Issue 1 JULY–DECEMBER 2002 ISSN 1083-0464

SOUTHWEST JOURNAL OF PURE AND APPLIED MATHEMATICS

CAMERON UNIVERSITY
DEPARTMENT OF MATHEMATICAL SCIENCES
LAWTON, OK 73505, U.S.A

Southwest Journal
of
Pure and Applied Mathematics

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SWJPAM
Cameron University
Department of Mathematical Sciences
Lawton, OK 73505, U.S.A.

Publication Program

2 issues per year
July, December

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SOUTHWEST JOURNAL
OF
PURE AND APPLIED MATHEMATICS
ISSUE 1, DECEMBER–JULY 2002

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ISSN 1083-0464

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 1, July 2002, pp. 1–5.
Submitted: April 1, 2001. Published: July 1 2002.

A SAMPLING SCHEME FOR RELIABILITY ESTIMATION

Mohamed Djerdjour and Kamel Rekab

Abstract. A Three-stage sampling scheme for estimating the reliability of a series system when the component reliabilities are modeled as beta random variables is proposed. The sampling scheme specifies how a total of T units can be allocated to two independent components. It is shown that the proposed sampling scheme is nearly optimal.

1991 A.M.S. (MOS) Subject Classification Codes. 62l05, 62l10

Key Words and Phrases. Bayes risk, martingales, uniform integrability.

1. Introduction

Suppose that a system is composed of two independent components connected in series and R_1 and R_2 are the corresponding reliabilities. Then the product $R_1 R_2$ is the reliability of the system, and the problem of estimating this reliability is the problem of estimating the product of means of two independent Bernoulli populations.

We take a Bayesian approach to this problem, assuming that prior information of some sort is available and can be modeled in the form of beta priors on R_1 and R_2.

Although selecting the form of an optimal estimator of the reliability of the system is critical, the real difficulty lies in determining an optimal sampling scheme that specifies how to allocate a total number of T units to two independent components. It is assumed that the total number of units to be tested is fixed, but that the number of units of the first component and of the second component to be tested may be random.

In [1], a frequentist approach was used to examine fixed and sequential sampling schemes for estimating the reliability of a series system. A fully sequential sampling scheme was proposed and shown to perform much better than the balanced sampling scheme. In [2], a Bayesian approach was used to examine the efficiency

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of the myopic sampling scheme. In [3], a two-stage sampling scheme was proposed and shown to be nearly optimal. In this article, we propose a three-stage sampling scheme to [3]. Although the focus here is on estimating the product of means instead of the difference between means. In next section, we define the framework used to formulate the problem.

2. Bayes Risk

Since each component either fails or functions, a Bernoulli formulation of this problem is necessary. Let X_{1j} denote the outcome of the j^{th} unit of the first component and X_{2j} denote the outcome of the j^{th} unit of the second component. Then $X_{1j} = 0$ with probability $1 - R_1$, if the j^{th} unit of type one component fails, and $X_{1j} = 1$ with probability R_1, otherwise. Similarly, let $X_{2j} = 0$ with probability $1 - R_2$, if the j^{th} unit of type two component fails, and $X_{2j} = 1$ with probability R_2, otherwise.

Let E_T denote the data generated by the random variables

$$X_{11}, \ldots X_{1M}, X_{21}, \ldots X_{2N},$$

where $M + N = T$. Let the component reliabilities R_1 an R_2 be independent beta distributed with parameters (a_1, b_1) and (a_2, b_2).

The Bayes estimator of the system reliability is the posterior mean

$$\hat{R}(M,N) = \left[\frac{a_1 + \sum_{j=1}^{M} X_{1j}}{M + r_1}\right]\left[\frac{a_2 + \sum_{j=1}^{N} X_{2j}}{N + r_2}\right], \tag{1}$$

where $r_1 = a_1 + b_1$ and $r_2 = a_2 + b_2$. The Bayes risk is given by

$$\begin{aligned} B(M,N) = {} & E[(R_1)^2]E[(R_2)^2] \\ & - E\left[[R_1]^2 - \frac{R_1(1-R_1)}{M + r_1}\right] E\left[[R_2]^2 - \frac{R_2(1-R_2)}{N + r_2}\right]. \end{aligned}$$

3. Three-Stage Sampling Scheme

For random sampling schemes, M, N are random positive integers. The Bayes risk can be approximated by

$$E\left[\frac{R_1(1-R_1)}{M + r_1}[R_2]^2\right] + E\left[\frac{R_2(1-R_2)}{N + r_2}[R_1]^2\right]$$

which can also be written as

$$E\left(\frac{U_M}{M + r_1} + \frac{V_N}{N + r_2}\right),$$

where

$$U_M = E\left\{R_1(1-R_1)R_2^2 \middle| E_T\right\}$$

and

$$V_N = E\left\{ R_2(1-R_2)R_1^2 \middle| E_T \right\}.$$

The approximated Bayes risk is minimized by choosing M such that

$$\frac{M+r_1}{T+r_1+r_2} = \frac{\sqrt{U_M}}{\sqrt{U_M}+\sqrt{V_N}} \tag{2}$$

and $N = T - M$.

Simple calculations yield

$$U_M = \frac{a_{1M}b_{1M}}{(M+r_1)(M+r_1+1)} \frac{a_{2N}(a_{2N}+1)}{(N+r_2)(N+r_2+1)}$$

and

$$V_N = \frac{a_{2N}b_{2N}}{(N+r_2)(N+r_2+1)} \frac{a_{1M}(a_{1M}+1)}{(M+r_1)(M+r_1+1)},$$

where

$$a_{1M} = a_1 + \sum_{j=1}^{M} X_{1j}, \quad b_{1M} = b_1 + M - \sum_{j=1}^{M} X_{1j},$$

$$a_{2N} = a_2 + \sum_{j=1}^{N} X_{2j} \quad \text{and} \quad b_N = b_2 + N - \sum_{j=1}^{N} X_{2j}.$$

There are many sampling schemes that may satisfy (2) for large values of T. To describe the three-stage sampling scheme, let $r = r_T$ and $s = s_T$ be chosen such that

$$r + s < \frac{T}{2},$$

and

$$\frac{r+s}{T} \to 0,$$

as $T \to \infty$.

In the first batch test r units from each component. In the second batch test $m-r$ units from the first component and $n-r$ units from the second component, where

$$m = max\{r, (T-2s)\hat{C}_{r,r}\}$$

where

$$\hat{C}_{r,r} = \frac{\sqrt{U_r}}{\sqrt{U_r}+\sqrt{V_r}},$$

and

$$n = T - 2s - m.$$

In the third batch test $2s$ units of which $M-m$ are from the first component and $N-n$ are from the second component, where

$$M = max\{m, T\hat{C}_{m,n}\}$$

where

$$\hat{C}_{m,n} = \frac{\sqrt{U_m}}{\sqrt{U_m} + \sqrt{V_n}},$$

and

$$N = T - M.$$

Theorem Let R_T denote the Bayes risk incurred by the three-stage sampling scheme. Then

$$\lim_{T\to\infty} MR_T = \frac{a_1 a_2[b_1(a_2+1) + b_2(a_1+1)]}{(a_1+b_1)(a_1+b_1+1)(a_2+b_2)(a_2+b_2+1)} + r_0,$$

where

$$r_0 = \frac{2(a_1+.5)(a_2+.5)\Gamma(a_1+.5)\Gamma(b_1+.5)\Gamma(a_2+.5)\Gamma(b_2+.5)}{(a_1+b_1)(a_1+b_1+1)(a_2+b_2)(a_2+b_2+1)\Gamma(a_1)\Gamma(b_1)\Gamma(a_2)\Gamma(b_2)}.$$

Proof. Let the scaled approximated Bayes risk be denoted by $(T+r+s)R_T$. It is the sum of two positive terms

$$E[(\sqrt{U_M} + \sqrt{V_N})^2]$$

and

$$E\left[\frac{[(N+r_2)\sqrt{U_M} - (M+r_1)\sqrt{V_N}]^2}{(M+r_1)(N+r_2)}\right].$$

The proof will follow by showing that

$$U_M \to R_1(1-R_1)R_2^2 \quad \text{and} \quad V_N \to R_2(1-R_2)R_1^2$$

as $T\to\infty$, and that

$$E\left[\frac{[(N+r_2)\sqrt{U_M} - (M+r)\sqrt{V_N}]^2}{(M+r_1)(N+r_2)}\right] \to 0 \quad \text{as} \quad T\to\infty.$$

Since $M\to\infty$ and $N\to\infty$ and U_M and V_N are uniformly integrable martingale,

$$U_M \to R_1(1-R_1)R_2^2 \quad \text{and} \quad V_N \to R_2(1-R_2)R_1^2$$

by the martingale convergence theorem. Therefore

$$\hat{C}_{M,N} \to C,$$

in probability as $T\to\infty$ where

$$C = \frac{R_2\sqrt{R_1(1-R_1)}}{R_2\sqrt{R_1(1-R_1)} + R_1\sqrt{R_2(1-R_2)}}.$$

From the conditions on r and s it follows that

$$\frac{M}{N} \to \frac{R_2\sqrt{[R_1(1-R_1)]}}{R_1\sqrt{[R_2(1-R_2)]}} \quad \text{almost surely as} \quad T \to \infty.$$

The proof of the theorem follows by showing that

$$\frac{[(N+r_2)\sqrt{U_M} - (M+r)\sqrt{V_N}]^2}{(M+r_1)(N+r_2)} \quad \text{is uniformly integrable.}$$

Since

$$\frac{[(N+r_2)\sqrt{U_M} - (M+r)\sqrt{V_N}]^2}{(M+r_1)(N+r_2)} \leq 2\left[\frac{N+r_2}{M+r_1}U_M + \frac{M+r_1}{N+r_2}V_N\right],$$

uniform integrability follows by showing that

$$\frac{N+r_2}{M+r_1}U_M \quad \text{and} \quad \frac{M+r_1}{N+r_2}V_N \quad \text{are both uniformly integrable.}$$

The rest of the proof is similar to [3]. A sketch of it follows. Using the three-stage sampling scheme,

$$\frac{T}{M} \leq \frac{1}{\hat{C}_{m,n}},$$

therefore

$$\frac{N+r_2}{M+r_1}U_M \leq \frac{U_M}{\hat{C}_{m,n}} + U_M(r_1+r_2).$$

Since U_M is uniformly integrable and

$$\frac{U_M}{\hat{C}_{m,n}} \leq max_{1\leq k\leq T}(U_k + \sqrt{U_k V_k}),$$

uniform integrability is established by applying Doob's inequality for nonnegative submatingales. A completely anlogous argument for $\frac{M+r_1}{N+r_2}V_N$ yields the theorem.

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Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 1, July 2002, pp. 6–32.
Submitted: May 12, 2002. Published: July 1 2002.

FINDING ELLIPSES AND HYPERBOLAS TANGENT TO TWO, THREE, OR FOUR GIVEN LINES

Alan Horwitz

Abstract. Given lines $L_j,\ j=1,2,3,4$, in the plane, such that no three of the lines are parallel or are concurrent, we want to find the locus of centers of ellipses tangent to the L_j. In the case when the lines form the boundary of a four sided convex polygon R, let M_1 and M_2 be the midpoints of the diagonals of R. Let L be the line thru M_1 and M_2, let Z be the open line segment connecting M_1 and M_2, let Y be the closed line segment connecting M_1 and M_2, and let X be the open line segment which is the part of L lying inside R. It is well known that if an ellipse E is **inscribed** in R, then the center of E must lie on Z(see [1] and [2]). We prove(Theorem 11) that every point of Z is the center of some ellipse inscribed in R, which implies that the locus of centers of ellipses inscribed in R is precisely equal to Z. In addition, we prove(Theorem 11) that there is a hyperbola tangent to each of the L_j and with center $(h,k)\in R$ if and only if $(h,k)\in X-Y$. More generally, any ellipse tangent to the L_j(and not just inscribed ones) must have its center on L.

