matos-Pereira [14], Pohozaev [19].

The historical evolution of the equation (1.3) with an excellent bibliography was presented by Medeiros at al [16], and a new model associated with Kirchhoff equation (1.3) was also deduced, which we denominate Medeiros-Kirchhoff equation, namely

$$u_{tt}(x,t) - \Big(f(t) + g(t) \int_{\Omega_t} |u_x(t)|^2 dx\Big) u_{xx}(x,t) = 0 \text{ in } Q. \tag{1.4}$$

where the coefficients $f(t) = \dfrac{\tau_0}{m} + \dfrac{k\big(\gamma(t) - \gamma_0\big)}{m\gamma_0}$ and $g(t) = \dfrac{k}{2m\gamma(t)}$ are defined in §1 of Medeiros at al [16], $\Omega_t$ and $Q_t$ are defined by

$$\Omega_t = \{x \in \mathbb{R} : \alpha(t) < x < \beta(t) \quad \text{and} \quad t \in [0, \infty[\,\},$$
$$Q_t = \big\{(x,t) \in \mathbb{R}^2 : x \in \Omega_t \quad \text{and } t \in [0, \infty[\,\big\},$$

$\alpha$ and $\beta$ are real functions, $\gamma = \beta - \alpha$, $\gamma_0 = \gamma(0)$, and $\tau_0$, $m$, and $k$ are physical constants. The boundary of $Q_t$ is defined by

$$\Sigma_t = \bigcup_{0<t<\infty} \Big(\alpha(t) \times \beta(t), t\Big).$$

Medeiros at al [16] proved that the mixed problem associated with the equation (1.4) has local solutions. Besides, whether the equation (1.4) is considered over the action of a weak-internal damping $u_t(x,t)$, and the initial data are bounded, then the total energy of the system is asymptotically stable. All these results were got supposing $\Omega_t$ increasing.

The goal of this paper is to establish an analysis qualitative of the solutions of the equation (1.1) inside of a non-cylindrical domain $Q_t$ in two different contexts. In the equation (1.1) we can see two linear internal damping, that is, $\sqrt{\rho}\delta u_t$ and $\nu u_{xxxxt}$. Motivated for this physical situation, firstly we neglect in the equation (1.1) the linear-strong damping, that is, we assume the aerodynamic constant $\nu$ is negligible. Thus, we have the weak-damped equation

(1.5)
$$u_{tt}(x,t) - \Big(\zeta_0 + \zeta_1 \int_{\Omega_t} |u_x(t)|^2 dx + \sigma \int_{\Omega_t} u_x(t) u_{xt}(t) dx\Big) u_{xx}(x,t) + \sqrt{\rho}\delta u_t(x,t) + u_{xxxx}(x,t) = 0 \text{ in } Q_t.$$

In the second situation we omit the linear-weak damping, assuming $\rho$ is now negligible. This way, we have the strong damped equation

(1.6)
$$u_{tt}(x,t) - \Big(\zeta_0 + \zeta_1 \int_{\Omega_t} |u_x(t)|^2 dx + \sigma \int_{\Omega_t} u_x(t) u_{xt}(t) dx\Big) u_{xx}(x,t) + u_{xxxx}(x,t) + \nu u_{xxxxt}(x,t) = 0 \text{ in } Q_t.$$

We will make this choice to exalt the different difficulties technique caused by the effect of each damping when the equations are considerate in non-cylindrical domain $Q_t$. Perhaps, these difficulties happen due to the used method- Galerkin's method. But in this case, we believe that such method is the most natural, because the equations (2.4) and (3.10) obtained from the equations (1.5) and (1.6) respectively, through the variable change (1.7) have variable coefficients.

Comparing the equations (1.5) and (1.6) with the equation (1.4) we can conclude that $\zeta_0$ and $\zeta_1$ depend continuously on the lenght of domain $\Omega_t$. Therefore, depend on the variable time $t$. But, to simplify our analysis we will assume $\zeta_0$ and $\zeta_1$ constants.

It will be considerate for the equations (1.5) and (1.6) the fully clamped boundary conditions

$$u(x,t) = u_x(x,t) = 0 \text{ on } \Sigma_t.$$

The strategy to obtain existence of solutions to the mixed problems associated with equations (1.5) and (1.6) in a non-cylindrical domain is to transform such problems into another, defined over a cylindrical domain whose sections are not time dependent. The technique that transfoms the equation from moving boundary into fixing boundary, which we will also use, was initialy utilized in nonlinear-hyperbolic equation by Ferrel-Medeiros [10] and Medeiros at al [16]. That is, we will use a suitable change of variable

$$v(y,t) = u(x,t), \tag{1.7}$$

where $y = \frac{x-\alpha(t)}{\gamma(t)}$. We can see from definition of $\Omega_t$ that $y \in (0,1)$.

The results of existence of solutions are obtained assuming the hypothesis

$$\alpha, \quad \beta \quad \text{belong to} \quad W^{3,1}([0, \infty[; \mathbb{R}). \tag{1.8}$$

Existence of global solutions within of fixed domain for the mixed problem associated with equation (1.1) was investigated by Hughes & Marsden [12]. With respect to asymptotic stability, Holmes & Marsden [11] supposing some restrictive hypotheses about the aerodynamic pressure $\rho$ concluded that the derivative of solution is negative.

This paper is divided in two sections. The section 2 is devoted to investigate the existence and uniqueness of strong global solutions of the mixed problem associated with the equation (1.5). Furthermore, the asymptotic behavior of solutions is proved directly inside of non-cylindrical domain. In the section 3 we prove the existence and uniqueness of weak global solutions for a mixed problem associated with the equation (1.6), provided that the initial data are bounded. Besides, the exponential decay for energy, due to technical difficulties, initially will be shown for approximate solutions inside of the fixed domain $Q$. Made this, we will obtain the same result for the energy inside of the non-cylindrical domain $Q_t$.

## 2. Flutter panel with linear-weak damping

### 2.1 Existence and uniqueness of solutions

In this section we consider the initial data- and boundary- value problem

(2.1)
$$u_{tt}(x,t) - \Big(\zeta_0 + \zeta_1 \int_{\Omega_t} |u_x(t)|^2 dx + \sigma \int_{\Omega_t} u_x(t) u_{xt}(t) dx\Big) u_{xx}(x,t) + \sqrt{\rho}\delta u_t(x,t) + u_{xxxx}(x,t) = 0 \quad \text{in } Q_t,$$

(2.2)
$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega_0,$$

(2.3)
$$u(x,t) = u_x(x,t) = 0 \quad \text{on } \Sigma_t.$$

Tanking into account (1.7) into the system (2.1)-(2.3) yields

$$\begin{aligned} &v_{tt} + a_1(y,t)v_y + a_2(y,t)v_{yt} + a_3(y,t)v_{yy} + \\ &b_1(t)\Big(\int_\Omega |v_y|^2 dy\Big) v_{yy} + b_2(t)\Big(\int_\Omega v_y v_{yt} dy\Big) v_{yy} + \\ &\Big(\int_\Omega a_4(y,t) v_y v_{yy} dy\Big) v_{yy} + b_3(t) v_{yyyy} + \sqrt{\rho}\delta v_t = 0 \ \text{ in } Q, \end{aligned} \tag{2.4}$$

$$v(y,0) = v_0(y), \quad v_t(y,0) = v_1(y) \ \text{ with } 0 < y < 1, \tag{2.5}$$

$$v(0,t) = v_y(0,t) = 0, \quad v(1,t) = v_y(1,t) = 0 \ \text{ for all } t \geq 0, \tag{2.6}$$

where the coefficients of equation (2.4) are given by

$$\begin{aligned} &a_1(y,t) = \frac{1}{\gamma^2}\left\{ (2\gamma' - \sqrt{\rho}\delta\gamma)(\alpha' + y\gamma') - \gamma(\alpha'' + y\gamma'') \right\}, \\ &a_2(y,t) = -\frac{2}{\gamma}(\alpha' + y\gamma'), \quad a_3(y,t) = \frac{1}{\gamma^2}\Big\{ (\alpha' + y\gamma')^2 - \zeta_0 \Big\}, \\ &a_4(y,t) = \frac{\sigma}{\gamma^5}(\alpha' + y\gamma'), \quad b_1(t) = \frac{1}{\gamma^4}(\sigma\gamma' - \zeta_1\gamma), \\ &b_2(t) = -\frac{\sigma}{\gamma^3}, \quad b_3(t) = \frac{1}{\gamma^4}. \end{aligned} \tag{2.7}$$

Thanks to variable chance (1.7) the function $u$ is a solution of the mixed problem (2.1)-(2.3) if, and only if $v$ is a solution of the mixed problem (2.4)-(2.6). Thus, we have that the two following theorems are equivalents

**Theorem 2.1.** *Let $\alpha$ and $\beta$ be satisfying the hypothesis (1.8). If the initial data $u_0$ belongs to $H_0^2(\Omega_0)\cap H^4(\Omega_0)$ and $u_1$ belongs to $H_0^2(\Omega_0)$, then the unique strong solution $u$ of the mixed problem (2.1)-(2.3) lies in*

$$u \in L^\infty\left(0,\infty; H_0^2\left(\Omega_t\right)\cap H^4\left(\Omega_t\right)\right), \tag{2.8}$$

$$u_t \in L^\infty\left(0,\infty; H_0^2\left(\Omega_t\right)\right), \tag{2.9}$$

$$u_{tt} \in L^\infty\left(0,\infty; L^2\left(\Omega_t\right)\right), \tag{2.10}$$

*and the equation (2.1) is verified in the sense of $L^\infty\left(0,\infty; L^2\left(\Omega_t\right)\right)$.*

**Theorem 2.2.** *We assume $\alpha$ and $\beta$ as in the Theorem 2.1, and*

$$-\frac{1}{8} \le \frac{\gamma'(t)}{\gamma(t)} \le \min\left\{\frac{1}{8}, \frac{2\zeta_1}{3\sigma}\right\} \quad \textit{for all } t\ge 0. \tag{2.11}$$

*Besides, if $\alpha'(0)=\beta'(0)=0$, then $v_0$ belongs to $H_0^2(\Omega)\cap H^4(\Omega)$, $v_1$ belongs to $H_0^2(\Omega)$, there exists a unique strong solution $v$ of the mixed problem (2.4)-(2.6) lies in*

$$v \in L^\infty\left(0,\infty; H_0^2(\Omega)\cap H^4(\Omega)\right), \tag{2.12}$$

$$v_t \in L^\infty\left(0,\infty; H_0^2(\Omega)\right), \tag{2.13}$$

$$v_{tt} \in L^\infty(0,\infty; L^2\left(\Omega\right)), \tag{2.14}$$

*and the equation (2.4) is verified in the sense of $L^\infty(0,\infty; L^2(\Omega))$.*

To show the existence of solutions of the problem (2.4)-(2.6) we will use the Galerkin's methods. Thus, by $V_m=[w_1,w_2,w_3,\cdots,w_m]$ we will denote the subspace of $H_0^2(\Omega)$ spanned by the $m$ first vectors of $(w_j)_{j\in\mathbb{N}}$, where $(w_j)_{j\in\mathbb{N}}$ is orthonormalized and complete in $L^2(\Omega)$. We assume that initial data in $V_m$ satisfy

$$\begin{aligned} v_0^m(y,0) = v_0^m &\longrightarrow v_0 \text{ strongly in } H_0^2(\Omega)\cap H^4(\Omega),\\ v_{0t}^m(y,0) = v_1^m &\longrightarrow v_1 \text{ strongly in } H_0^2(\Omega). \end{aligned} \tag{2.15}$$

**Proof of Theorem 2.2 - Existence.** The approximate problem associated with (2.4)-(2.6) is given, for all $\phi\in V_m$, by the following system of ordinary differential equations

$$\begin{aligned} &(v_{tt}^m(t),\phi) + \left(a_1(y,t)v_y^m(t),\phi\right) + \left(a_2(y,t)v_{yt}^m(t),\phi\right) + \\ &\sqrt{\rho}\delta\left(v_t^m(t),\phi\right) + b_3(t)\left(v_{yy}^m(t),\phi_{yy}\right) + \\ &b_2(t)\left(\int_\Omega v_y^m(t)v_{yt}^m(t)dy\right)\left(v_{yy}^m(t),\phi\right) + \left(a_3(y,t)v_{yy}^m(t),\phi\right) + \\ &b_4(t)\left(\int_\Omega \left|v_y^m(t)\right|^2 dy\right)\left(v_{yy}^m(t),\phi\right) = 0,\\ &v^m(y,0) = v_0^m \text{ and } v_t^m(y,0) = v_1^m, \end{aligned} \tag{2.16}$$

where $b_4(t) = \frac{1}{2\gamma^5(t)}\left\{3\sigma\gamma'(t) - 2\zeta_1\gamma(t)\right\}$ was determinate by definition of the $b_1(t)$, $a_4(y,t)$ and of the identity $\int_\Omega a_4(y,t)v_y^m(t)v_{yy}^m(t)dy = \frac{\sigma\gamma'(t)}{2\gamma^5(t)}\int_\Omega \left|v_y^m(t)\right|^2 dy$.

Henceforth $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. Note that $b_4(t)$ is not a positive function in view of hypothesis (2.11).

Under the assumption (2.15), we have that the system (2.16) has a local solution $v^m$ on the interval $[0, t_m[$. This interval will be extended to any interval $[0, \infty[$ thanks to estimates below. To simplify our analysis we will write, throughout this paper, $v$ instead of $v^m$. Our next task is to determine estimative on $v$ in order to take the limit in the system (2.16).

- **Estimate I.** Substituting $\phi$ by $2v_t(t)$ into $(2.16)_1$ and denoting by $\widehat{M}$ the function $\widehat{M}(\xi) = \int_0^\xi M(\tau)d\tau$ we get

$$
\begin{aligned}
&\frac{d}{dt}\left|v_t(t)\right|^2 - b_4(t)\frac{d}{dt}\widehat{M}\Big(\left|v_y(t)\right|^2\Big) + b_3(t)\frac{d}{dt}\left|v_{yy}(t)\right|^2 + \\
&2\sqrt{\rho}\delta\left|v_t(t)\right|^2 + 2\left(a_2(y,t)v_{yt}(t), v_t(t)\right) - \\
&2b_2(t)\Big(\int_\Omega v_y(t)v_{yt}(t)dy\Big)^2 = \\
&-2\left(a_1(y,t)v_y(t), v_t(t)\right) - 2\left(a_3(y,t)v_{yy}(t), v_t(t)\right),
\end{aligned}
\tag{2.17}
$$

where $|\cdot|^2$ denotes the norm in $L^2(\Omega)$. Denoting by

$$
b_5(t) = \sup_{y\in[0,1]}\left|a_1(y,t)\right|_{\mathbb{R}}, \quad b_6(t) = \sup_{y\in[0,1]}\left|a_3(y,t)\right|_{\mathbb{R}},
$$

observing the relations

$$
2(a_2(y,t)v_{yt}(t), v_t(t)) = \frac{2\gamma'(t)}{\gamma(t)}\left|v_t(t)\right|^2, \quad -2b_2(t)\left(\int_\Omega v_y(t)v_{yt}(t)dt\right)^2 \geq 0.
$$

we get from (2.17) that

$$
\begin{aligned}
&\frac{d}{dt}E_1(t) + 2\Big(\sqrt{\rho}\delta + \frac{\gamma'(t)}{\gamma(t)}\Big)\left|v_t(t)\right|^2 \leq \\
&\left|b_4'(t)\right|\widehat{M}\left(\left|v_y(t)\right|^2\right) + \left|b_3'(t)\right|\left|v_{yy}(t)\right|^2 + \\
&2b_5(t)\left|v_y(t)\right|\left|v_t(t)\right| + 2b_6(t)\left|v_{yy}(t)\right|\left|v_t(t)\right|,
\end{aligned}
\tag{2.18}
$$

where $E_1$ is the function

$$
E_1(t) = \left|v_t(t)\right|^2 - b_4(t)\widehat{M}\left(\left|v_y(t)\right|^2\right) + b_3(t)\left|v_{yy}(t)\right|^2.
$$

The two last terms of right-hand side of (2.18) satisfy

$$
2b_5(t)\left|v_y(t)\right|\left|v_t(t)\right| \leq \frac{b_5^2(t)}{\sqrt{\rho}\delta}\left|v_{yy}(t)\right|^2 + \sqrt{\rho}\delta\left|v_t(t)\right|^2,
$$

$$
2b_6(t)\left|v_{yy}(t)\right|\left|v_t(t)\right| \leq \frac{b_6^2(t)}{\sqrt{\rho}\delta}\left|v_{yy}(t)\right|^2 + \sqrt{\rho}\delta\left|v_t(t)\right|^2.
$$

Taking into account the last two inequalities into (2.18) yields

$$\frac{d}{dt}E_1(t) \le b_7(t)\left\{|v_t(t)|^2 + \widehat{M}\left(|v_y(t)|^2\right) + |v_{yy}(t)|^2\right\}, \tag{2.19}$$

where

$$b_7(t) = 2\frac{|\gamma'(t)|}{\gamma(t)} + |b_4'(t)| + |b_3'(t)| + \frac{1}{\sqrt{\rho}\delta}\left[b_5^2(t) + b_6^2(t))\right].$$

Integrating (2.19) from $0$ to $t$ and observing that $E_1(0)$ is bounded, yields

$$E_1(t) \le K_0 + \int_0^t b_7(s)\left\{|v_s(s)|^2 + \widehat{M}\left(|v_y(s)|^2\right) + |v_{yy}(s)|^2\right\} ds, \tag{2.20}$$

where $K_0$ is a real positive constant that depends only on the initial data. Using the hypotheses (1.8), (2.11), and definition of coefficients in (2.7) we obtain from (2.20) and by using of Gronwall's inequality that

$$\begin{aligned} &(v^m) \text{ is bounded in } L^\infty\left(0,\infty; H_0^2(\Omega)\right), \\ &(v_t^m) \text{ is bounded in } L^\infty\left(0,\infty; L^2(\Omega)\right). \end{aligned} \tag{2.21}$$

- **Estimate II.** Differentiating the equation $(2.16)_1$ with respect to $t$, replacing $\phi$ by $2v_{tt}(t)$, using the identities

$$2\left(a_2(y,t)v_{ytt}(t), v_{tt}(t)\right) = \frac{2\gamma'(t)}{\gamma(t)}|v_{tt}(t)|^2,$$

$$2b_3'(t)\left(v_{yy}(t), v_{yytt}(t)\right) = 2b_3'(t)\frac{d}{dt}\left(v_{yy}(t), v_{yyt}(t)\right) - 2b_3'(t)|v_{yyt}(t)|^2,$$

and denoting by $E_2$ the function

$$E_2(t) = |v_{tt}(t)|^2 + 2b_3'(t)\left(v_{yy}(t), v_{yyt}(t)\right) + b_3(t)|v_{yyt}(t)|^2,$$

we get

$$\begin{aligned} &\frac{d}{dt}E_2(t) + 2\Big(\sqrt{\rho}\delta + \frac{\gamma'(t)}{\gamma(t)}\Big)|v_{tt}(t)|^2 = \\ &2b_3''(t)\left(v_{yy}(t), v_{yyt}(t)\right) + 3b_3'(t)|v_{yyt}(t)|^2 - \\ &2\left(a_{1t}(y,t)v_y(t), v_{tt}(t)\right) - 2b_2(t)\left(\int_\Omega v_y(t)v_{ytt}(t)dy\right)\left(v_{yy}(t), v_{tt}(t)\right) - \\ &2\left(a_1(y,t)v_{yt}(t), v_{tt}(t)\right) - 2b_2(t)\left(\int_\Omega v_y(t)v_{yt}(t)dy\right)\left(v_{yyt}(t), v_{tt}(t)\right) - \\ &2\left(a_{2t}(y,t)v_{yt}(t), v_{tt}(t)\right) - 2b_4'(t)\left(\int_\Omega |v_y(t)|^2 dy\right)\left(v_{yy}(t), v_{tt}(t)\right) - \\ &2\left(a_{3t}(y,t)v_{yy}(t), v_{tt}(t)\right) - 2b_4(t)\left(\int_\Omega |v_y(t)|^2 dy\right)\left(v_{yyt}(t), v_{tt}(t)\right) - \end{aligned}$$

$$2\left(a_3(y,t)v_{yyt}(t),v_{tt}(t)\right)-2b_2(t)\left(\int_\Omega |v_{yt}(t)|^2\,dy\right)(v_{yy}(t),v_{tt}(t))-$$
$$2\left\{2b_4(t)+b_2'(t)\right\}\left(\int_\Omega v_y(t)v_{yt}(t)dy\right)(v_{yy}(t),v_{tt}(t)).$$

Hence, proceeding as estimate I and using (2.21), hypothesis (1.8) and usual inequalities we obtain after some simplifications that

$$\frac{d}{dt}E_2(t)\le b_8(t)+b_9(t)\Big\{\,|v_{tt}(t)|^2+|v_{yyt}(t)|^2\,\Big\}, \tag{2.22}$$

where $b_8(t)$ and $b_9(t)$ are $L^1(0,\infty)-$ function defined from coefficients (2.7).