1991 A.M.S. (MOS) Subject Classification Codes. 51.

Key Words and Phrases. Ellipse, hyperbola, tangent, quadrilateral.

Introduction

Given finitely lines $L_j,\ j=1,2,3,...,N$ in the plane, such that no three of the lines are parallel or are concurrent, we want to find the locus of centers of ellipses tangent to the L_j. Our main results concern four given lines (see §3), though we need results for two or three given tangents (see §1, §2, and §4). Most of the theorems extend easily to hyperbolas as well. It is useful to make the following definition.

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Definition. Given a finite set of distinct lines $L_1, L_2, ..., L_N$ in the plane, and an angle α, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, suppose that there is an ellipse with center (h,k) and rotation angle α which is tangent to each of the L_j. Then we say that (h,k) is α **admissible** . If $\alpha = 0$ we just call (h,k) admissible. We allow the angle of rotation to **vary**, and then look at the union, over α, of the α admissible centers, S_α. In the case when the lines form the boundary of a four sided convex polygon R, let M_1 and M_2 be the midpoints of the diagonals of R. Let L be the line thru M_1 and M_2, let Z be the open line segment connecting M_1 and M_2, let X be the open line segment which is the part of L lying inside R, and let Y equal the **closed** line segment connecting M_1 and M_2. It is well known that if an ellipse E is **inscribed** in R, then the center of E must lie on Z(see [1] and [2]). We prove the stronger result(Theorem 11) that there is an ellipse inscribed in R with center (h,k) if and only if $(h,k) \in Z$. In addition, we prove(Theorem 11) that there is a hyperbola tangent to each of the L_j and with center $(h,k) \in R$ if and only if $(h,k) \in X - Y$.
Remark : Due to a miscommunication the paper "Finding an Ellipse Tangent to finitely many given Lines"[3], which appeared in SWJPM in December of 2000, was not the final version to be published. Section 2 is a modification and simplification of results which appeared in [3]. Section 4 contains a simplified and abridged version of material which appeared in [3]. It also contains corrections for numerous errors in the original version. All of our results in Section 3 for four given tangents are new.

§1. Preliminary Material

Our results are stated and proved first for ellipses with major and minor axes parallel to the x and y axes. Later we indicate how to extend the results easily to rotated ellipses. We use the fact that at any point of the ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, $dy/dx = \left(-\frac{b^2}{a^2}\right)\left(\frac{x-h}{y-k}\right)$.

Lemma 1. Let (h,k) be given. Suppose that L is a non-vertical line with equation $y - k = m(x-h) + B$
(i) If (u,v) is a **positive** solution of the equation $m^2u + v = B^2$, then $B \neq 0$, and L is tangent to the ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ at (r,s), where $a^2 = u, b^2 = v$, $r = h - \frac{a^2 m}{B}$, and $s = k + \frac{b^2}{B}$.
(ii) If L is tangent to the ellipse E: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ at (r,s), then $B \neq 0, s \neq k$, and $m^2a^2 + b^2 = B^2$.

Proof. Since $u,v > 0$, $B \neq 0$. $\frac{(r-h)^2}{a^2} + \frac{(s-k)^2}{b^2} = \frac{a^2m^2 + b^2}{B^2} = 1$ and $m(r-h) + B = m\left(-\frac{a^2m}{B}\right) + B = \frac{-a^2m^2 + B^2}{B} = \frac{b^2}{B} = s - k$. Thus (r,s) lies on both E and L. Note that $s \neq k$, and since $B \neq 0$, $\frac{r-h}{s-k} = \frac{-a^2m}{b^2}$. Thus $\left[\frac{dy}{dx}\right]_{x=r, y=s} = \frac{-b^2(r-h)}{a^2(s-k)} = m$, and L is tangent to E at (r,s), which proves (i). If $B = 0$, then L passes thru (h,k), which is impossible if L is tangent to an ellipse with center

(h,k). If $s = k$, then L is vertical. Since $dy/dx = \left(-\frac{b^2}{a^2}\right)\left(\frac{x-h}{y-k}\right)$ at any point of $E(y \neq k)$, we have

$$m = -\frac{b^2}{a^2}\frac{r-h}{s-k} \tag{1}$$

Also,

$$s - k = m(r-h) + B \tag{2}$$

Using (1) and substituting (2) gives $a^2 m\,(m(r-h)+B) + b^2(r-h) = 0$, which implies

$$r = h - \frac{a^2 mB}{a^2m^2+b^2} \tag{3}$$

By (2) we also have

$$s = k + \frac{b^2 B}{a^2m^2+b^2} \tag{4}$$

(3), (4), and $\frac{(r-h)^2}{a^2} + \frac{(s-k)^2}{b^2} = 1$ imply that $\frac{a^2m^2B^2 + b^2B^2}{(a^2m^2+b^2)^2} = 1$, which implies that $B^2 = a^2m^2 + b^2$. That proves (ii).

Proposition 1. Suppose that L_1 and L_2 are non-vertical lines which are tangent to the ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ at (r_j, s_j), $j = 1,2$. Suppose that L_j has equation

$$y - k = m_j(x-h) + b_j \tag{5}$$

$j = 1,2$, with $m_1^2 \neq m_2^2$. Then

$$a^2 = \frac{b_2^2 - b_1^2}{m_2^2 - m_1^2},\; b^2 = \frac{b_1^2 m_2^2 - b_2^2 m_1^2}{m_2^2 - m_1^2} \tag{6}$$

Proof. By Lemma 1, part (ii), $a^2 = u$ and $b^2 = v$, where u and v satisfy the **nonsingular** linear system

$$\begin{aligned} m_1^2 u + v &= b_1^2 \\ m_2^2 u + v &= b_2^2 \end{aligned} \tag{7}$$

Substituting back for the unique solution (u,v) yields (6).

Remark. Throughout the paper we suppress, in our notation, the fact that if $y = L_j(x) = m_j x + c_j$ and $y - k = m_j(x-h) + b_j$ represent the same line, then

$$b_j = L_j(h) - k = m_j h + c_j - k \tag{8}$$

is really a function of h and k.

Theorem 1. Let h and k be given real numbers, and let L_1 and L_2 be distinct, non–vertical lines with equations $y-k=m_j(x-h)+b_j,\ j=1,2,\ m_1^2\neq m_2^2$.
Part 1: If

$$\frac{b_2^2-b_1^2}{m_2^2-m_1^2}>0 \text{ and } \frac{b_1^2m_2^2-b_2^2m_1^2}{m_2^2-m_1^2}>0 \tag{9}$$

then there is a unique ellipse E(non-rotated), with center (h,k), which has L_1 and L_2 as tangents. Furthermore, E has equation $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$, where a^2 and b^2 are given by (6).
Part 2: (9) is also a *necessary* condition for L_1 and L_2 to be tangent to some ellipse(non-rotated), with center (h,k).
Proof. Part 1: Since (9) holds, (7) has the unique positive solution (u,v) with $u=a^2$ and $v=b^2$, a^2 and b^2 defined as in (6). By Lemma 1, part (i), the L_j are tangent, at (r_j,s_j), to the ellipse E with equation $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$, where $r_j=h-\dfrac{a^2m_j}{b_j}$ and $s_j=k+\dfrac{b^2}{b_j}$. Now if $\tilde{E}$ is *any* ellipse with center (h,k) and tangent to the L_j, then the equation of $\tilde{E}$ is $\dfrac{(x-h)^2}{\tilde{a}^2}+\dfrac{(y-k)^2}{\tilde{b}^2}=1$. By Proposition 1, $\tilde{a}$ and $\tilde{b}$ are given by (6), which implies that $\tilde{a}=a^2$ and $\tilde{b}^2=b^2$. That proves uniqueness.
Part 2: Follows immediately from Proposition 1.
Given L_1 and L_2, we can now give an equivalent characterization of the set of admissible centers.

Corollary 1. Let $y=L_1(x)$ and $y=L_2(x)$ be distinct, non–vertical lines with equations $y=L_j(x)=k+m_j(x-h)+b_j, j=1,2$ and $|m_1|<|m_2|$. Then there is an ellipse E(non-rotated) with center (h,k), which has L_1 and L_2 as tangents if and only if

$$\left|\frac{m_1}{m_2}\right||b_2|<|b_1|<|b_2| \tag{10}$$

or equivalently

$$\left|\frac{m_1}{m_2}\right|<\left|\frac{L_1(h)-k}{L_2(h)-k}\right|<1 \tag{11}$$

Proof. Since $|m_1|<|m_2|$, (10) is equivalent to (9). Since $b_j=L_j(h)-k$, (10) is equivalent to (11) and the fact that $b_2\neq 0\Leftrightarrow L_2(h)-k\neq 0$.
Example. $L_1:\ y=x+3$ and $L_2:\ y=2x-3$ are given. Then L_1 and L_2 are tangents to a non-rotated ellipse E with center (h,k) if and only if $\dfrac{1}{2}<\left|\dfrac{h+3-k}{2h-3-k}\right|<1$.

Remark. One can also obtain results similar to Proposition 1 or Theorem 1 if one of the L_j(but not **both**) is *vertical* . If L_1 has equation $x-h=b_1$, then $b_1=a$ or $-a$, which implies that $a^2=b_1^2$. Since $m_2^2a^2+b^2=b_2^2$ still holds, one then solves to get b^2.
We shall have occasion to use the following theorem, which follows immediately from Lemma 1, Proposition 1, and Theorem 1.

Theorem 2. Given N non-vertical lines $L_1,...,T_N$ with equations $y-k=m_j(x-h)+b_j,\ j=1,...,N$.
(i) There is an ellipse E(non-rotated), with center (h,k), tangent to each of the L_j if and only if the linear system

$$\begin{aligned} m_1^2u+v&=b_1^2\\ m_2^2u+v&=b_2^2\\ &\vdots\\ m_N^2u+v&=b_N^2 \end{aligned} \tag{12}$$

has a **positive** solution (u,v). If $m_i^2\neq m_j^2$ whenever $i\neq j$, then E is unique and has equation $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$, where $a^2=u$ and $b^2=v$.
(ii) If $m_i^2\neq m_j^2$ whenever $i\neq j$ and (12) is consistent, then for any $i\neq j$, the unique solution of (12) is

$$u=\frac{b_j^2-b_i^2}{m_j^2-m_i^2}\text{ and }v=\frac{b_i^2m_j^2-b_j^2m_i^2}{m_j^2-m_i^2} \tag{13}$$

Remark. If $m_1^2=m_2^2$, then the existence-uniqueness result above fails. We discuss this for two given lines in the next theorem.

Theorem 3. Let h and k be given real numbers, and let L_1 and L_2 be distinct, non–vertical lines with equations $y-k=m_j(x-h)+b_j,\ j=1,2$. Suppose that $m_1^2=m_2^2$.
(i) If $|b_1|=|b_2|\neq 0$, then L_1 and L_2 are tangent to any ellipse $E\in F=\left\{\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1\right\}$, where a^2 and b^2 are any positive real numbers satisfying the equation

$$a^2m_1^2+b^2=b_1^2 \tag{14}$$

(ii) If $|b_1|\neq|b_2|$ or $b_1=0=b_2$, then L_1 and L_2 are not tangent to any ellipse(non-rotated) with center (h,k).
Proof. Since the line with equation $m_1^2u+v=b_1^2$ has a positive v intercept and a non-positive slope, $m_1^2u+v=b_1^2$ has infinitely many positive solutions in the unknowns u and v. If $a^2=u,b^2=v$ for some positive solution (u,v), then (14) holds. Also, $a^2m_2^2+b^2=b_2^2$ since $m_1^2=m_2^2$ and $b_1^2=b_2^2$. (i) then follows from Lemma 1, part (i). To prove (ii), suppose that L_1 and L_2 are tangent to an ellipse E with equation $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$. By Lemma 1, part (ii), $b_j^2=a^2m_j^2+b^2,\ j=1,2$.