Defining by $E_3$ the function $E_3(t)=E_1(t)+E_2(t)$ and using (2.19), (2.22) yields

$$\begin{aligned}\frac{d}{dt}E_3(t)\le b_8(t)+(b_7(t)+b_9(t))\Big\{&|v_t(t)|^2+\widehat{M}\left(|v_y(t)|^2\right)+\\ &|v_{yy}(t)|^2+|v_{tt}(t)|^2+|v_{yyt}(t)|^2\,\Big\}.\end{aligned} \tag{2.23}$$

Integrating (2.23) from $0$ to $t$ and observing that $v_{tt}(0)$ is bounded in $L^2(\Omega)$ we have, thanks to Gronwall's inequality, that there exists a real positive constant $K_1$ such that

$$E_3(t)\le K_1\ \text{ for all }\ t\ge 0. \tag{2.24}$$

By usual inequality we get the following inequalities

$$\begin{aligned}&|2b_3'(t)\left(v_{yy}(t),v_{yyt}(t)\right)|\le\frac{8|\gamma'(t)|}{\gamma^5(t)}\,|v_{yy}(t)|\,|v_{yyt}(t)|\le\\ &\frac{4|\gamma'(t)|}{\gamma(t)}\Big(\frac{1}{\gamma^4(t)}\,|v_{yy}(t)|^2+\frac{1}{\gamma^4(t)}\,|v_{yyt}(t)|^2\,\Big)\le\\ &\frac12 b_3(t)\,|v_{yy}(t)|^2+\frac12 b_3(t)\,|v_{yyt}(t)|^2,\end{aligned} \tag{2.25}$$

where we have used in the last inequality above the hypothesis (2.11), i.e., $\frac{4|\gamma'(t)|}{\gamma(t)}\le\frac12$. From definition of $E_3$ and (2.25) we have

$$\begin{aligned}&|v_t(t)|^2+b_4(t)\widehat{M}\left(|v_y(t)|^2\right)+|v_{tt}(t)|^2+\\ &\frac12 b_3(t)\,|v_{yy}(t)|^2+\frac12 b_3(t)\,|v_{yyt}(t)|^2\le E_3(t)\ \text{ for all }\ t\ge 0.\end{aligned} \tag{2.26}$$

Thus, from (2.24) and (2.26) we obtain

$$\begin{aligned}&(v_t^m)\ \text{ is bounded in }\ L^\infty\left(0,\infty;H_0^2(\Omega)\right),\\ &(v_{tt}^m)\ \text{ is bounded in }\ L^\infty\left(0,\infty;L^2(\Omega)\right).\end{aligned} \tag{2.27}$$

- **Limit of the approximate solutions.** From estimates obtained in (2.21) and (2.27) we can take the limit in the nonlinear system (2.16). In fact, from

(2.21) and (2.27) we have, in particular, that the sequences $(v^m)$, $(v_t^m)$ and $(v_{tt}^m)$ are bounded in $L^2\left(0,T;H_0^1(\Omega)\right)$, $L^2\left(0,T;H_0^1(\Omega)\right)$ and $L^2\left(0,T;L^2(\Omega)\right)$, respectively. Thus, by compact injection of $H_0^1\left(\Omega\times]0,T[\right)$ into $L^2\left(\Omega\times]0,T[\right)$ it follows by Lions-Aubin's theorem that there exists a subsequence of $(v^m)_{m\in\mathbb{N}}$, which we denote as the original sequence, such that

$$\begin{aligned}
&v^m \longrightarrow v \text{ strong in } \; L^2\left(0,T;H_0^1(\Omega)\right) \text{ as } m\longrightarrow\infty,\\
&v_t^m \longrightarrow v_t \text{ strong in } \; L^2\left(0,T;H_0^1(\Omega)\right) \text{ as } m\longrightarrow\infty.
\end{aligned} \tag{2.28}$$

We still can obtain from (2.21) and (2.27) the following convergence

$$\begin{aligned}
&v^m \longrightarrow v \text{ weak star in } \; L^\infty\left(0,\infty;H_0^2(\Omega)\right) \text{ as } m\longrightarrow\infty,\\
&v_t^m \longrightarrow v_t \text{ weak star in } \; L^\infty\left(0,\infty;H_0^2(\Omega)\right) \text{ as } m\longrightarrow\infty,\\
&v_{tt}^m \longrightarrow v_{tt} \text{ weak star in } \; L^\infty\left(0,\infty;L^2(\Omega)\right) \text{ as } m\longrightarrow\infty.
\end{aligned} \tag{2.29}$$

The convergence (2.28) and $(2.29)_1$ are sufficient to take the limit in the nonlinear terms of the approximate system (2.16). Thus, there exists a strong solution $v$ defined on $\Omega\times[0,\infty[$ with values in $\mathbb{R}$ satisfying (2.4). The statements (2.13) and (2.14) are given by convergence $(2.29)_{2,3}$. The regularity (2.12) is guaranteed by $(2.29)_1$ and by results on elliptic regularity (see, for instance, Nirenberg [17]) in view of

$$\begin{aligned}
&-v_{yyyy} = \gamma^4\left\{v'' + a_1 v_y + a_2 v_{yt} + a_3 v_{yy} + \sqrt{\rho}\delta v_t\right\} +\\
&\gamma^4\left\{b_1\int_\Omega |v_y|^2 dy + b_2\int_\Omega v_y v_{yt} dy + \int_\Omega a_4 v_y v_{yy} dy\right\} v_{yy},
\end{aligned}$$

to be obtained in the sense of $L^2\left(0,\infty;L^2(\Omega)\right)$.

• **Uniqueness.** Let $w = v - \tilde{v}$ be, where $v$ and $\tilde{v}$ are two solutions of the problem (2.4)-(2.6) on $[0,\infty[$. Thus, for all $\phi\in H_0^2(\Omega)$, the solution $w$ satisfies:

$$\begin{aligned}
&(w_{tt}(t),\phi) + (a_1(y,t)w_y(t),\phi) + (a_2(y,t)w_{yt}(t),\phi) + b_3(t)\left(w_{yy}(t),\phi_{yy}\right) +\\
&b_2(t)\left(\int_\Omega w_y(t)v_{yt}(t)dy + \int_\Omega w_{yt}(t)\tilde{v}_y(t)dy\right)(w_{yy}(t),\phi) + \sqrt{\rho}\delta\left(w_t(t),\phi\right) +\\
&(a_3(y,t)w_{yy}(t),\phi) + b_4(t)\left(\int_\Omega |v_y(t)+\tilde{v}_y(t)|\,|w_y(t)|\,dy\right)(w_{yy}(t),\phi) = 0,\\
&w(y,0) = w_t(y,0) = 0 \text{ for all } y\in]0,1[,\\
&w(0,t) = w_y(0,t) = 0 \text{ for all } t\geq 0,\\
&w(1,t) = w_y(1,t) = 0 \text{ for all } t\geq 0.
\end{aligned}$$

Replacing $\phi$ by $w_t$ in the preceding system yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|w_t(t)|^2 + \frac{1}{2}b_3(t)\frac{d}{dt}|w_{yy}(t)|^2 + \sqrt{\rho}\delta\,|w_t(t)|^2 = -\\
&(a_1(y,t)w_y(t),w_t(t)) - (a_2(y,t)w_{yt}(t),w_t(t)) - (a_3(y,t)w_{yy}(t),w_t(t)) -\\
&b_2(t)\left(\int_\Omega w_y(t)v_{yt}(t)dy + \int_\Omega w_{yt}(t)\tilde{v}_y(t)dy\right)(w_{yy}(t),w_t(t)) -\\
&b_4(t)\left(\int_\Omega |v_y(t)+\tilde{v}_y(t)|\,|w_y(t)|\,dy\right)(w_{yy}(t),w_t(t)).
\end{aligned}$$

Applying usual inequalities and $(2.21)_1$ in the right-hand side of preceding identity we obtain

$$\frac{1}{2}\frac{d}{dt}\left\{|w_t(t)|^2 + b_3(t)\,|w_{yy}(t)|^2\right\} + \sqrt{\rho}\delta\,|w_t(t)|^2 \leq |w_t(t)|^2 + b_{10}(t)\,|w_{yy}(t)|^2, \tag{2.30}$$

where $b_{10}(t)$ is a positive function defined on $[0,\infty[$, which depends on the coefficients of the equation (2.4). Integrating (2.30) from $0$ to $t$, using the hypothesis (1.8), the initial-boundary conditions, and Gronwall's inequality we obtain $w = 0$ for all $t \in [0,\infty[$. Thus, we have concluded the proof of Theorem 2.2. Therefore, the Theorem 2.1 is got from Theorem 2.2 and (1.7) □