Lemma 1, part (ii), $m_1^2 = m_2^2$,, and $b_1^2 = b_2^2$, then imply that $b_1^2 \neq 0 \neq b_2^2$. That contradicts the assumptions in (ii).

Rotated Ellipses

For simplicity of exposition, we have only considered non-rotated ellipses. However, the previous results extend with little effort to ellipses whose axes are *rotated* clockwise about $(0,0)$ thru a specified angle $\alpha, -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$. Let (x_α, y_α) denote the new coordinates, and suppose that we are given (h,k) in xy coordinates. Assume throughout that no given line is parallel to the y_α axis(i.e., all lines are non–vertical in the rotated coordinates). It follows easily that the line $y - k = m(x-h) + b$ becomes

$$y_\alpha - k_\alpha = \left(\frac{m + \tan\alpha}{1 - m\tan\alpha}\right)(x_\alpha - h_\alpha) + \frac{b\sec\alpha}{1 - m\tan\alpha} \tag{15}$$

where $h_\alpha = h\cos\alpha - k\sin\alpha, k_\alpha = h\sin\alpha + k\cos\alpha$ and $1 - m\tan\alpha \neq 0$. We find it useful to introduce the following notation. For given $m_{i,\alpha}, b_{i,\alpha}, m_{j,\alpha}, b_{j,\alpha}$, and α:

$$S_{ij} = \frac{b_{j,\alpha}^2 - b_{i,\alpha}^2}{m_{j,\alpha}^2 - m_{i,\alpha}^2}, T_{ij} = \frac{b_{i,\alpha}^2 m_{j,\alpha}^2 - b_{j,\alpha}^2 m_{i,\alpha}^2}{m_{j,\alpha}^2 - m_{i,\alpha}^2} \tag{16}$$

We now state the following generalization of Theorem 1 without proof.

Theorem 4. Let h_α and k_α be given real numbers, and let L_1 and L_2 be distinct lines with equations $y_\alpha - k_\alpha = m_{j,\alpha}(x_\alpha - h_\alpha) + b_{j,\alpha},\ j = 1,2$. Suppose that $m_{2,\alpha}^2 \neq m_{1,\alpha}^2$.
Part 1: If

$$S_{12} > 0 \text{ and } T_{12} > 0 \tag{17}$$

then there is a unique ellipse E, with equation $\frac{(x_\alpha - h_\alpha)^2}{a^2} + \frac{(y_\alpha - k_\alpha)^2}{b^2} = 1$, which has L_1 and L_2 as tangents, where $a^2 = S_{12}$ and $b^2 = T_{12}$
Part 2: (17) is also a *necessary* condition for L_1 and L_2 to be tangent to some ellipse with center (h,k) and rotation angle α.
Remark. As in the non-rotated case, if $|m_{1,\alpha}| < |m_{2,\alpha}|$, then (17) is equivalent to

$$\left|\frac{m_{1,\alpha}}{m_{2,\alpha}}\right| |b_{2,\alpha}| < |b_{1,\alpha}| < |b_{2,\alpha}| \tag{18}$$

It is also interesting to allow the rotation angle to *vary* . This leads to the following
Question: Let L_1 and L_2 be two given nonparallel lines. Is any point $(h,k) \notin L_1 \cup L_2$ α admissible for some α, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$? The following theorem answers this question in the affirmative.

Theorem 5. Let h and k be given real numbers, and let L_1 and L_2 be distinct, non–vertical lines with equations $y - k = m_j(x-h) + b_j,\ j = 1,2$. Assume that $m_1 \neq m_2$ and that $(h,k) \notin L_1 \cup L_2$. Then there is an angle $\alpha, -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, such that the ellipse E with center (h,k) and rotation angle α has L_1 and L_2 as tangents.

Proof. Note that $(h,k) \notin L_1 \cup L_2 \Rightarrow b_1 \neq 0 \neq b_2$. We want to apply (10) of Corollary 1 in the *new* coordinates x_α and y_α. Using (15) , we replace m_j by $m_{j,\alpha}$ and b_j by $b_{j,\alpha}$, where $m_{j,\alpha} = \dfrac{m_j + \tan\alpha}{1 - m_j \tan\alpha}$ and $b_{j,\alpha} = \dfrac{b_j \sec\alpha}{1 - m_j \tan\alpha}$ Since we are allowing the angle of rotation to vary, we may rotate the coordinate axes so that in the new coordinates, $m_1 = 0 < m_2$. Then $\left|\dfrac{m_1}{m_2}\right| |b_2| < |b_1|$. If $|b_1| < |b_2|$, then by the corollary to Theorem 1, there is a non-rotated ellipse tangent to the L_j. So assume now that $|b_2| \leq |b_1|$, and let $\alpha_0 = \arctan(1/m_2)$. Note that $\dfrac{m + \tan\alpha}{1 - m\tan\alpha}$ is a strictly increasing function of m, which implies that $m_{1,\alpha} < m_{2,\alpha}$ for any given α. As $\alpha \to \alpha_0$, $\dfrac{|b_{1,\alpha}|}{|m_{1,\alpha}|} = \left|\dfrac{b_1 \sec\alpha}{\tan\alpha}\right| \to \left|\dfrac{b_1 \sec\alpha_0}{\tan\alpha_0}\right| = |b_1| \sqrt{1 + m_2^2}$, and $\dfrac{|b_{2,\alpha}|}{|m_{2,\alpha}|} = \left|\dfrac{b_2 \sec\alpha}{m_2 + \tan\alpha}\right| \to \left|\dfrac{b_2 \sec\alpha_0}{m_2 + \tan\alpha_0}\right| = \dfrac{|b_2|}{\sqrt{1 + m_2^2}}$. Note that $\tan\alpha_0 = \dfrac{1}{m_2} \Rightarrow m_2 + \tan\alpha_0 \neq 0$. Now $\dfrac{|b_{2,\alpha}|}{|m_{2,\alpha}|} < \dfrac{|b_{1,\alpha}|}{|m_{1,\alpha}|}$ if and only if $\dfrac{|b_2|}{\sqrt{1 + m_2^2}} < |b_1| \sqrt{1 + m_2^2}$ if and only if $\left|\dfrac{b_2}{b_1}\right| < 1 + m_2^2$, which holds since $\left|\dfrac{b_2}{b_1}\right| \leq 1 < 1 + m_2^2$. Also, $\lim_{\alpha \to \alpha_0} |b_{2,\alpha}| = \lim_{\alpha \to \alpha_0} \left|\dfrac{b_2 \sec\alpha}{1 - m_2 \tan\alpha}\right| = \infty$, while $\lim_{\alpha \to \alpha_0} |b_{1,\alpha}| = \lim_{\alpha \to \alpha_0} |b_1 \sec\alpha| \neq \infty$. Thus for α sufficiently close to α_0, $\left|\dfrac{m_{1,\alpha}}{m_{2,\alpha}}\right| |b_{2,\alpha}| < |b_{1,\alpha}| < |b_{2,\alpha}|$, and the theorem follows from the remark after Theorem 4.

An Aside on Hyperbolas

For **hyperbolas** rather than ellipses tangent to L_1 and L_2, one would want a solution of (7) with $uv < 0$. In Lemma 1, part (ii), if T is tangent to the hyperbola $H : \dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = \pm 1$ at (r,s), then $B \neq 0, s \neq k$, and $m^2 a^2 - b^2 = \pm B^2$. It then follows that a necessary and sufficient condition to have a hyperbola tangent to L_1 and L_2 is $\dfrac{b_2^2 - b_1^2}{m_2^2 - m_1^2} \dfrac{b_1^2 m_2^2 - b_2^2 m_1^2}{m_2^2 - m_1^2} < 0$. A version of Theorem 4 for hyperbolas follows immediately.

Theorem 6. Let h_α and k_α be given real numbers, and let L_1 and L_2 be distinct lines with equations $y_\alpha - k_\alpha = m_{j,\alpha}(x_\alpha - h_\alpha) + b_{j,\alpha}$, $j = 1, 2$. Assume that $m_{2,\alpha}^2 \neq m_{1,\alpha}^2$.

Part 1: If

$$S_{12} T_{12} < 0 \tag{19}$$

then there is a unique hyperbola H, with center (h,k) and rotation angle α, which has L_1 and L_2 as tangents.

Part 2: (19) is also a *necessary* condition for a hyperbola with center (h,k) and rotation angle α to exist which has L_1 and L_2 as tangents.

§2 Rotated Versions of Key Theorems from §1
Assume throughout, unless stated otherwise, that no given line is parallel to the y_α axis. The line L_j with equation $y = m_j x + c_j$ becomes

$$y_\alpha = m_{j,\alpha} x_\alpha + c_{j,\alpha} \tag{20}$$

where

$$m_{j,\alpha} = \frac{m_j + \tan\alpha}{1 - m_j \tan\alpha} \text{ and } c_{j,\alpha} = \frac{c_j \sec\alpha}{1 - m_j \tan\alpha} \tag{21}$$

$1 - m_j \tan\alpha \neq 0$. Rewrite L_j in the form

$$y_\alpha - k_\alpha = m_{j,\alpha}(x_\alpha - h_\alpha) + b_{j,\alpha}, b_{j,\alpha} = m_{j,\alpha} h_\alpha + c_{j,\alpha} - k_\alpha \tag{22}$$

In xy coordinates, $h_\alpha = h\cos\alpha - k\sin\alpha$ and $k_\alpha = h\sin\alpha + k\cos\alpha$ for some real numbers h and k. Dividing thru by $\cos\alpha$ and using the substitution

$$w = \tan\alpha$$

yields

$$h_\alpha = \frac{1}{\sqrt{1+w^2}}(h - kw), k_\alpha = \frac{1}{\sqrt{1+w^2}}(hw + k) \tag{23}$$

which gives

$$b_{j,\alpha} = \frac{m_{j,\alpha}}{\sqrt{1+w^2}}(h - kw) + c_{j,\alpha} - \frac{1}{\sqrt{1+w^2}}(hw + k) \tag{24}$$

We shall always speak of α admissible centers in xy coordinates. Throughout, given $L_1, ..., L_N$, $\Re_\alpha$ denotes the set of α admissible centers, i.e., the set of all (h, k) such that the ellipse, $\frac{(x_\alpha - h_\alpha)^2}{a^2} + \frac{(y_\alpha - k_\alpha)^2}{b^2} = 1$, is tangent to each of the L_j. If L_i and L_j are not parallel, we let $i(L_i, L_j) = (x_{l,\alpha}, y_{l,\alpha}), i \neq l \neq j$ denote their point of intersection(in rotated coordinates). Theorems 14-17(proved in §4) and Theorem 2 are easily extendable to rotated ellipses, which we state here.

Theorem 7. Suppose one is given N lines $L_1, ..., L_N$ with equation given by (22) and an angle $\alpha, -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$. Then
(i) There is an ellipse E with center (h, k) and rotation angle α tangent to each of the L_j if and only if the linear system

$$\begin{aligned} m_{1,\alpha}^2 u + v &= b_{1,\alpha}^2 \\ m_{2,\alpha}^2 u + v &= b_{2,\alpha}^2 \\ &\vdots \\ m_{N,\alpha}^2 u + v &= b_{N,\alpha}^2 \end{aligned} \tag{25}$$

has a positive solution (u,v). If $m_{i,\alpha}^2 \neq m_{j,\alpha}^2$ whenever $i \neq j$, then E is unique and the equation of E in the new rotated coordinates is $\dfrac{(x_\alpha - h_\alpha)^2}{a^2} + \dfrac{(y_\alpha - k_\alpha)^2}{b^2} = 1$, where $a^2 = u$ and $b^2 = v$. (ii) If $m_{i,\alpha}^2 \neq m_{j,\alpha}^2$ whenever $i \neq j$ and (25) is consistent, then for any $i \neq j$, the unique solution of (25) is (see (16))

$$u = S_{ij} \text{ and } v = T_{ij} \tag{26}$$

Expanding the determinants and the left-hand side of (36) in Theorem 12 yields

Theorem 8. Let $L_j,\ j = 1,2,3$, be distinct, non-concurrent lines with equations given by (20), $i \neq j \Rightarrow m_{i,\alpha}^2 \neq m_{j,\alpha}^2$. Let $d_\alpha = \prod\limits_{j>i} (m_{j,\alpha} - m_{i,\alpha})$,
$a_{1,\alpha} = m_{1,\alpha}c_{1,\alpha}\left(m_{3,\alpha}^2 - m_{2,\alpha}^2\right) - m_{2,\alpha}c_{2,\alpha}\left(m_{3,\alpha}^2 - m_{1,\alpha}^2\right) + m_{3,\alpha}c_{3,\alpha}\left(m_{2,\alpha}^2 - m_{1,\alpha}^2\right)$,
$a_{2,\alpha} = -c_{1,\alpha}\left(m_{3,\alpha}^2 - m_{2,\alpha}^2\right) + c_{2,\alpha}\left(m_{3,\alpha}^2 - m_{1,\alpha}^2\right) - c_{3,\alpha}\left(m_{2,\alpha}^2 - m_{1,\alpha}^2\right)$,
$a_{3,\alpha} = (c_{1,\alpha}^2(m_{3,\alpha}^2 - m_{2,\alpha}^2) - c_{2,\alpha}^2(m_{3,\alpha}^2 - m_{1,\alpha}^2) + c_{3,\alpha}^2(m_{2,\alpha}^2 - m_{1,\alpha}^2))/2$.
Then (25), with $N = 3$, has a unique solution if and only if (h_α, k_α) lies on the curve γ with equation

$$d_\alpha h_\alpha k_\alpha + a_{1,\alpha}h_\alpha + a_{2,\alpha}k_\alpha + a_{3,\alpha} = 0 \tag{27}$$

Theorem 9. Let L_1 and L_2 be distinct lines with equations given by (20), $0 \neq m_{1,\alpha}^2 \neq m_{2,\alpha}^2$. Let L_3 be the line with equation $x_\alpha = c_{3,\alpha}$. Let γ be the curve with equation
$(h_\alpha - x_{3,\alpha})(k_\alpha - a_{1,\alpha}) = -\frac{1}{2}(m_{1,\alpha} + m_{2,\alpha})(c_{3,\alpha} - x_{3,\alpha})^2$, where
$a_{1,\alpha} = L_2(c_{3,\alpha}) + L_1(c_{3,\alpha}) - y_{3,\alpha}$. Let $q_3(k_\alpha) = \prod\limits_{j=1}^{3} (k_\alpha - y_{j,\alpha})$,
$q_1(k_\alpha) = m_{1,\alpha}^2(k_\alpha - a_{1,\alpha})$. Then $(h,k) \in \Re_\alpha \Leftrightarrow (h_\alpha, k_\alpha) \in \gamma,\ h_\alpha \neq c_{3,\alpha}$ and $\dfrac{q_3(k_\alpha)}{q_1(k_\alpha)} > 0$. For each $(h,k) \in \Re_\alpha$, there is a unique ellipse E with equation $\dfrac{(x_\alpha - h_\alpha)^2}{a^2} + \dfrac{(y_\alpha - k_\alpha)^2}{b^2} = 1$ which is tangent to each of the L_j, with $a^2 = (h_\alpha - c_{3,\alpha})^2$ and $b^2 = \dfrac{q_3(k_\alpha)}{q_1(k_\alpha)}$.

§3 Main Results for Four Given Tangents

We now discuss the locus of centers of ellipses tangent to **four** given lines, where the angle of rotation is allowed to vary. In particular, using the methods of this paper we are able to strengthen and extend some known results about ellipses inscribed in quadrilaterals. Given four lines in the plane, $L_j,\ j = 1,2,3,4$, such that no three of the lines are parallel or are concurrent, we want to find the locus of centers of ellipses tangent to the lines. In the case when the lines form the boundary of a four sided convex polygon R, let M_1 and M_2 be the midpoints of the diagonals of R. Let L be the line thru M_1 and M_2, let Z be the open line segment connecting M_1 and M_2, let Y be the **closed** line segment connecting M_1 and M_2, and let X be the open line segment which is the part of L lying inside R. It is well known that if an ellipse E is **inscribed** in R, then the center of E must lie on Z(see [1] and [2]). The proof of this result actually goes back to Newton. It is stated in [2] that the locus of centers of ellipses inscribed in R actually **equals** Z, but Newton only proved that the center of E must lie on Z, as is noted in [1]. Indeed, it is not even clear that an ellipse even **exists** which is inscribed in R, let alone whether **every point** of Z is the center of such an ellipse. We prove(Theorem 11) that every point of Z is the center of some ellipse inscribed in R, which implies that the locus of centers of ellipses inscribed in R is precisely equal to Z. In addition, we prove (Theorem 11) that there is a hyperbola tangent to each of the L_j and with center $(h,k) \in R$ if and only if $(h,k) \in X - Y$. More generally, any ellipse tangent to the L_j(and not just inscribed ones) must have its center on L.

Theorem 11. Let R be a four sided **convex** polygon in the xy plane which is not a parallelogram, and let M_1 and M_2 be the midpoints of the diagonals of R. Let L be the line thru M_1 and M_2, and let L_j, $j = 1,2,3,4$ denote the lines which make up the boundary of R. Let Z be the open line segment connecting M_1 and M_2, let X be the open line segment which is the part of L lying inside R, and let Y equal the **closed** line segment connecting M_1 and M_2.
(i) If an ellipse E or hyperbola H is tangent to each of the L_j, then the center of E or of H must lie on L.
(ii) There is an ellipse E, **inscribed** in R and with center (h,k), if and only if $(h,k) \in Z$.
(iii) There is a hyperbola tangent to each of the L_j and with center $(h,k) \in R$, if and only if $(h,k) \in X - Y$.
Remark. If R is a parallelogram, then an ellipse E which is inscribed in R must have center equal to the center of R. There is no hyperbola tangent to each of the L_j and with center $(h,k) \in R$.
Proof. We shall assume that no two sides of R are parallel. The proof when at least two sides of R are parallel is similar, but somewhat simpler, than the proof given here. By using an affine transformation, we may assume that the vertices of R are $(0,0), (1,0)$, $(0,1)$, and (s,t) for some real numbers s and t. Since R is convex, it follows easily that

$$s > 0, t > 0 \text{ and } s + t \geq 1$$

Since R is four sided and no two sides of R are parallel,

$$s + t > 1 \text{ and } s \neq 1 \neq t$$

Write $L_1 : y = 0, L_2 : x = 0, L_3 : y = 1 + \left(\dfrac{t-1}{s}\right)x, L_4 : y = \dfrac{t}{s-1}(x-1)$. For fixed α, (21) gives $m_{1,\alpha} = w$, $m_{3,\alpha} = \dfrac{t-1+ws}{s-(t-1)w}$, $m_{4,\alpha} = \dfrac{t+w(s-1)}{s-1-wt}$, $c_{1,\alpha} = 0$, $c_{3,\alpha} = \dfrac{s\sqrt{1+w^2}}{s-(t-1)w}$, $c_{4,\alpha} = -t\dfrac{\sqrt{1+w^2}}{s-1-wt}$. The equation of L_2 in x_α and y_α coordinates is given by (20), with $m_{2,\alpha} = -\dfrac{1}{w}$ and $c_{2,\alpha} = 0$. Writing L_j in the form $y_\alpha - k_\alpha = m_{j,\alpha}(x_\alpha - h_\alpha) + b_{j,\alpha}$ yields, by (24), $b_{1,\alpha} = -k\sqrt{1+w^2}$,
$b_{2,\alpha} = -h\dfrac{\sqrt{1+w^2}}{w}$, $b_{3,\alpha} = \dfrac{((t-1)h-(k-1)s)\,w^2 + (t-1)h - (k-1)s}{(s-(w-1)t)\sqrt{1+w^2}}$,

$b_{4,\alpha} = \dfrac{(t(h-1)-k(s-1))\,w^2 + t(h-1) - k(s-1)}{(s-1-wt)\sqrt{1+w^2}}$. In general, $m_{j,\alpha}^2 - m_{i,\alpha}^2 = c_{i,j}\left(1+w^2\right)\dfrac{p_{i,j}(w)}{q_{i,j}(w)}$, where the $c_{i,j}$ are nonzero constants depending on s and t, and $p_{i,j}$ and $q_{i,j}$ are polynomials depending on s and t, with deg $p_{i,j} = 2$ and deg $q_{i,j} \le 4$. In particular, $m_{2,\alpha}^2 - m_{1,\alpha}^2 = \dfrac{1-w^4}{w^2}$. Now $m_{j,\alpha}$ and $c_{j,\alpha}$ are not defined, for $j \ge 2$, when

$$w \in F_1 = \{0, \frac{s}{t-1}, \frac{s-1}{t}\}$$

Let

$$F_2 = \{w : m_{j,\alpha}^2 - m_{i,\alpha}^2 = 0 \text{ for some } i \neq j\}$$

and let

$$F = F_1 \cup F_2, G = \{w : -\infty < w < \infty\} - F$$

We must assume, for now, that $w \in G$, and in particular we want $m_{j,\alpha}^2 - m_{i,\alpha}^2 \neq 0$ so that we can apply Theorem 8. For s and t given, F is a *finite* set. By Theorem 8, applied to L_1, L_2, L_4 and L_2, L_3, L_4, respectively, if $w \in G$, then an admissible center (h,k) must lie on the curves with equations

$$d_\alpha h_\alpha k_\alpha + a_{1,\alpha} h_\alpha + a_{2,\alpha} k_\alpha + a_{3,\alpha} = 0 \tag{28}$$

and

$$D_\alpha h_\alpha k_\alpha + A_{1,\alpha} h_\alpha + A_{2,\alpha} k_\alpha + A_{3,\alpha} = 0 \tag{29}$$

where $m_{4,\alpha}$ replaces $m_{3,\alpha}$ when Theorem 8 is applied to L_1, L_2, and L_4, and $m_{4,\alpha}$ replaces $m_{1,\alpha}$ when Theorem 8 is applied to L_2, L_3, and L_4. In (29) we used capital letters to better distinguish the two equations. Using (23), we can rewrite (28) and (29) in the original xy coordinate system to obtain

$$d_\alpha \tfrac{1}{1+w^2}(h-kw)(hw+k) + a_{1,\alpha}\tfrac{1}{\sqrt{1+w^2}}(h-kw) + a_{2,\alpha}\tfrac{1}{\sqrt{1+w^2}}(hw+k) + a_{3,\alpha} = 0 \tag{30}$$

and

$$D_\alpha \tfrac{1}{1+w^2}(h-kw)(hw+k) + A_{1,\alpha}\tfrac{1}{\sqrt{1+w^2}}(h-kw) + A_{2,\alpha}\tfrac{1}{\sqrt{1+w^2}}(hw+k) + A_{3,\alpha} = 0 \tag{31}$$