## 2.2 Asymptotic behavior

Our goal is to show that the total energy of the system (2.1)-(2.3) decay exponentially as $t \to +\infty$. We will show this property directly in non-cylindrical domain $Q_t$. The structure of non-cylindrical domain has a particular importance in the prove of the asymptotic stability of the energy. This way, we assume that the non-cylindrical $Q_t$ is increasing. It means, the functions $\alpha$ and $\beta$ satisfy the following properties

$$\alpha,\ \beta \in W^{1,1}\big([0,\infty[;\mathbb{R}\big), \quad \alpha'(t) < 0 \text{ and } \beta'(t) > 0. \tag{2.31}$$

Besides, we suppose that the non-cylindrical domain is bounded, e.g., there exist real positive constants $\alpha_0$ and $\beta_0$ such that

$$\alpha(t) > -\alpha_0 \text{ and } \beta(t) < \beta_0 \text{ for all } t \geq 0. \tag{2.32}$$

As $u(x,t) = u_x(x,t) = 0$ on $\Sigma_t$ then we have, for all $t \geq 0$, the following relations to be used to proceed

$$\begin{aligned}
&u_t\left(\alpha(t),t\right) = -\alpha'(t)u_x\left(\alpha(t),t\right) = 0,\\
&u_t\left(\beta(t),t\right) = -\beta'(t)u_x\left(\beta(t),t\right) = 0,\\
&u_{xt}\left(\alpha(t),t\right) = -\alpha'(t)u_{xx}\left(\alpha(t),t\right),\\
&u_{xt}\left(\beta(t),t\right) = -\beta'(t)u_{xx}\left(\beta(t),t\right),\\
&\left(\int_{\Omega_t} u_x(t)u_{xt}(t)dx\right)^2 \geq 0.
\end{aligned} \tag{2.33}$$

Taking the scalar product on $L^2(\Omega_t)$ of both sides of (2.1) with $u_t$, integrating by parts the second and third term of left-hand side, observing the identities (2.33) and the Leibnitz's rule[1] yields

$$\frac{d}{dt}E(t) + \frac{1}{2}\beta'(t)u_{xx}^2(\beta(t),t) - \frac{1}{2}\alpha'(t)u_{xx}^2(\alpha(t),t) \leq -\sqrt{\rho}\delta\,|u_t(t)|^2 \text{ for all } t \geq 0,$$

[1]
$$\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(x,t)dx = \int_{\alpha(t)}^{\beta(t)} \frac{\partial}{\partial t}f(x,t)dx + f(\beta(t),t)\beta'(t) - f(\alpha(t),t)\alpha'(t).$$

where

$$E(t)=\frac{1}{2}\left\{|u_t(t)|^2+\zeta_0\,|u_x(t)|^2+\frac{\zeta_1}{2}\,|u_x(t)|^4+|u_{xx}(t)|^2\right\}. \tag{2.34}$$

From hypothesis $(2.31)_2$ we have

$$\frac{1}{2}\beta'(t)u_{xx}^2(\beta(t),t)-\frac{1}{2}\alpha'(t)u_{xx}^2(\alpha(t),t)\geq 0 \text{ for all } t\geq 0.$$

Thus, we obtain

$$\frac{d}{dt}E(t)\leq -\sqrt{\rho}\delta\,|u_t(t)|^2 \text{ for all } t\geq 0. \tag{2.35}$$

The asymptotic behavior of the energy is given by the following theorem

**Theorem 2.3.** *If the functions $\alpha$ and $\beta$ satisfy the hypotheses (2.31) and (2.32), then the energy defined in (2.34) associated with the solution of the system (2.1)-(2.3) satisfies*

$$E(t)\leq K_2\exp(-\omega t) \text{ for all } t\geq 0, \tag{2.36}$$

*where $\omega=\dfrac{\epsilon}{2\vartheta_2}$ is a real-positive number, $\vartheta_2$ is defined in (2.42), $K_2=E(0)+\epsilon F(0)$, the function $F(t)$ and the constant $\epsilon$ are defined by*

$$F(t)=\frac{1}{4}\left\{\sqrt{\rho}\delta\,|u(t)|^2+\frac{\sigma}{2}\,|u_x(t)|^4\right\}, \tag{2.37}$$
$$0<\epsilon=\min\left\{\frac{1}{\sqrt{\vartheta_1}},\ \sqrt{\rho}\delta\right\}.$$

*From hypothesis (2.32) the constant $\vartheta_1=\max\limits_{t\geq 0}\gamma(t)$ there exists and represents the real-positive constant of immersion of $H_0^1(\Omega_t)$ into $L^2(\Omega_t)$. Note that $\vartheta_1$ depend only of the length of the interval $\big[\alpha(t),\beta(t)\big]$.*

**Proof.** For each $\epsilon>0$ we consider the auxiliary function

$$E_\epsilon(t)=E(t)+\frac{\epsilon}{2}\int_{\Omega_t}u_t(t)u(t)dx \text{ for all } t\geq 0. \tag{2.38}$$

As $H_0^1(\Omega_t)$ is continuously embedded in $L^2(\Omega_t)$ we obtain after application of usual inequalities that, there exist a constant $\vartheta_1$

$$\frac{\epsilon}{2}\int_{\Omega_t}u_t(t)u(t)dx\leq\frac{1}{4}\,|u_t(t)|^2+\frac{\epsilon^2\vartheta_1}{4}\,|u_x(t)|^2. \tag{2.39}$$

Hence, and hypothesis $(2.37)_2$ we get

$$E_\epsilon(t)\leq\frac{3}{2}E(t) \text{ for all } t\geq 0.$$

On the other hand, we get from (2.38) and $(2.37)_2$ that

$$E_\epsilon(t) \geq E(t) - \frac{\epsilon}{2}\Big|\int_{\Omega_t} u_t(t)u(t)dx\Big| \geq \frac{1}{2}E(t) \text{ for all } t \geq 0.$$

Thus, the function $E_\epsilon$ satisfies the both inequalities

$$\frac{1}{2}E_\epsilon(t) \leq E(t) \leq \frac{3}{2}E_\epsilon(t) \text{ for all } t \geq 0. \tag{2.40}$$

Now, differentiating the function $E_\epsilon$ with respect to $t$ yields

$$E_\epsilon'(t) = E'(t) + \frac{\epsilon}{2}\int_{\Omega_t} u_{tt}(t)u(t)dx + \frac{\epsilon}{2}\left|u_t(t)\right|^2 \text{ for all } t \geq 0.$$

Replacing $u_{tt}$ by $\left\{\zeta_0 + \zeta_1 \int_{\Omega_t} |u_x|^2dx + \int_{\Omega_t} u_x u_{xt} dx\right\} u_{xx} - u_{xxxx} - \sqrt{\rho}\delta u_t$ in the second term of right-hand side of the last identity above and using the definition of the function $F$ yields, for all $t \geq 0$

$$E_\epsilon'(t) + \epsilon F'(t) = E'(t) + \frac{\epsilon}{2}\left|u_t(t)\right|^2 - \frac{\epsilon}{2}\left\{\zeta_0\left|u_x(t)\right|^2 + \zeta_1\left|u_x(t)\right|^4 + \left|u_{xx}(t)\right|^2\right\}.$$

Hence, (2.35) and $(2.37)_2$ we obtain

$$E_\epsilon'(t) + \epsilon F'(t) \leq -\frac{\epsilon}{2}E(t) \text{ for all } t \geq 0. \tag{2.41}$$

Using the definitions of the functions $E$, $E_\epsilon$ and $F$ it is easy to see that there exists a real positive constant $\vartheta_2$ dependent only of $\epsilon$ such that

$$E_\epsilon(t) + \epsilon F(t) \leq \vartheta_2 E(t) \text{ for all } t \geq 0. \tag{2.42}$$

Thus, from inequalities (2.41) and (2.42) there exists a suitable real positive constant $\omega$ dependent of $\epsilon$ and $\vartheta_2$ such that

$$E_\epsilon'(t) + \epsilon F'(t) - \omega\left\{E_\epsilon(t) + \epsilon F(t)\right\} \leq 0 \text{ for all } t \geq 0.$$

Therefore,

$$E_\epsilon(t) + \epsilon F(t) \leq \left\{E_\epsilon(0) + \epsilon F(0)\right\}\exp\left(-\omega t\right) \text{ for all } t \geq 0.$$

Hence, and inequality (2.40) we can conclude that the inequality (2.36) holds. Thus, the demonstration of Theorem 2.3 is completed □

## 3. Flutter panel with linear-strong damping

### 3.1 Existence and uniqueness of solutions

To reach our goal mentioned previously we consider in this section the initial data- and boundary- value problem

(3.1)

$$u_{tt}(x,t) - \Big(\zeta_0 + \zeta_1\int_{\Omega_t}|u_x(t)|^2dx + \sigma\int_{\Omega_t}u_x(t)u_{xt}(t)dx\Big)u_{xx}(x,t) +$$
$$u_{xxxx}(x,t) + \nu u_{xxxxt}(x,t) = 0 \text{ in } Q_t,$$

(3.2)

$$u(x,0) = u_0(x) \text{ and } u_t(x,0) = u_1(x) \text{ in } \Omega_0,$$

(3.3)

$$u(x,t) = u_x(x,t) = 0 \text{ on } \Sigma_t.$$

The global solutions of the system (3.1)-(3.3) will be obtained if the initial data are bounded. The structure of the non-cylindrical domain $Q_t$ has an important perform in this section. We will fix to follow the hypotheses necessary to state the results associated with the problem (3.1)-(3.3). The functions $\alpha$, $\beta$, $\gamma$, and its derivatives, satisfying for all $t \geq 0$, the following conditions

$$\varepsilon = \max\left\{|\alpha'(t)|, \quad |\alpha''(t)|, \quad |\beta'(t)|, \ |\beta''(t)|\right\}. \tag{3.4}$$

$$\frac{\gamma'(t)}{\gamma(t)} \leq \frac{\zeta_1}{\sigma}, \quad \frac{\gamma'(t)}{\gamma(t)} \leq \frac{1}{16\nu}. \tag{3.5}$$

As a consequence of (3.4) we have

$$|\gamma'(t)| \leq 2\varepsilon, \quad |\gamma''(t)| \leq 2\varepsilon. \tag{3.6}$$

We also assume that there exist real constants $\gamma_0$ and $K_3$ such that

$$0 < \gamma_0 \leq \gamma(t) \leq K_3. \tag{3.7}$$

$$\zeta_0 K_3^4 < \frac{1}{4}. \tag{3.8}$$

Finally, we choose $\varepsilon$ small enough such that

$$\frac{\zeta_1}{4} \geq \varepsilon\Gamma_0, \ \frac{1}{16} \geq \varepsilon\Gamma_1, \ \frac{\nu}{8} \geq \varepsilon\Gamma_2 \ \text{ and } \ \frac{3}{8} \geq \varepsilon\Gamma_3, \tag{3.9}$$

where $\Gamma_0$, $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ are defined in (3.37).

As done in the subsection 2.1, the function $u$ is a solution of the mixed problem (3.1)-(3.3) if and only if $v$ is a solution of the mixed problem

$$\begin{aligned} & v_{tt} + c_1(y,t)v_y + c_2(y,t)v_{yt} + \\ & c_3(y,t)(t)v_{yy} + d_1(t)\Big(\int_\Omega |v_y|^2 dy\Big)v_{yy} + \\ & d_2(t)\Big(\int_\Omega v_y v_{yt} dy\Big)v_{yy} + \Big(\int_\Omega c_4(y,t)v_y v_{yy} dy\Big)v_{yy} + \\ & d_3(t)v_{yyyy} + c_5(y,t)v_{yyyyy} + d_4(t)v_{yyyyt} = 0 \ \text{ in } \ Q, \end{aligned} \tag{3.10}$$

$$v(y,0) = v_0(y), \quad v_t(y,0) = v_1(y) \ \text{ with } \ 0 < y < 1, \tag{3.11}$$

$$\begin{aligned} & v(0,t) = v_y(0,t) = 0 \ \text{ for all } \ t \geq 0, \\ & v(1,t) = v_y(1,t) = 0 \ \text{ for all } \ t \geq 0, \end{aligned} \tag{3.12}$$

where the coefficients of equation (3.10) are given by

$$\begin{aligned} & c_1(y,t) = \frac{1}{\gamma^2}\left\{2\gamma'\left(\alpha' + y\gamma'\right) - \gamma\left(\alpha'' + y\gamma''\right)\right\}, \\ & c_2(y,t) = -\frac{2}{\gamma}\left(\alpha' + y\gamma'\right), \quad c_3(y,t) = \frac{1}{\gamma^2}\left\{\left(\alpha' + y\gamma'\right)^2 - \zeta_0\right\}, \\ & c_4(y,t) = \frac{\sigma}{\gamma^4}\left(\alpha' + y\gamma'\right), \quad c_5(y,t) = -\frac{\nu}{\gamma^5}\left(\alpha' + y\gamma'\right), \\ & d_1(t) = \frac{1}{\gamma^4}\left(\sigma\gamma' - \zeta_1\gamma\right), \quad d_2(t) = -\frac{\sigma}{\gamma^3}, \\ & d_3(t) = \frac{1}{\gamma^5}\left(\gamma - 4\nu\gamma'\right), \quad d_4(t) = \frac{\nu}{\gamma^4}. \end{aligned} \tag{3.13}$$

Thus, we can state the first result of this section

**Theorem 3.1.** *Under the hypotheses (3.4)-(3.9). If the initial data* $u_0$, $u_1$ *belong to* $H_0^2(\Omega_0)\cap H^4(\Omega_0)$, $H_0^2(\Omega_0)$ *respectively, and*

$$\begin{aligned}\frac{\nu^2\gamma^4(0)}{32K_3^4}\left|u_{0xxxx}\right|^2+\frac{\nu^2}{32K_3^4}\left|u_{0xx}\right|^2+\frac{\sigma\gamma(0)}{4}\left|u_{0x}\right|^4+\\ \frac{1}{4}\left|\left[\alpha'(0)+\frac{\gamma'(0)}{\gamma(0)}\big(x-\alpha(0)\big)\right]u_{0x}+u_1\right|^2<\frac{1}{4K_4},\end{aligned}\tag{3.14}$$

*then the unique weak solution* $u$ *of the mixed problem (3.1)-(3.3) lies in*

$$u\in L^\infty\left(0,\infty;H_0^2\left(\Omega_t\right)\cap H^4\left(\Omega_t\right)\right),\tag{3.15}$$

$$u_t\in L^\infty\left(0,\infty;H_0^2\left(\Omega_t\right)\right),\tag{3.16}$$

$$u_{tt}\in L^2\left(0,\infty;H^{-2}\left(\Omega_t\right)\right).\tag{3.17}$$

*The equation (3.1) is verified in the sense of* $L^2\left(0,\infty;H^{-2}\left(\Omega_t\right)\right)$. *The real positive constant* $K_4$ *is defined in (3.40).*

As done before, the Theorem 3.1 is equivalent to

**Theorem 3.2.** *Under the assumptions of Theorem 3.1, if* $\alpha'(0)=\beta'(0)=0$ *then* $v_0$, $v_1$ *belong to* $H_0^2(\Omega)\cap H^4(\Omega)$, $H_0^2(\Omega)$ *respectively,*

$$K_4E_5(0)<\frac{1}{4},\tag{3.18}$$

*and, in these conditions there exists a unique weak solution* $v$ *of the mixed problem (3.10)-(3.12) lies in*

$$v\in L^\infty\left(0,\infty;H_0^2(\Omega)\cap H^4(\Omega)\right),\tag{3.19}$$

$$v_t\in L^\infty\left(0,\infty;H_0^2(\Omega)\right),\tag{3.20}$$

$$v_{tt}\in L^2(0,\infty;H^{-2}\left(\Omega\right)).\tag{3.21}$$

*The equation (3.10) is verified in the sense of* $L^2(0,\infty;H^{-2}(\Omega))$ *and* $E_5(t)$ *is defined by*

$$\begin{aligned}E_5(t)=\frac{1}{4}\left|v_t(t)\right|^2+\frac{\sigma}{4\gamma^3(t)}\left|v_y(t)\right|^4+\\ \frac{\nu^2}{32K_3^4\gamma^4(t)}\left|v_{yy}(t)\right|^2+\frac{\nu^2}{32K_3^4\gamma^4(t)}\left|v_{yyyy}(t)\right|^2.\end{aligned}\tag{3.22}$$

**Proof of Theorem 3.2 - Existence.** We again will use the Galerkin's method. But, that time we are going to work with the special basis built with eigenfunctions of the operator $\mathbb{A}=\partial_y{}^4$ defined in $H_0^2(\Omega)\cap H^4(\Omega)$. Thus, by $V_m=[w_1,w_2,w_3,\cdots,w_m]$ we will denote the subspace of $H_0^2(\Omega)$ spanned by the $m$ first eigenvectors of $\mathbb{A}$. We will assume the initial data in $V_m$ satisfy

$$\begin{aligned}&v^m(y,0)=v_0^m\longrightarrow v_0\ \text{ strongly in }\ H_0^2(\Omega)\cap H^4(\Omega),\\ &v_t^m(y,0)=v_1^m\longrightarrow v_1\ \text{ strongly in }\ H_0^2(\Omega).\end{aligned}\tag{3.23}$$

For $v^m(t)$ belongs to $V_m$ we have, for all $\phi \in V_m$, that the system of ordinary differential equations

$$
\begin{aligned}
&(v^m_{tt}(t),\phi) + \big(c_1(y,t)v^m_y(t),\phi\big) + \big(c_2(y,t)v^m_{yt}(t),\phi\big) + \\
&\big(c_3(y,t)v^m_{yy}(t),\phi\big) + d_1(t)\Big(\int_\Omega \left|v^m_y(t)\right|^2 dy\Big)\left(v^m_{yy}(t),\phi\right) + \\
&\Big(\int_\Omega c_4(y,t)v^m_y(t)v^m_{yy}(t)dy\Big)\left(v^m_{yy}(t),\phi\right) + \\
&d_2(t)\Big(\int_\Omega v^m_y(t)v^m_{yt}(t)dy\Big)\left(v^m_{yy}(t),\phi\right) + d_3(t)\left(v^m_{yyyy}(t),\phi\right) + \\
&\big(c_5(y,t)v^m_{yyyyy}(t),\phi\big) + d_4(t)\left(v^m_{yyyyt}(t),\phi\right) = 0, \\
&v^m(y,0) = v^m_0 \ \text{ and } \ v^m_t(y,0) = v^m_1,
\end{aligned}
\tag{3.24}
$$

has a local solution $v^m$ provided that the hypothesis (3.23) holds. Again, throughout this subsection, we will write $v$ instead of $v^m$. Our next task is to determine estimative on $v$ sufficient to take the limit in the system (3.24).