Note that d_α and D_α have the factor $m_{4,\alpha} - m_{2,\alpha}$ in common. Thus multiplying (30) by $(m_{4,\alpha} - m_{3,\alpha})(m_{3,\alpha} - m_{2,\alpha})$, multiplying (31) by $(m_{4,\alpha} - m_{1,\alpha})(m_{2,\alpha} - m_{1,\alpha})$, and subtracting yields

$$\left(1+w^2\right)^4 st(t-1+s)(w^2(s-1)+2wt-w^2+1)\frac{2k(s-1)-2h(t-1)-(s-t)}{2(s-w(t-1))^2 w^3 (s-1-wt)^3} = 0$$

which implies that $2k(s-1) - 2h(t-1) - (s-t) = 0$, or

$$k = L(h) = \frac{1}{2}\frac{s-t+2h(t-1)}{s-1} \tag{32}$$

Now $M_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $M_2 = \left(\frac{1}{2}s, \frac{1}{2}t\right)$, so that $L(x)$ from (32) is the line L thru M_1 and M_2. Note that the **same equations** must be satisfied in order to have a **hyperbola** tangent to all four lines. Also, while we have assumed that $w \in G$, (32) actually holds for any $w \in (-\infty, \infty)$. One can prove this directly for $w \in F_1$ using Theorem 9. It also follows easily that for any given w, there is *at most one pair* $i \neq j$ with $m_{j,\alpha}^2 - m_{i,\alpha}^2 = 0$. One can then choose two sets of three of the L_j so that L_i and L_j never appear in the same set. (32) can then be derived in the same way as above. That proves (i). We now attempt to solve (30) with $k = L(h)$. First, the LHS of (30) factors into $\frac{1}{4}\frac{\left(1+w^2\right)^2 t}{w^2\left(s-1-tw\right)^2\left(s-1\right)}P(w,h)$, where

$$P(w,h) = A(h)w^2 + B(h)w - A(h)$$

$A(h) = 2\left(s-1\right)\left(\left(2t-2\right)h^2 + \left(s+2-t\right)h - s\right)$ and
$B(h) = \left(1-2h\right)\left(s-t\right)\left(\left(2s-4+2t\right)h + s - t\right)$. Of course the coefficients of $P(w,h)$ depend on s and t as well, but we suppress that in our notation.

Claim: If $k = L(h)$, then $A(h)$ and $B(h)$ cannot both be 0 for the same value of h.

Proof of Claim: Suppose that $A(h_0) = B(h_0) = 0$ for some h_0. Now $B(h_0) = 0$ if and only if $h_0 = \frac{1}{2}$ or $h_0 = \frac{1}{2}\frac{t-s}{s+t-2}$. If $h_0 = \frac{1}{2}$, then $\left(2t-2\right)h_0^2 + \left(s+2-t\right)h_0 - s = \frac{1}{2} - \frac{1}{2}s \neq 0$ since $s \neq 1$. If $h_0 = \frac{1}{2}\frac{t-s}{s+t-2}$, then $k_0 = L(h_0) = \frac{1}{2}\frac{s-t}{s-2+t} = -h_0$, which implies that h_0 and k_0 cannot both be positive. That proves the claim.

By the claim, P has two **real** roots $w = w_0$ and $w = -\frac{1}{w_0}$ for each given h, whenever $w_0 \neq 0$, i.e., whenever $\left(2t-2\right)h^2 + \left(s+2-t\right)h - s \neq 0$(note that $s-1 \neq 0$). Of course this just reflects the fact that one can rotate the coordinates axes by α or by $\alpha - \frac{\pi}{2}$ and obtain the same axes of the ellipse. The LHS of (31), with $k = L(h)$, factors into $-\frac{1}{4}\frac{s\left(1+w^2\right)^2\left(s-1+t\right)}{w^2\left(s-wt+w\right)^2\left(s-1-wt\right)^2\left(s-1\right)}P(w,h)$. Thus (30) and (31) hold simultaneously for the same h, k, and w if and only if $P(w,h) = 0$ and $k = L(h)$. Since $w \in G$, $m_{j,\alpha}^2 - m_{i,\alpha}^2 \neq 0$ for $i \neq j$. By Theorem 8, (28) and (29) hold simultaneously for the same h_α, k_α, and w if and only if (25) has a solution, with $N = 4$. Then, by Theorem 7, part (ii), (25) has a solution, with $N = 4$, if and only if

$$S_{ij} = S_{lm},\ T_{ij} = T_{lm} \text{ for any distinct } i,j,l,m \in \{1,2,3,4\} \tag{33}$$

Of course, (28) and (29) are equivalent to (30) and (31) in xy coordinates. Now an ellipse exists tangent to all four lines if and only if the same ellipse can be found which is tangent to L_1, L_2, and L_4, and to L_2, L_3, and L_4. Thus Theorem 7 and Theorem 8 imply that an ellipse exists tangent to all four lines if and only if there are $w \geq 0$ and h such that $P(w,h) = 0$, $S_{ij} > 0$, and $T_{ij} > 0$ for some $i \neq j$, with $k = L(h)$. It is easiest to work with $i = 1, j = 2$, which is possible except for $w = 0$ or 1. We want to prove that if $(h,k) \in Z$, with $P(w,h) = 0$, then S_{12} and T_{12} are **both positive** if $w \notin \{0,1\}$. We shall prove that at least one of S_{12} and T_{12} are positive at M_1 and M_2, and that neither can be zero on Z. Let

$$I = \text{open interval with endpoints } \frac{1}{2} \text{ and } \frac{1}{2}s$$

For each given $h \in I$, we shall always assume that w_0 denotes the **nonnegative** root of $P(w,h)$. Then w_0 is a continuous function of h, which we denote by $f(h)$. Letting $w = f(h)$ and $k = L(h)$, S_{ij} and T_{ij} are then functions of h on I. $S_{12}(h)$ and $T_{12}(h)$ are continuous at any point of I if $w = f(h) \neq 0$ or 1. We now show how to extend S_{12} and T_{12} to be continuous at those points as well. First, $(h,k) \in Z$ if and only if $h = \lambda\frac{1}{2} + (1-\lambda)\frac{1}{2}s, 0 < \lambda < 1$ and $k = L(h)$. We now break the rest of the proof up into two cases:

Case 1: $s \neq t$

$w = 1$ is a solution of $P(w,h) = 0$ if and only if $B(h) = 0$, which holds if and only if $h = \frac{1}{2}$ or $h = \frac{1}{2}\frac{t-s}{s+t-2}$. If $h = \frac{1}{2}$, then $(h,k) \notin Z$ and $(h,k) \notin X - Y$(we shall use the latter fact when proving (iii)). If $h = \frac{1}{2}\frac{t-s}{s+t-2}$, then $k = L(h) = \frac{1}{2}\frac{s-t}{s-2+t} = -h$, which implies that h and k cannot both be positive. Hence $w = 1$ **cannot** occur as a solution of $P(w,h) = 0$ for $(h,k) \in Z \cup X - Y$. However, $P(0,h_0)$ can equal 0 for $h_0 \in I$ and certain values of s and t. This is not a problem since, for general h and $k, S_{12} = \frac{k^2w^2 - h^2}{w^2-1}$ and $T_{12} = \frac{h^2w^2-k^2}{w^2-1}, w \notin \{0,1\}$. Thus for *any* fixed h and k, $\lim_{w\to 0} S_{12} = h^2$ and $\lim_{w \to 0} T_{12} = k^2$. S_{12} and T_{12} are then continuous at h_0 if we define, with $k = L(h)$, $S_{12}(h_0) = h_0^2$ and $T_{12}(h) = (L(h_0))^2$. It is also possible that as $h \to h_0 \in I$, the positive root of $P(w,h)$ approaches ∞. Since $\lim_{w\to\infty} S_{12} = k^2$ and $\lim_{w\to\infty} T_{12} = h^2$, defining, with $k = L(h)$, $S_{12}(h_0) = (L(h_0))^2$ and $T_{12}(h_0) = h_0^2$ makes S_{12} and T_{12} continuous at h_0 in that case.

Case 2: $s = t$

In this case the line L is $y = x$, which contains one of the diagonals. The proof is very similar, but simpler, than the proof above for $s \neq t$. Either $(2t-2)\,h^2 + 2h - t = 0$, which gives an inscribed circle, or $w = \pm 1$ and $h = k$ can take on any value between $\frac{1}{2}$ and $\frac{1}{2}t$.

Thus we have proven that S_{12} and T_{12} are continuous at any point of I. Now consider the endpoints of I.

- If $h = \frac{1}{2}s$, then $P(w,h) = -(s-1)^2\,(ws+t)\,(s-tw) = 0 \Leftrightarrow w = \frac{s}{t} \neq 1$ since $s \neq t$. If $w = \frac{s}{t}$, then $S_{12} = 0$ and $T_{12} = \frac{1}{4}t^2 + \frac{1}{4}s^2$, and thus S_{12} and T_{12} are continuous and nonnegative at $\frac{1}{2}s$.
- If $h = \frac{1}{2}$, then $P(w,h) = -2(w^2-1)\,(s-1)^2 = 0 \Leftrightarrow w = 1$ since $s \neq 1$. S_{12} and T_{12} are undefined when $w = 1$. However, if $w = 1$ and $h = \frac{1}{2}$, then $S_{13} = \frac{1}{2}$ and $T_{13} = 0$. By (33), this implies that $\lim_{h\to 1/2} S_{12}(h) = \frac{1}{2}$ and $\lim_{h\to 1/2} T_{12}(h) = 0$. Thus defining $S_{12}\left(\frac{1}{2}\right) = \frac{1}{2}$ and $T_{12}\left(\frac{1}{2}\right) = 0$ extends S_{12} and T_{12} to be continuous