- **Estimate I.** Substituting $\phi$ by $v_t(t)$ into $(3.24)_1$, integrating by parts from the third to the tenth terms we get the following relations

$$
\begin{aligned}
(c_2(y,t)v_{yt}(t),v_t(t)) &= \frac{2\gamma'(t)}{\gamma(t)}\left|v_t(t)\right|^2, \\
(c_3(y,t)v_{yy}(t),v_t(t)) &= \Big(\frac{1}{\gamma^2(t)}\big(\alpha'(t)+y\gamma'(t)\big)^2 v_{yy}(t),v_t(t)\Big) + \frac{\zeta_0}{2\gamma^2(t)}\frac{d}{dt}\left|v_y(t)\right|^2, \\
d_1(t)\Big(\int_\Omega \left|v_y(t)\right|^2 dy\Big)(v_{yy}(t),v_t(t)) &+ \Big(\int_\Omega c_4(y,t)v_y(t)v_{yy}(t)dy\Big)(v_{yy}(t),v_t(t)) = \\
&-\frac{1}{4}\Big(-\frac{\zeta_1\gamma(t)}{\gamma^4(t)} + \frac{\sigma\gamma'(t)}{2\gamma^4(t)}\Big)\frac{d}{dt}\left|v_y(t)\right|^4, \\
d_2(t)\Big(\int_\Omega v_y(t)v_{yt}(t)dy\Big)(v_{yy}(t),v_t(t)) &= \frac{\sigma}{\gamma^3(t)}\Big(\int_\Omega v_y(t)v_{yt}(t)dy\Big)^2 \geq 0, \\
d_3(t)\left(v_{yyyy}(t),v_t(t)\right) &+ d_4(t)\left(v_{yyyyt}(t),v_t(t)\right) = \\
&\frac{1}{2\gamma^5(t)}\Big(\gamma(t)-4\nu\gamma'(t)\Big)\frac{d}{dt}\left|v_{yy}(t)\right|^2 + \frac{\nu}{\gamma^4(t)}\left|v_{yyt}(t)\right|^2, \\
(c_5(y,t)v_{yyyyy}(t),v_t(t)) &= -\left(v_{yyyy}(t),\frac{\nu\gamma'(t)}{\gamma^5(t)}v_t(t) + \frac{\nu}{\gamma^5(t)}\big(\alpha'(t)+y\gamma'(t)v_{yt}(t)\big)\right).
\end{aligned}
$$

Thus, using the six precedents relations yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|v_t(t)\right|^2 + \frac{\zeta_0}{2\gamma^2(t)}\frac{d}{dt}\left|v_y(t)\right|^2 + \frac{1}{4}\Big(\frac{\zeta_1}{\gamma^3(t)} - \frac{\sigma\gamma'(t)}{2\gamma^4(t)}\Big)\frac{d}{dt}\left|v_y(t)\right|^4 + \\
&\frac{1}{2\gamma^5(t)}\Big(\gamma(t)-4\nu\gamma'(t)\Big)\frac{d}{dt}\left|v_{yy}(t)\right|^2 + \frac{\nu}{\gamma^4(t)}\left|v_{yyt}(t)\right|^2 \leq \\
&\frac{2|\gamma'(t)|}{\gamma(t)}\left|v_t(t)\right|^2 + (c_1(y,t)v_y(t),v_t(t)) + \\
&\Big(\frac{1}{\gamma^2(t)}\big(\alpha'(t)+y\gamma'(t)\big)v_{yy}(t),v_t(t)\Big) + \\
&\left(v_{yyyy}(t),\frac{\nu\gamma'(t)}{\gamma^5(t)}v_t(t) + \frac{\nu}{\gamma^5(t)}\big(\alpha'(t)+y\gamma'(t)v_{yt}(t)\big)\right).
\end{aligned}
\tag{3.25}
$$

Using the hypotheses (3.4), (3.5), (3.7) and usual inequalities we get estimates for the right-hand side of (3.25). Thus, denoting by $E_1(t)$ the function

(3.26)
$$\begin{aligned} &E_1(t) = \frac{1}{2}\left|v_t(t)\right|^2 + \frac{\zeta_0}{2\gamma^2(t)}\left|v_y(t)\right|^2 + \\ &\frac{1}{4}\Big(\frac{\zeta_1}{\gamma^3(t)} - \frac{\sigma\gamma'(t)}{2\gamma^4(t)}\Big)\left|v_y(t)\right|^4 + \frac{1}{2\gamma^5(t)}\Big(\gamma(t) - 4\nu\gamma'(t)\Big)\left|v_{yy}(t)\right|^2, \end{aligned}$$

yields

$$\begin{aligned} &\frac{d}{dt}E_1(t) + \frac{1}{\gamma^4(t)}\Big(\frac{\nu}{2} - 4\varepsilon K_3^3\Big)\left|v_{yyt}(t)\right|^2 \leq \\ &\frac{\varepsilon C_0}{\gamma^2(t)}\left|v_y(t)\right|^4 + \frac{\varepsilon C_1}{\gamma^4(t)}\left|v_{yy}(t)\right|^2 + \frac{10\nu\varepsilon^2}{\gamma^6(t)}\left|v_{yyyy}(t)\right|^2, \end{aligned} \tag{3.27}$$

where $C_0$ and $C_1$ depends on $\zeta_1$, $K_3$, $\sigma$, $\epsilon$, $\gamma_0$ and $\zeta_0$, $K_3$, $\nu$, $\epsilon$, $\gamma_0$, respectively.

- **Estimate II.** Substituting $\phi$ by $v(t)$ into $(3.24)_1$ and proceeding as in the estimate I and denoting by $E_2(t)$ the function

$$E_2(t) = (v_t(t), v(t)) + \frac{\nu}{2\gamma^4(t)}\left|v_{yy}(t)\right|^2 + \frac{\sigma}{4\gamma^3(t)}\left|v_y(t)\right|^4, \tag{3.28}$$

we can write

(3.29)
$$\begin{aligned} &\frac{d}{dt}E_2(t) + \frac{\zeta_0}{\gamma^2(t)}\left|v_y(t)\right|^2 + \frac{1}{\gamma^4(t)}\left|v_{yy}(t)\right|^2 + \frac{\zeta_1}{\gamma(t)}\left|v_y(t)\right|^4 = \\ &\left|v_t(t)\right|^2 + \frac{6\nu\gamma'(t)}{\gamma^5(t)}\left|v_{yy}(t)\right|^2 + \frac{\sigma\gamma'(t)}{4\gamma^4(t)}\big(4\gamma^2(t)+1\big)\left|v_y(t)\right|^4 - \\ &(c_2(y,t)v_{yt}(t), v(t)) - \Big(\frac{1}{\gamma^2(t)}\big(\alpha'(t) + y\gamma'(t)\big)^2 v_{yy}(t), v(t)\Big) - \\ &\bigg(v_{yyyy}(t), \frac{\nu\gamma'(t)}{\gamma^5(t)}v(t) + \frac{\nu}{\gamma^5(t)}\big(\alpha'(t)+y\gamma'(t)\big)v_y(t)\bigg). \end{aligned}$$

Using the hypotheses (3.4), (3.6), (3.7) and Poincarè's inequality we obtain estimates for the right-hand side of (3.29) yielding

$$\begin{aligned} &\frac{d}{dt}E_2(t) + \frac{\zeta_0}{\gamma^2(t)}\left|v_y(t)\right|^2 + \frac{1}{\gamma(t)}\Big(\zeta_1 - \varepsilon C_2\Big)\left|v_y(t)\right|^4 + \\ &\frac{1}{\gamma^4(t)}\Big(\frac{1}{2} - \varepsilon C_3\Big)\left|v_{yy}(t)\right|^2 \leq \\ &\big(1 + 2\varepsilon^2 K_3^2\big)\left|v_{yyt}(t)\right|^2 + \frac{1}{4\gamma^4(t)}\left|v_{yyyy}(t)\right|^2. \end{aligned} \tag{3.30}$$

where $C_2$ and $C_3$ depends on $K_3$, $\sigma$, $\gamma_0$ and $K_3$, $\nu$, $\epsilon$, $\gamma_0$, respectively.

- **Estimate III.** Substituting $\phi$ by $v_{yyyy}(t)$ into $(3.24)_1$, proceeding as in the estimates I and II and denoting by $E_3(t)$ the function

$$E_3(t) = (v_t(t), v_{yyyy}(t)) + \frac{\nu}{2\gamma^4(t)} |v_{yyyy}(t)|^2 , \tag{3.31}$$

we obtain

$$\begin{aligned}
&\frac{d}{dt}E_3(t) + \frac{1}{\gamma^4(t)} |v_{yyyy}(t)|^2 = \\
&|v_{yyt}(t)|^2 + \frac{11\nu\gamma'(t)}{2\gamma^5(t)} |v_{yyyy}(t)|^2 - (c_1(y,t)v_y(t), v_{yyyy}(t)) - \\
&(c_2(y,t)v_{yt}(t), v_{yyyy}(t)) - (c_3(y,t)v_{yy}(t), v_{yyyy}(t)) - \\
&d_1(t)\Big(\int_\Omega |v_y(t)|^2\, dy\Big)(v_{yy}(t), v_{yyyy}(t)) - \\
&\Big(\int_\Omega c_4(y,t)v_y(t)v_{yy}(t)dy\Big)(v_{yy}(t), v_{yyyy}(t)) - \\
&d_2(t)\Big(\int_\Omega v_y(t)v_{yt}(t)dy\Big)(v_{yy}(t), v_{yyyy}(t)) .
\end{aligned} \tag{3.32}$$

Again, using the hypotheses (3.4), (3.6), (3.7), Poincarè's inequality we obtain estimates for the right-hand side of (3.34), and this way yields

$$\begin{aligned}
&\frac{d}{dt}E_3(t) + \frac{1}{\gamma^4(t)} |v_{yyyy}(t)|^2 \leq \\
&\Big[2\varepsilon\Big(17\varepsilon^2 + 8\varepsilon K_3 + K_3^2 + 2\zeta_0\Big) + 2\zeta_0^2\Big] |v_{yy}(t)|^2 + \\
&\Big(1 + 8\varepsilon^2 K_3^2\Big) |v_{yyt}(t)|^2 + \frac{\varepsilon}{\gamma^4(t)}\Big(\frac{11\nu}{\gamma_0} + \frac{3}{4\gamma_0}\Big) |v_{yyyy}(t)|^2 + \\
&\frac{1}{\gamma^4(t)}\Big[\varepsilon\big(\zeta_1 + \sigma\big) |v_{yy}(t)|\, |v_{yyyy}(t)| + \sigma K_3 |v_t(t)|\, |v_{yyyy}(t)| \Big] |v_{yy}(t)|^2 .
\end{aligned}$$

Applying Cauchy-Schwartz's inequality in the last term of precedent inequality and making some modifications yields

$$\begin{aligned}
&\frac{d}{dt}E_3(t) + \frac{1}{\gamma^4(t)}\Big[1 - \varepsilon\Big(\frac{11\nu}{\gamma_0} + \frac{3}{4\gamma_0}\Big)\Big] |v_{yyyy}(t)|^2 \leq \\
&\Big(\varepsilon C_4 + 2\zeta_0^2\Big) |v_{yy}(t)|^2 + \Big(1 + 8\varepsilon^2 K_3^2\Big) |v_{yyt}(t)|^2 + \\
&\frac{1}{2\gamma^4(t)}\Big[ |v_t(t)|^2 + \varepsilon\big(\zeta_1 + \sigma\big) |v_{yy}(t)|^2 + C_5 |v_{yyyy}(t)|^2 \Big] |v_{yy}(t)|^2 ,
\end{aligned} \tag{3.33}$$

where $C_4$ and $C_5$ depende on $\zeta_0$, $K_3$, $\epsilon$, and $\zeta_1$, $K_3$, $\sigma$, $\epsilon$, respectively.