and nonnegative at $\frac{1}{2}$.
Thus S_{12} and T_{12} are continuous(or can be extended to be continuous) on the closure of I, and nonnegative at the endpoints of I.
Claim: $S_{12}(h)$ and $T_{12}(h)$ cannot equal 0 for $h \in I$.
Proof of Claim: We can assume that $w \neq 1$ since we have shown that $w = 1$ **cannot** occur as a solution of $P(w, h) = 0$ for $h \in I$. If $P(0, h) = 0$, then $S_{12}(h)$ and $T_{12}(h)$ cannot equal 0 since $h^2 \neq 0$, $L^2(h) \neq 0$ on I. So assume also that $w \neq 0$. Then $S_{12} = \frac{k^2w^2 - h^2}{w^2 - 1} = 0$ if and only if $h = kw(h \neq -kw$ since $h > 0, k > 0)$. If $h = kw$ along with $k = L(h)$, then $h = \frac{1}{2}w\frac{s-t}{s-1-(t-1)w}$ if $w \neq \frac{s-1}{t-1}$, which implies that $P(w, h) = -2(w^2-1)(s-1)^2\frac{s-tw}{s-1-(t-1)w}$, which equals 0 if and only if $w = \frac{s}{t}$, which implies that $h = \frac{1}{2}s \notin I$. If $h = kw$ along with $k = L(h)$ and $w = \frac{s-1}{t-1}$, then $h = \frac{1}{2}\frac{s-t+2h(t-1)}{t-1}$, which has no solution since $s - t \neq 0$. $T_{12} = 0$ if and only if $k = wh$. If $k = wh$ along with $k = L(h)$, then $h = \frac{1}{2}\frac{s-t}{w(s-1)-(t-1)}$ if $w \neq \frac{t-1}{s-1}$. This implies that $P(w, h) = -2(w+1)(w-1)^2(s-1)^2\frac{(s^2-s)w+t-t^2}{(w(s-1)-(t-1))^2} = 0$ if and only if $w = \frac{t(t-1)}{s(s-1)}$, which implies that $h = -\frac{1}{2}\frac{s}{t-1}$. Now $h > 0 \Rightarrow t < 1$ and $w \geq 0 \Rightarrow s < 1$. Also, $(h, k) \in Z$ if and only if $h = \lambda\frac{1}{2} + (1-\lambda)\frac{1}{2}s, 0 < \lambda < 1$ and $k = L(h)$. Thus $-\frac{1}{2}\frac{s}{t-1} = \lambda\frac{1}{2} + (1-\lambda)\frac{1}{2}s \Rightarrow \lambda = \frac{st}{(t-1)(s-1)}$. Since $s + t \geq 1$, $t \geq 1 - s$ and $1 - t \leq s \Rightarrow \lambda = \frac{s}{1-s}\frac{t}{1-t} \geq \frac{s}{1-s}\frac{1-s}{s} = 1$, which implies that $h \notin I$. If $k = wh$, along with $k = L(h)$ and $w = \frac{t-1}{s-1}$, then again there is no solution since $s - t \neq 0$. That proves the claim.
Since $S_{12}(h)$ and $T_{12}(h)$ are nonnegative at the endpoints of I, positive at at least one of the endpoints of I, and nonzero on the interior of I, by the Intermediate Value Theorem $S_{12}(h) > 0$ and $T_{12}(h) > 0$ when $h \in I$. Now let $H = \{h \in I : f(h) \in G\}$. $(-\infty, \infty) - G$ is a finite set and $P(w, h) = A(h)(w^2 + C(h)w - 1)$,where $C(h)$ is a nontrivial rational function. Thus $P(w_0, h_0) = 0$ if and only if $C(h_0) = \frac{1}{w_0} - w_0$, and by the definition of f, $w_0 = f(h_0)$. It then follows easily that $f^{-1}\{(-\infty, \infty) - G\}$ is also a finite set, which implies that $I - H$ is a finite set. We have proven that if $h \in H$ and $k = L(h)$, then there is an ellipse with equation $\frac{(x_\alpha - h_\alpha)^2}{a^2} + \frac{(y_\alpha - k_\alpha)^2}{b^2} = 1$ which is tangent to each of the $L_j, j = 1, 2, 3, 4$, where $a^2 = S_{12}(h)$ and $b^2 = T_{12}(h)$. Since $S_{12}(h)$ and $T_{12}(h)$ are both positive on $int(I)$,if $h_0 \in I - H$, one can obtain an ellipse with center $(h_0, L(h_0))$ by taking a limit of the ellipses obtained for $h \in H$. That proves that if $(h, k) \in Z$, then there is an ellipse inscribed in R and with center (h, k).
Rest of the proof of (ii) and (iii)

Note that $(h,k) \in X \Rightarrow h,k > 0$ and that $\{h : (h,k) \in X - Y\}$ consists of two disjoint open intervals, $J_1 \cup J_2$. The particular endpoints of X depend on whether $s > 1, s < 1, t > s$, etc.

Claim: $S_{12}(h)$ and $T_{12}(h)$ cannot equal 0 for $h \in J_1 \cup J_2$.

Proof of Claim: The proof follows exactly as in the proof above that $S_{12}(h)$ and $T_{12}(h)$ cannot equal 0 for $h \in I$, except for showing that $T_{12}(h) \neq 0$ when $w \neq \dfrac{t-1}{s-1}$, $h = \dfrac{1}{2}\dfrac{s}{1-t}$, and $s,t < 1$. We omit the proof when $s = t$ and consider the following two cases.

Case 1: $s < t$

Then $L(x)$ intersects ∂R in the lines L_2 and $L_4, J_1 = \left(0, \dfrac{1}{2}s\right)$, and $J_2 = \left(\dfrac{1}{2}, \dfrac{1}{2}s + \dfrac{1}{2}t\right)$. Now $\dfrac{1}{2}s + \dfrac{1}{2}t < \dfrac{1}{2}\dfrac{s}{1-t}$ since $s + t > 1$, which implies that $h = \dfrac{1}{2}\dfrac{s}{1-t} \notin J_1 \cup J_2$ since $\dfrac{1}{2}s + \dfrac{1}{2}t$ is the largest value of $h \in J_1 \cup J_2$.

Case 2: $s > t$

Then $L(x)$ intersects ∂R in the lines L_1 and $L_3, J_1 = \left(\dfrac{1}{2}\dfrac{t-s}{t-1}, \dfrac{1}{2}s\right)$, and $J_2 = \left(\dfrac{1}{2}, \dfrac{1}{2}s\dfrac{s-2+t}{t-1}\right)$. Again, $s + t > 1$ implies that $\dfrac{1}{2}s\dfrac{s-2+t}{t-1} < \dfrac{1}{2}\dfrac{s}{1-t}$, which implies that $h = \dfrac{1}{2}\dfrac{s}{1-t} \notin J_1 \cup J_2$ since $\dfrac{1}{2}s\dfrac{s-2+t}{t-1}$ is the largest value of $h \in J_1 \cup J_2$. That proves the claim.

We want to show that $S_{12}(h)$ and $T_{12}(h)$ change sign at $h = \dfrac{1}{2}s$ and at $h = \dfrac{1}{2}$, respectively. Recall that $w = f(h) \Leftrightarrow P(w,h) = A(h)w^2 + B(h)w - A(h) = 0$ and $w \geq 0$, where $A(h) = 2(s-1)\left((2t-2)h^2 + (s+2-t)h - s\right)$ and $B(h) = (1-2h)(s-t)((2s-4+2t)h + s - t)$. Thus implicit differentiation gives $w' = \dfrac{dw}{dh} = -\dfrac{A'(h)w^2 + B'(h)w - A'(h)}{2wA(h) + B(h)}$. Assume throughout that $k = L(h)$. Now suppose that $S'_{12}(h) = 0$. We shall derive a contradiction. Since $S_{12} = \dfrac{k^2w^2 - h^2}{w^2 - 1}$,

$$(h^2 - k^2)ww' + (kk'w^2 - h)(w^2 - 1) = 0 \tag{34}$$

where $k' = \dfrac{dk}{dh} = L'(h) = \dfrac{t-1}{s-1}$. First, if $h = \dfrac{1}{2}s$, then $w = \dfrac{s}{t} \Rightarrow \dfrac{dw}{dh} = 2(s-t)\dfrac{s^2 - 2s + t^2 - 2t}{t^2(s-1)(s^2+t^2)}$, and (34) becomes $w(2(s-t)\frac{s^2-2s+t^2-2t}{t^2(s-1)(s^2+t^2)})(h^2 - k^2) + (w^2-1)(k(\frac{t-1}{s-1})w^2 - h) = -(s+t)^2(s-t)^2\dfrac{s}{t^3(s^2+t^2)(s-1)} = 0$, which it cannot since $s \neq t$. Thus $S'_{12}(\frac{1}{2}s) \neq 0$, which implies that $S_{12}(h)$ changes sign at $h = \dfrac{1}{2}s$.

Second, if $h = \dfrac{1}{2}$, then $w = 1$, and $T_{12}(h)$ is undefined when $w = 1$. However, consider $T'_{12}(h) = \dfrac{(w^2-1)(w^2h - kk') + ww'k^2}{(w^2-1)^2}$. Now $w'(\frac{1}{2}) = -\dfrac{B'(\frac{1}{2})}{2A(\frac{1}{2}) + B(\frac{1}{2})} = -2\dfrac{s-t}{s-1}$, which implies that $\lim\limits_{h \to 1/2, w \to 1}(w^2-1)(w^2h - kk') + ww'k^2 = \dfrac{1}{4}w'(\frac{1}{2}) = \dfrac{1}{2}\dfrac{t-s}{s-1} \neq 0$. Thus $T'_{12}(h)$ has the same sign on either side of $h = \dfrac{1}{2}$, which implies that $T_{12}(h)$ also changes sign at $h = \dfrac{1}{2}$. Since $S_{12}(\frac{1}{2}s) = T_{12}(\frac{1}{2}) = 0$, $S_{12}(\frac{1}{2}) > 0$,, $T_{12}(\frac{1}{2}s) > 0$, and $S_{12}(h), T_{12}(h) > 0$ on I, by the claim above and the Intermediate Value Theorem, $S_{12}(h)T_{12}(h) < 0$ on $J_1 \cup J_2$. Let $J = \{h \in J_1 \cup J_2 : f(h) \in G\}$. We have shown that there is a hyperbola with center (h, k) tangent to each of the L_j if $h \in J$. Again one can use a limiting argument, as earlier, if $h \notin J$. That proves that if $(h, k) \in X - Y$, then there is a hyperbola with center (h, k) tangent to each of the L_j. To finish the proof of (ii) and (iii): In the proof of (ii) given above, look at the **non-positive** roots w of $P(w, h)$ instead. Everything follows in the same way. Thus, if $h \in I$, then $S_{ij}(h) > 0$ and $T_{ij}(h) > 0$ for either choice of the root w. Thus there cannot be a hyperbola with center $(h, k) \in Z$ which is tangent to each of the L_j. Similarly, in the proof just given that if $(h, k) \in X - Y$, then there is a hyperbola with center (h, k) tangent to each of the L_j, again look at the non-positive roots w of $P(w, h)$. If $h \in J_1 \cup J_2$, then $S_{ij}(h)T_{ij}(h) < 0$, which implies that if $(h, k) \in X - Y$, then there cannot be an ellipse inscribed in R and with center (h, k).

Remark. The fact that if $(h, k) \in X - Y$, then there cannot be an ellipse inscribed in R and with center (h, k) follows from the result in [1] mentioned earlier. However, we wanted to give a self contained proof which does not use orthogonal projections.

It is possible that one is just given the four lines L_j, $j = 1, 2, 3, 4$ and not the polygon R of which they form the boundary. One still might want to know if there is some ellipse tangent to the given lines, and to characterize the locus of centers of ellipses tangent to the given lines. The following proposition gives a partial answer.

Proposition 2. Let L_1, L_2, L_3, L_4 be four given lines in the xy plane, such that no three of the L_j are concurrent. Suppose also that one of the following holds:
(i) No **two** of the L_j are parallel, or
(ii) Exactly two of the L_j are parallel, and the intersection point of the other two lines does not lie between the two parallel lines.
Then the L_j form the boundary of a four sided convex polygon R.

Proof. (i) Pick a line L. The other lines intersect L at three distinct points since no three lines have a common intersection point. Let M be the line which intersects L between the other two intersection points. The interior of the triangle T formed by the three lines not equal to M is cut by M. The two regions into which M divides T are each convex, since they are the intersections of convex regions(a triangle and a half plane). One of these regions is R.
(ii) Suppose that L_1 and L_2 are parallel and let P_{ij} = point of intersection of L_i and $L_j, (i, j) \neq (1, 2)$. Let R be the polygon with vertices P_{13}, P_{14}, P_{23}, and P_{24}. Since P_{34} does not lie between L_1 and L_2, it follows easily that R is a four sided convex polygon.
Theorem 11 then gives

Corollary 2. If L_1, L_2, L_3, L_4 satisfy (i) or (ii) of Proposition 2, then there is an ellipse, E, tangent to each of the L_j, and the center of E must lie on the line thru the midpoints of the diagonals of R.

§4. Three Given Tangents

The rest of the paper is a modification and simplification of results which appeared in [3]. It also contains corrections for several errors in the original version.

Throughout, given $L_1, ..., L_N$, $\Re$ denotes the set of admissible centers, i.e., the set of all (h,k) such that the **non-rotated** ellipse, $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, is tangent to each of the L_j. For convenience, we use the following notation from (16) for $\alpha = 0$. For given m_i, b_i, m_j, and b_j:

$$S_{ij} = \frac{b_j^2 - b_i^2}{m_j^2 - m_i^2}, T_{ij} = \frac{b_i^2 m_j^2 - b_j^2 m_i^2}{m_j^2 - m_i^2}$$

We want to characterize $\Re$ when $N = 3$ and also give an explicit formula for E. Several cases must be considered depending upon whether $m_i^2 = m_j^2$ and/or $m_i = m_j$ for some $i \neq j$. In each case the admissible centers lie on a hyperbola or a straight line(a degenerate hyperbola). The precise curve is obtained by finding conditions on the coefficient matrix of the linear system (12) with $N = 3$. If L_i and L_j are not parallel, we let

$$(x_l, y_l) = \left(\frac{c_j - c_i}{m_i - m_j}, \frac{m_i c_j - m_j c_i}{m_i - m_j} \right) \tag{35}$$

$i \neq l \neq j$, denote their point of intersection.

§4.1 No two of the tangents have slopes equal in absolute value
We first state our results when none of the L_j is horizontal or vertical.

Theorem 12. Let L_1, L_2, L_3 be distinct, non-concurrent, and non–vertical lines with equations $y = L_j(x) = m_j x + c_j$, $j = 1, 2, 3$. Assume that $i \neq j \Rightarrow m_i^2 \neq m_j^2$ and let $D = \prod_{j>i} (m_j - m_i)$, $M = \begin{pmatrix} m_1 & 1 & c_1 \\ m_2 & 1 & c_2 \\ m_3 & 1 & c_3 \end{pmatrix}$, $a_1 = \begin{vmatrix} 1 & m_1^2 & m_1 c_1 \\ 1 & m_2^2 & m_2 c_2 \\ 1 & m_3^2 & m_3 c_3 \end{vmatrix}$, $a_2 = -\begin{vmatrix} 1 & m_1^2 & m_1 \\ 1 & m_2^2 & m_2 \\ 1 & m_3^2 & m_3 \end{vmatrix}$, $p_3(h) = \prod_{j=1}^{3} (h - x_j)$, $p_1(h) = h + \frac{a_2}{D}$, $q_3(k) = \prod_{j=1}^{3} (k - y_j)$, $q_1(k) = k + \frac{a_1}{D}$, $C = -\frac{|M|^2}{2D^2} \prod_{j>i} (m_j + m_i)$. Let γ be the curve with equation

$$p_1(h) q_1(k) = C \tag{36}$$

(i) Then

$$\Re = \left\{ (h,k) \in \gamma : \frac{p_3(h)}{p_1(h)} > 0 \text{ and } \frac{q_3(k)}{q_1(k)} > 0 \right\} \tag{37}$$

For each $(h,k) \in \Re$, there is a unique ellipse $E : \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ which is tangent to each of the L_j, with $a^2 = \frac{p_3(h)}{p_1(h)}$ and $b^2 = \frac{q_3(k)}{q_1(k)}$.
(ii) Write equation (36) in the form $k = f(h)$ or $h = g(k)$.
(a) Suppose that $m_3 \neq 0$, let $w_j = g(y_j)$, $j = 1, 2, 3$, and define the cubic polynomial

$$r_3(h) = -2 \frac{x_2 - x_3}{m_3 - m_2} \prod_{j=1}^{3} m_j \prod_{j=1}^{3} (h - w_j)$$

Then

$$\Re = \left\{(h, f(h)) : \frac{p_3(h)}{p_1(h)} > 0 \text{ and } r_3(h) > 0\right\} \tag{38}$$

with a^2 as in (i) and $b^2 = \dfrac{r_3(h)}{(p_1(h))^2}$.
(b) Suppose that $m_3 = 0$. Then

$$\Re = \left\{(h, k) \in \gamma : \frac{p_3(h)}{p_1(h)} > 0 \text{ and } h \neq \frac{1}{2}(x_1 + x_2)\right\} \tag{39}$$

with a^2 as in (i) and $b^2 = (f(h) - c_3)^2$
(iii) Also, given any $0 \leq e_0 < 1$, there is an $(h, k) \in \Re$ such that E has eccentricity e_0.

Proof. Note that $\operatorname{rank}(M) \geq 2$. If $|M| = 0$, then $\operatorname{rank}(M) = 2 = \operatorname{rank}\begin{pmatrix} m_1 & 1 \\ m_2 & 1 \\ m_3 & 1 \end{pmatrix}$, which would imply that the system of equations $m_j x + c_j = y, j = 1, 2, 3$ has a solution (x, y). Since we assumed that the L_j are **not** concurrent, $|M| \neq 0$. This implies that $C \neq 0$. Since $|M| = (m_l - m_i)(m_j - m_i)(x_j - x_l)$ for any distinct $\{i, j, l\} \subset \{1, 2, 3\}$,

$$x_i \neq x_j \text{ for } i \neq j \tag{40}$$

It is also useful to note that

$$|M| = -c_2 m_3 + c_2 m_1 + c_1 m_3 + c_3 m_2 - c_3 m_1 - c_1 m_2 \neq 0 \tag{41}$$

Given (h, k), write L_j in the form $y - k = m_j(x - h) + b_j$, $j = 1, 2, 3$. By Theorem 2 with $N = 3$, there is an ellipse, with center (h, k), tangent to **all three** lines if and only if

$$\begin{aligned} m_1^2 u + v &= b_1^2 \\ m_2^2 u + v &= b_2^2 \\ m_3^2 u + v &= b_3^2 \end{aligned} \tag{42}$$

has a positive solution (u, v). Since $\operatorname{rank}\begin{pmatrix} m_1^2 & 1 \\ m_2^2 & 1 \\ m_3^2 & 1 \end{pmatrix} = 2$, $\operatorname{rank}\begin{pmatrix} m_1^2 & 1 & b_1^2 \\ m_2^2 & 1 & b_2^2 \\ m_3^2 & 1 & b_3^2 \end{pmatrix} \geq 2$, and thus (42) has a solution(not necessarily positive) if and only if $\begin{vmatrix} m_1^2 & 1 & b_1^2 \\ m_2^2 & 1 & b_2^2 \\ m_3^2 & 1 & b_3^2 \end{vmatrix} = 0$, which gives

$$(m_3^2 - m_2^2)b_1^2 - (m_3^2 - m_1^2)b_2^2 + (m_2^2 - m_1^2)b_3^2 = 0 \tag{43}$$

By Theorem 2 again, if (43) holds, then the solution of (42) is unique and is given by

$$u = S_{12} = S_{23} = S_{13} \tag{44}$$

and

$$v = T_{12} = T_{23} = T_{13} \tag{45}$$

Hence, by Theorem 2, with $N = 3$,

$$S_{12} > 0 \text{ and } T_{12} > 0 \tag{46}$$

and (43) are necessary and sufficient conditions for the existence of an ellipse E tangent to all three lines. Using (8), (43) becomes $\begin{vmatrix} m_1^2 & (m_1h + c_1 - k)^2 & 1 \\ m_2^2 & (m_2h + c_2 - k)^2 & 1 \\ m_3^2 & (m_3h + c_3 - k)^2 & 1 \end{vmatrix} = 0.$ Multiplying column 1 by $-h^2$ and adding to column 2 yields $\begin{vmatrix} m_1^2 & 2m_1h(c_1 - k) + (c_1 - k)^2 & 1 \\ m_2^2 & 2m_2h(c_2 - k) + (c_2 - k)^2 & 1 \\ m_3^2 & 2m_3h(c_3 - k) + (c_3 - k)^2 & 1 \end{vmatrix} = 0$, which is equivalent to (36). Solving (36) for k and for h, respectively, yields

$$k = f(h) = \frac{C}{p_1(h)} - \frac{a_1}{D} \tag{47}$$

and

$$h = g(k) = \frac{C}{q_1(k)} - \frac{a_2}{D} \tag{48}$$

It is easy to show that, for any distinct $\{i, j, l\} \subset \{1, 2, 3\}$,

$$Dx_j + a_2 = (m_i + m_l)|M| \tag{49}$$

and

$$Dy_j + a_1 = -m_j(m_i + m_l)|M| \tag{50}$$

By (40),(49), and (50), $Dx_l + a_2 \neq 0$ and $Dy_l + a_1 \neq 0$ for any l. Hence by (47) and (48), $v_j = f(x_j)$ and $w_j = g(y_j)$ are each *finite* . By (8), $S_{12} = p(h, k)$, where p is a polynomial which is monic and quadratic in h, and linear in k. Also by (8),

$$b_j^2 - b_i^2 = (L_j(h) - T_i(h))\,(L_j(h) + T_i(h) - 2k) \tag{51}$$

Thus $h = x_3 \Rightarrow L_2(h) = L_1(h) \Rightarrow b_2^2 - b_1^2 = 0 \Rightarrow p(x_3, k) = 0$ for any k. Now write L_j in the form $x - h = \frac{1}{m_j}(y - k) - \frac{b_j}{m_j}$. Substituting $\frac{1}{m_j}$ for m_j and $-\frac{b_j}{m_j}$ for b_j in S_{12} yields T_{12}. By interchanging h and k, this easily implies that $T_{12} = q(h, k)$, where q is a polynomial which is monic and quadratic in k, and linear in h. Also, $q(h, y_3) = 0$ for any h. Finally, by (44) and (45), $p(x_j, k) = q(h, y_j) = 0$ for any h and k, and for $j = 1, 2, 3$. For $(h, k) \in \gamma$, $p(h, k) = p(h, f(h)) = \frac{p_3(h)}{p_1(h)}$, where p_3 is a monic polynomial of degree ≤ 3 and $p_1(h) = h + \frac{a_2}{D}$, by (47). $p(x_j, f(x_j)) = 0$ since $p(x_j, k) = 0$ for any k, which implies that $p_3(h) =$