Multiplying the inequalities (3.30) and (3.33) for $\dfrac{\nu}{8K_3^4}$ and, to the result it is added to the (3.27) yields

$$\begin{aligned}
&\frac{d}{dt}E_4(t) + \frac{\nu}{8K_3^6}\Big[\frac{\zeta_1}{2} - \varepsilon\Big(C_2 + \frac{8K_3^6 C_0}{\nu\gamma_0^2}\Big)\Big] |v_y(t)|^4 + \\
&\frac{\nu}{8K_3^4}\frac{1}{\gamma^4(t)}\Big[\frac{1}{8} - 2\zeta_0^2 K_3^4 + \frac{1}{8} - \varepsilon\Big(C_3 + C_4 + \frac{8K_3^4 C_1}{\nu\gamma_0^4}\Big)\Big] |v_{yy}(t)|^2 +
\end{aligned}$$

$$\begin{aligned}\frac{1}{\gamma^4(t)}\Big[\frac{\nu}{4}-\frac{\varepsilon}{K_3^2}\Big(4K_3^5+\nu\varepsilon\Big)\Big]\,|v_{yyt}(t)|^2 + \\ \frac{\nu}{8K_3^8}\Big[\frac{3}{4}-\varepsilon\Big(\frac{11\nu}{\gamma_0}+\frac{3}{4\gamma_0}+\frac{80\varepsilon K_3^4}{\gamma_0^6}\Big)\Big]\,|v_{yyyy}(t)|^2 + \\ \frac{\nu}{8K_3^4}\frac{1}{\gamma^4(t)}\Big(\frac{1}{4}-\Lambda(t)\Big)\,|v_{yy}(t)|^2 \leq 0,\end{aligned} \tag{3.34}$$

where

$$E_4(t) = E_1(t) + \frac{\nu}{8K_3^4}E_2(t) + \frac{\nu}{8K_3^4}E_3(t), \tag{3.35}$$

$$\Lambda(t) = \frac{1}{2}\left\{|v_t(t)|^2 + \frac{\varepsilon}{\gamma_0^4}\,(\zeta_1+\sigma)\,|v_{yy}(t)|^2 + C_5\,|v_{yyyy}(t)|^2\right\}. \tag{3.36}$$

Denoting by

$$\begin{aligned}\Gamma_0 = C_2 + \frac{8K_3^6C_0}{\nu\gamma_0^2},\quad \Gamma_1 = C_3 + C_4 + \frac{8K_3^4C_1}{\nu\gamma_0^4}\\ \Gamma_2 = \frac{1}{K_3^2}\Big(4K_3^5+\nu\varepsilon\Big),\quad \Gamma_3 = \frac{11\nu}{\gamma_0}+\frac{3}{4\gamma_0}+\frac{80\varepsilon K_3^4}{\gamma_0^6},\end{aligned} \tag{3.37}$$

and observing the hypotheses (3.8) and (3.9) we get from (3.24) that

$$\frac{d}{dt}E_4(t) + \frac{\nu}{8K_3^4}\frac{1}{\gamma^4(t)}\Big(\frac{1}{4}-\Lambda(t)\Big)\,|v_{yy}(t)|^2 \leq 0. \tag{3.38}$$

Our next task is to prove that

$$E_4(t) \leq E_4(0) \quad \text{for all} \quad t \geq 0. \tag{3.39}$$

In fact, from (3.22), (3.35) and (3.36) is not hard to see that there exist real positive constants $K_4$ and $K_5$ such that

$$\Lambda(t) \leq K_5E_4(t) \leq K_4E_5(t) \ \text{ for all } \ t \geq 0. \tag{3.40}$$

From (3.18) and (3.40) yields $\Lambda(0) < \dfrac{1}{4}$. Hence, as $\Lambda(t)$ is continuous on $[0,\infty[$ then there exists a positive $t_m$ such that

$$\Lambda(t) < \frac{1}{4} \ \text{ for all } \ t \in [0, t_m[. \tag{3.41}$$

From (3.41) we affirm that there exists $t^*$ such that

$$\Lambda(t) < \frac{1}{4} \ \text{ for all } \ t \in [0, t^*[. \tag{3.42}$$

Besides, we suppose that

$$\Lambda(t^*) = \frac{1}{4}. \tag{3.43}$$

Thus, integrating (3.38) from $0$ to $t^*$ and in addition observing (3.42) yields

$$E_4(t^*) \leq E_4(0). \tag{3.44}$$

From (3.40), (3.44) and (3.18) yields

$$\Lambda(t^*) \leq K_5 E_4(t^*) \leq K_5 E_4(0) \leq K_4 E_5(0) < \frac{1}{4}. \tag{3.45}$$

Thus, from (3.43) and (3.45) we obtain a contradiction $\rightarrow \mid \leftarrow$. Therefore, we have that (3.39) is true.

On the other hand, the constant $\dfrac{\nu}{8K_3^4}$ is such that

$$\begin{aligned} &\frac{1}{4}\left|v_t(t)\right|^2 + \frac{\nu^2}{32K_3^4\gamma^4(t)}\left|v_{yy}(t)\right|^2 + \frac{\nu^2}{32K_3^4\gamma^4(t)}\left|v_{yyyy}(t)\right|^2 \leq \\ &\frac{1}{2}\left|v_t(t)\right|^2 + \frac{\nu}{8K_3^4}\left(v_t(t), v(t)\right) + \frac{\nu}{8K_3^4}\left(v_t(t), v_{yyyy}(t)\right) + \\ &\frac{\nu^2}{16K_3^4\gamma^4(t)}\left|v_{yy}(t)\right|^2 + \frac{\nu^2}{16K_3^4\gamma^4(t)}\left|v_{yyyy}(t)\right|^2. \end{aligned} \tag{3.46}$$

In fact, using the hypothesis (3.7), the inequality (3.46) is a consequence of

$$\frac{\nu}{8K_3^4}\left(v_t(t), v(t)\right) \geq -\frac{1}{8}\left|v_t(t)\right|^2 - \frac{\nu^2}{32K_3^4\gamma^4(t)}\left|v_{yy}(t)\right|^2,$$

$$\frac{\nu}{8K_3^4}\left(v_t(t), v_{yyyy}(t)\right) \geq -\frac{1}{8}\left|v_t(t)\right|^2 - \frac{\nu^2}{32K_3^4\gamma^4(t)}\left|v_{yyyy}(t)\right|^2.$$

Using the left-hand side of (3.46) we define the function $E_6(t)$ by

$$E_6(t) = \frac{1}{4}\left|v_t(t)\right|^2 + \frac{\nu^2}{32K_3^4\gamma^4(t)}\left|v_{yy}(t)\right|^2 + \frac{\nu^2}{32K_3^4\gamma^4(t)}\left|v_{yyyy}(t)\right|^4. \tag{3.47}$$

From (3.47), (3.46), (3.35), (3.22), (3.5) and neglecting some positive terms we obtain

$$E_4(t) \geq E_5(t) \geq E_6(t). \tag{3.48}$$

Thus, from (3.39) and (3.48) yields

$$\begin{aligned} &(v^m) \ \text{ is bounded in } \ L^\infty\left(0,\infty; H^4(\Omega)\right), \\ &(v_t^m) \ \text{ is bounded in } \ L^\infty\left(0,\infty; L^2(\Omega)\right). \end{aligned} \tag{3.49}$$

Integrating with respect to $t$ the inequalities (3.34) we can see without difficulty that

$$(v_t^m) \ \text{ is bounded in } \ L^2\left(0,\infty; H^2(\Omega)\right). \tag{3.50}$$

From (3.49) and (3.50) we can take the limit in the approximate solutions of the system (3.24). It is made by a similar way of subsection 2.1, and the limit solution $v$ is got in the weak sense of distributions. Still, from $(3.49)_1$ and (3.50) we can affirm that $v_{yyyyy}$ and $v_{yyyyt}$ belong to $L^2\left(0,\infty; H^{-1}(\Omega)\right)$ and $L^2\left(0,\infty; H^{-2}(\Omega)\right)$ respectively. Hence, we can conclude that (3.10) is verified in the sense of $L^2\left(0,\infty; H^{-2}(\Omega)\right)$. Consequently (3.21) is verified, and (3.19), (3.20) are guaranteed by $(3.49)_1$ and (3.50) respectively.

The uniqueness is obtained by energy's methods. This way the demonstration of Theorem 3.2 is concluded. Finally, as said before, the demonstration of Theorem 3.1 is established via Theorem 3.2 and (1.7) $\square$

### 3.2 Asymptotic behavior

We start this subsection pointing out the difficulty to establish the asymptotic behavior of energy directly inside of the non-cylindrical domain. In fact, taking the scalar product on $L^2(\Omega_t)$ of both sides of (3.1) with $u_t$, integrating by parts, assuming the hypothesis $(2.31)_2$, and observing the boundary conditions we get

$$\frac{d}{dt}E(t) \leq u_{xxt}u_{xt}\Big|_{\alpha(t)}^{\beta(t)} - \nu\,|u_{xxt}(t)|^2 \quad \text{for all } t \geq 0, \tag{3.51}$$

where

$$E(t) = \frac{1}{2}\left\{|u_t(t)|^2 + \zeta_0\,|u_x(t)|^2 + \frac{\zeta_1}{2}\,|u_x(t)|^4 + |u_{xx}(t)|^2\right\}. \tag{3.52}$$

Note that the term of greater order in the energy has two spacial derivatives, while the pointwise term in (3.51) has three. Thus, is not easy to build a Lyapunov functional as made in the subsection 2.2. Therefore, we initially will show the exponential decay of the energy associated with the solution $v$ guaranteed by Theorem 3.2.

**Lemma 3.1.** *If $v$ is the solution of the system (3.10)-(3.12), then*

$$E_5(t) \leq K_6 \exp\Big\{-\frac{t}{\Upsilon}\Big\} \quad \textit{for all } t \geq 0, \tag{3.53}$$

*where $E_5(t)$ is defined in (3.22), $K_6 = E_4(0)$, and $\Upsilon$ is real positive constant determinate in (3.61).*

**Proof.** The inequalities (3.53) will be obtained as limit of the approximate solutions $v^m$. The convergence obtained in the proof of Theorem 3.2 allow us to conclude such result. Therefore, we will also denote henceforth the function $v^m$ by $v$. From (3.34) and (3.9) we have for all $t \geq 0$ that

$$\begin{aligned}\frac{d}{dt}E_4(t) + \frac{\nu\zeta_1}{32K_3^6}\,|v_y(t)|^4 + \frac{\nu}{128K_3^4\gamma^4(t)}\,|v_{yy}(t)|^2 +&\\ \frac{\nu}{8\gamma^4(t)}\,|v_{yyt}(t)|^2 + \frac{3\nu}{64K_3^8}\,|v_{yyyy}(t)|^2 \leq 0.&\end{aligned} \tag{3.54}$$

On the other hand, observing that $\dfrac{\nu}{128K_3^4\gamma^4(t)} = \dfrac{\nu}{256K_3^4\gamma^4(t)} + \dfrac{\nu}{256K_3^4\gamma^4(t)}$, and by Poicarè's inequality we get the following inequalities

$$\frac{1}{2}\,|v_t(t)|^2 \leq \Upsilon_0\frac{\nu}{8\gamma^4(t)}\,|v_{yyt}(t)|^2 \quad \text{where } \Upsilon_0 = \frac{4K_3^4}{\nu}. \tag{3.55}$$

$$\frac{\zeta_0}{2\gamma^2(t)}\,|v_y(t)|^2 \leq \Upsilon_1\frac{\nu}{256K_3^4\gamma^4(t)}\,|v_{yy}(t)|^2 \quad \text{and } \Upsilon_1 = \frac{128\zeta_0K_3^6}{\nu}. \tag{3.56}$$

$$\Big(\frac{\zeta_1}{4\gamma^3(t)} - \frac{\sigma\gamma'(t)}{8\gamma^4(t)} + \frac{\nu}{8K_3^4}\frac{\sigma}{4\gamma^3(t)}\Big)\,|v_y(t)|^4 \leq \Upsilon_2\frac{\nu\zeta_1}{32K_3^6}\,|v_y(t)|^4, \tag{3.57}$$

where $\Upsilon_2 = \dfrac{K_3^3}{\nu\zeta_1\gamma_0^4}\Big(8\zeta_1 K_3^4 + 4K_3^3\sigma\varepsilon_0 + \nu\sigma\Big)$ and $\varepsilon_0 = \inf_{t\geq 0}|\gamma'(t)|$.

$$\begin{aligned}&\Big[\frac{\nu^2}{16K_3^4\gamma^4(t)} + \frac{1}{2\gamma^5(t)}\Big(\gamma(t) - 4\nu\gamma'(t)\Big)\Big]\left|v_{yy}(t)\right|^2 \leq \\ &\Upsilon_3 \frac{\nu}{256K_3^4\gamma^4(t)}\left|v_{yyy}(t)\right|^2 \quad \text{where} \\ &\Upsilon_3 = \frac{K_3}{\nu\gamma_0}\Big(\nu^2 + 8K_3^4 + 32\nu\varepsilon_0 K_3^3\Big).\end{aligned} \tag{3.58}$$

From similar way we can affirm that there exist real positive constants $\Upsilon_4$ and $\Upsilon_5$ such that

$$\begin{aligned}\frac{\nu}{8K_3^4}\Big[(v_t(t), v(t)) + (v_t(t), v_{yyyy}(t)) + \frac{\nu}{\gamma^4(t)}\left|v_{yyyy}(t)\right|^2\Big] &\leq \\ \Upsilon_4\left|v_{yyt}(t)\right|^2 + \Upsilon_5\frac{3\nu}{64K_3^2}\left|v_{yyyy}(t)\right|^2&\end{aligned} \tag{3.59}$$

From (3.55)-(3.59) and (3.35) we can write

$$\begin{aligned}E_4(t) \leq \Upsilon\frac{\nu\zeta_1}{32K_3^6}\left|v_y(t)\right|^4 + \Upsilon\frac{\nu}{128K_3^4\gamma^4(t)}\left|v_{yy}(t)\right|^2 &+ \\ \Upsilon\frac{\nu}{8\gamma^4(t)}\left|v_{yyt}(t)\right|^2 + \Upsilon\frac{3\nu}{64K_3^8}\left|v_{yyyy}(t)\right|^2&,\end{aligned} \tag{3.60}$$

where

$$\Upsilon = \max\big\{\Upsilon_0,\ \Upsilon_1,\ \Upsilon_2,\ \Upsilon_3,\ \Upsilon_4,\ \Upsilon_5,\big\}. \tag{3.61}$$

From (3.54) and (3.60) we obtain

$$\frac{d}{dt}E_4(t) + \frac{1}{\Upsilon}E_4(t) \leq 0 \ \text{ for all } \ t \geq 0.$$

It implies that

$$E_4(t) \leq E_4(0)\exp\Big\{-\frac{t}{\Upsilon}\Big\} \ \text{ for all } \ t \geq 0. \tag{3.65}$$

Using (3.46) and the definitions of $E_4(t)$ and $E_5(t)$ in (3.35) and (3.22) respectively, we obtain

$$E_4(t) \geq E_5(t) \ \text{ for all } \ t \geq 0.$$

Hence, we conclude (3.53) □

To conclude this subsection we will prove that the energy of the system (3.1)-(3.3) given by (3.52) has the same behavior of the function $E_5(t)$, i.e.,

**Theorem 3.3.** *If $u$ is the solution of the system (3.1)-(3.3), then*

$$E(t) \leq K_7\exp\Big\{-\frac{t}{\Upsilon}\Big\} \ \textit{for all } \ t \geq 0, \tag{3.63}$$

*where $E(t)$, $K_7$ and $\Upsilon$ are defined in (3.52), (3.69) and (3.61) respectively.*

**Proof.** From (1.7) we have

$$u_t = v_t - \frac{1}{\gamma}\big(\alpha' + y\gamma'\big)v_y, \quad \partial_x^i u = \frac{1}{\gamma^i}\partial_y^i v \ \text{ for } \ i \in \mathbb{N}. \tag{3.64}$$

From $(3.64)_1$ we obtain

$$|u_t(x,t)|^2_{\mathbb{R}} \leq 2\,|v_t(y,t)|^2_{\mathbb{R}} + \frac{2}{\gamma^2(t)}\big(\,|\alpha'(t)| + |\gamma'(t)|\,\big)^2\,|u_y(y,t)|^2_{\mathbb{R}}\,.$$

Integrating this inequalities from $\alpha(t)$ to $\beta(t)$ and observing that $y = \frac{x-\alpha(t)}{\gamma(t)}$ and $dy = \frac{1}{\gamma(t)}dx$ then we get by using (3.4), (3.6) and (3.7) that

$$|u_t(t)|^2_{L^2(\Omega_t)} \leq 2K_3\,|v_t(t)|^2_{L^2(0,1)} + \frac{18\varepsilon^2}{\gamma_0^2}\,|v_y(t)|^2_{L^2(0,1)}\,. \tag{3.65}$$

Using $(3.64)_2$ and repeating the same arguments above we get

$$\left|\partial_x^i u\right|^2_{L^2(\Omega_t)} \leq \frac{1}{\gamma_0^i}\,\left|\partial_y^i v\right|^2_{L^2(0,1)} \quad \text{for } \ i \in \mathbb{N}. \tag{3.66}$$

From (3.5), (3.7) and definition of $E_5(t)$ we obtain

$$\begin{aligned} E_5(t) \geq \frac{1}{4}\,|v_t(t)|^2 + \frac{\sigma}{4K_3^3}\,|v_y(t)|^2 + {}\\ \frac{\nu^2}{64K_3^8}\,|v_y(t)|^4 + \frac{\nu^2}{64K_3^8}\,|v_{yy}(t)|^2\,. \end{aligned} \tag{3.67}$$

Using (3.65), (3.66) and observing the definition of $E(t)$ in (3.52) yields

$$\begin{aligned} E(t) \leq K_3\,|v_t(t)|^2 + \Big(\frac{9\varepsilon^2}{\gamma_0^2} + \frac{\zeta_0}{2\gamma_0}\Big)\,|v_y(t)|^2 + {}\\ \frac{\zeta_1}{4\gamma_0^3}\,|v_y(t)|^4 + \frac{1}{2\gamma_0^3}\,|v_{yy}(t)|^2\,. \end{aligned} \tag{3.68}$$

From (3.53), (3.67) and (3.68) we obtain (3.63) where

$$K_7 = K_6\Big[4K_3 + \Big(\frac{9\varepsilon^2}{\gamma_0^2} + \frac{\zeta_0}{2\gamma_0}\Big)\frac{4K_3^4}{\sigma} + \frac{16\zeta_1 K_3^5}{\nu^2} + \frac{32K_3^5}{\nu^2}\Big]. \tag{3.69}$$

Thus, the proof of Theorem 3.3 is completed $\square$

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