$\prod_{j=1}^{3}(h-x_j)$. Similarly, for $(h,k)\in\gamma$, $q(h,k)=q(g(k),k)=\dfrac{q_3(k)}{q_1(k)}$, where q_3 is a monic polynomial of degree ≤ 3 and $q_1(k)=k+\dfrac{a_1}{D}$, by (48). $q(g(y_j),y_j)=0$ since $q(h,y_j)=0$ for any h, which implies that $q_3(k)=\prod_{j=1}^{3}(k-y_j)$. Since $S_{12}=\dfrac{p_3(h)}{p_1(h)}$ and $T_{12}=\dfrac{q_3(k)}{q_1(k)}$, that proves (i). Substituting $h=g(k)$, one can write $\dfrac{p_3(h)}{p_1(h)}=\dfrac{\prod_{j=1}^{3}(g(k)-x_j)}{g(k)+\frac{a_2}{D}}$. By (49), $\prod_{j=1}^{3}(Dx_j+a_2)=|M|^3\prod_{j>i}(m_j+m_i)$. Using $f(x_j)=v_j$, it is not hard to show that $g(k)-x_j=-\dfrac{(Dx_j+a_2)(k-v_j)}{Dk+a_1}\Rightarrow\prod_{j=1}^{3}(g(k)-x_j)=-\dfrac{\prod_{j=1}^{3}(Dx_j+a_2)}{(Dk+a_1)^3}\prod_{j=1}^{3}(k-v_j)=-|M|^3\dfrac{\prod_{j>i}(m_j+m_i)}{(Dk+a_1)^3}\prod_{j=1}^{3}(k-v_j)$. Finally, it follows easily that $g(k)+\dfrac{a_2}{D}=|M|^2\dfrac{\prod_{j>i}(m_j+m_i)}{-2D(Dk+a_1)}$. Putting this altogether, and using $\dfrac{|M|}{D}=\dfrac{x_2-x_3}{m_3-m_2}$, one obtains $\dfrac{p_3(h)}{p_1(h)}=2\dfrac{x_2-x_3}{m_3-m_2}\dfrac{\prod_{j=1}^{3}(k-v_j)}{(k+\frac{a_1}{D})^2}=\dfrac{s_3(k)}{(k+\frac{a_1}{D})^2}$, where $s_3(k)=2\dfrac{x_2-x_3}{m_3-m_2}\prod_{j=1}^{3}(k-v_j)$. $\dfrac{p_3(h)}{p_1(h)}$ is positive, on γ, precisely when $s_3(k)>0$ since $k+\dfrac{a_1}{D}=0$ cannot yield a point on γ. Substituting $k=f(h)$, (50), and arguing as above, one can also show that $\dfrac{q_3(k)}{q_1(k)}=-\dfrac{x_2-x_3}{m_3-m_2}\prod_{j=1}^{3}m_j\dfrac{\prod_{j=1}^{3}(h-w_j)}{(h+\dfrac{a_2}{D})^2}$, which is positive, on γ, precisely when $r_3(h)>0$ since $h+\dfrac{a_2}{D}=0$ cannot yield a point on γ. Hence

$$\Re=\{(h,k)\in\gamma:r_3(h)>0\text{ and }s_3(k)>0\}\tag{52}$$

Note that $\Re\neq\emptyset$ since the incenter of T is admissible. One could also prove directly that $\Re$ is nonempty. The *uniqueness* follows from Theorem 1. Since $\dfrac{q_3(f(h))}{q_1(f(h))}=\dfrac{r_3(h)}{(p_1(h))^2}$, (38) follows from (52). That proves (ii)(a).

We now prove that there is an $(h,k)\in\Re$ such that E has eccentricity e_0,with $m_3\neq 0$. Using (35), it follows easily that for any set of distinct $i,j,l\in\{1,2,3\}$, $r_3(x_i)=\dfrac{1}{8}(m_j+m_l)^2\dfrac{(m_j-m_i)^2(m_l-m_i)^2(x_j-x_l)^4}{(m_j-m_l)^4}$, which implies that

$$r_3(x_i)>0\text{ for any }i\tag{53}$$

and $r_3\left(-\dfrac{a_2}{D}\right)=\dfrac{1}{8}(m_1+m_2)^2(x_3-x_2)^4(m_1+m_3)^2\dfrac{(m_3+m_2)^2}{(m_3-m_2)^4}$, which implies that

$$r_3\left(-\frac{a_2}{D}\right) > 0 \tag{54}$$

Let $0 < c \le 1$ be given. $\frac{a^2}{b^2} = c \Rightarrow \frac{p_1(h)p_3(h)}{r_3(h)} = c$, which holds if and only if $E(h) \equiv p_1(h)p_3(h) - cr_3(h) = 0, r_3(h) \ne 0$. By (53) and (54), $E(x_j) = -cr_3(x_j) < 0$ and $E\left(-\frac{a_2}{D}\right) = -cr_3\left(-\frac{a_2}{D}\right) < 0$. Let $x_{\max} = \max\{x_1, x_2, x_3, -a_2/D\}$. Then $E(x_{\max}) < 0$, and since $\lim_{h\to\infty} E(h) = \infty$, $E(h_0) = 0$ for some $h_0 > x_{\max}$. Since $p_1(h_0)p_3(h_0) > 0$ it follows that $r_3(h_0) \ne 0$ and $\frac{p_1(h_0)p_3(h_0)}{r_3(h_0)} = c > 0$ implies that $r_3(h_0) > 0$. Thus $(h_0, f(h_0)) \in \Re$ by (38), which implies, upon letting $a^2 = \frac{p_3(h_0)}{p_1(h_0)}$and $b^2 = \frac{r_3(h_0)}{(p_1(h_0))^2}$, that the equation $1 - \frac{a^2}{b^2} = e_0^2, 0 \le e_0 < 1$ always has a solution. To prove (ii)(b), if $m_3 = 0$ then it still follows that $S_{12} = \frac{p_3(h)}{p_1(h)}$. However, $y_1 = y_2 = c_3$, and $T_{12} = \frac{q_3(k)}{q_1(k)}$, where $q_3(k) = (k - c_3)^2(k - y_3)$. Since $\frac{a_1}{D} = \frac{m_1c_2 - m_2c_1}{m_2 - m_1} = -y_3$, $\frac{q_3(k)}{q_1(k)} = (k - c_3)^2 \Rightarrow T_{12} = \frac{q_3(f(h))}{q_1(f(h))} = (k - c_3)^2 > 0$ when $k \ne c_3$. Now $k \ne c_3$ if and only if $h \ne g(c_3) = \frac{m_1(c_3 - c_2) + m_2(c_3 - c_1)}{2m_1m_2} = \frac{1}{2}(x_1 + x_2)$. That proves (39). Again, the *uniqueness* follows from Theorem 1. The proof that there is an $(h, k) \in \Re$ such that E has eccentricity e_0 is similar to the proof above when $m_3 \ne 0$ and we omit it.

§4.1 One of the Tangents is Vertical

Theorem 13. Let L_1 and L_2 be distinct, non-concurrent, and non–vertical lines with equations $y = L_j(x) = m_jx + c_j$, $j = 1, 2$, and assume that $0 \ne m_1^2 \ne m_2^2$. Let L_3 be the vertical line with equation $x = c_3$, and let γ be the curve with equation $(h - x_3)(k - a_1) = -\frac{1}{2}(m_1 + m_2)(c_3 - x_3)^2$, where $a_1 = L_2(c_3) + L_1(c_3) - y_3$. Let $q_3(k) = \prod_{j=1}^{3}(k - y_j), q_1(k) = m_1^2(k - a_1)$. Then

(i) $\Re = \left\{(h, k) \in \gamma : h \ne c_3 \text{ and } \frac{q_3(k)}{q_1(k)} > 0\right\}$. For each $(h, k) \in \Re$, there is a unique ellipse $E : \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$ which is tangent to each of the L_j, with $a^2 = (h - c_3)^2$and $b^2 = \frac{q_3(k)}{q_1(k)}$.

(ii) Also, given any $0 \le e_0 < 1$, there is an $(h, k) \in \Re$ such that E has eccentricity e_0.

Proof. Given $C = (h, k)$, write L_j in the form $y - k = m_j(x - h) + b_j$, $j = 1, 2$, and write L_3 in the form $x - h = b_3$. Arguing as earlier, a necessary condition for E to be tangent to L_1 and L_2 is that (7) has the unique positive solution (a^2, b^2) which satisfies (6). Also, a necessary condition for E to be tangent to

the line $x = c_3$ is that $a \pm h = c_3 \Rightarrow a^2 = (c_3 \pm h)^2 = b_3^2 > 0 \Rightarrow h \neq c_3$. Thus $S_{12} = b_3^2 \Rightarrow b_2^2 - b_1^2 - (m_2^2 - m_1^2)(h - c_3)^2 = 0 \Rightarrow -2(m_2 - m_1)hk - 2(c_2 - c_1)k + 2(m_2c_2 - m_1c_1 + c_3(m_2^2 - m_1^2))h + c_2^2 - c_1^2 - c_3^2(m_2^2 - m_1^2) = 0$ $\Rightarrow hk - \dfrac{c_1 - c_2}{m_2 - m_1}k - \dfrac{m_2c_2 - m_1c_1 + c_3(m_2^2 - m_1^2)}{m_2 - m_1}h - \dfrac{c_2^2 - c_1^2 - c_3^2(m_2^2 - m_1^2)}{2(m_2 - m_1)} = 0$.
Now $\dfrac{m_2c_2 - m_1c_1 + c_3(m_2^2 - m_1^2)}{m_2 - m_1} = -m_1x_3 + c_2 + c_3(m_1 + m_2) = L_2(c_3) + L_1(c_3) - L_1(x_3) = a_1$, and thus we have $hk - x_3k - a_1h - a_3 = 0 \Rightarrow (h - x_3)(k - a_1) = a_1x_3 + a_3 = \dfrac{m_2c_2 - m_1c_1 + c_3(m_2^2 - m_1^2)}{m_2 - m_1}\dfrac{c_1 - c_2}{m_2 - m_1} + \dfrac{c_2^2 - c_1^2 - c_3^2(m_2^2 - m_1^2)}{2(m_2 - m_1)} = -\frac{1}{2}(m_1 + m_2)\dfrac{(-c_1 + c_2 + (m_2 - m_1)c_3)^2}{(m_2 - m_1)^2} = -\dfrac{1}{2}(m_1 + m_2)(c_3 - x_3)^2$, which gives the curve γ. $a^2 = b_3^2 \Rightarrow b^2 = b_1^2 - m_1^2a^2 = b_1^2 - m_1^2b_3^2 > 0 \Rightarrow \dfrac{b_1^2}{m_1^2} > b_3^2 \Rightarrow p(h) \equiv \dfrac{(m_1h + c_1 - k)^2}{m_1^2} - (h - c_3)^2 > 0$. Solving $S_{12} - (h - c_3)^2 = 0$ for h gives $h = \dfrac{1}{2}\dfrac{(2c_2 - 2c_1)k - c_2^2 + c_1^2 + c_3^2(m_2^2 - m_1^2)}{(m_1 - m_2)k + m_2c_2 - m_1c_1 + c_3(m_2^2 - m_1^2)}$. Substituting into $p(h)$ yields $q(k) = 0$, where $q(k) = (c_1 + m_1c_3 - k)(c_2 - k + m_2c_3)\dfrac{(m_2 - m_1)k + m_1c_2 - m_2c_1}{m_1^2((m_2 - m_1)k - (m_2c_2 - m_1c_1 + c_3(m_2^2 - m_1^2)))}$.
Thus $q(k) = \dfrac{q_3(k)}{q_1(k)}$, where $q_3(k) = \prod_{j=1}^{3}(k - y_j)$ and $q_1(k) = m_1^2(k - \dfrac{m_2c_2 - m_1c_1 + c_3(m_2^2 - m_1^2)}{m_2 - m_1}) = m_1^2(k - (-m_1x_3 + c_2 + c_3(m_1 + m_2))) = m_1^2(k - a_1)$. That implies that $\Re \subset \{(h,k) \in \gamma : h \neq c_3 \text{ and } \dfrac{q_3(k)}{q_1(k)} > 0\}$. Now if $h \neq c_3$ and $\dfrac{q_3(k)}{q_1(k)} > 0$, then letting $a^2 = (h - c_3)^2$and $b^2 = \dfrac{q_3(k)}{q_1(k)}$ and basically reversing the steps above, $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ is tangent to each of the L_j, $j = 1, 2, 3$. That proves (i). We omit the proof of (ii).

§4.2 Two of the tangents have slopes equal in absolute value

In this section we assume that $m_1^2 = m_2^2 \neq m_3^2$. Given $C = (h,k)$, write L_j in the form $y - k = m_j(x - h) + b_j$, $j = 1, 2, 3$. (42) now becomes

$$
\begin{aligned}
m_1^2u + v &= b_1^2 \\
m_1^2u + v &= b_2^2 \\
m_3^2u + v &= b_3^2
\end{aligned}
\tag{55}
$$

Since $\operatorname{rank}\begin{pmatrix} m_1^2 & 1 \\ m_1^2 & 1 \\ m_3^2 & 1 \end{pmatrix} = 2$, $\operatorname{rank}\begin{pmatrix} m_1^2 & 1 & b_1^2 \\ m_1^2 & 1 & b_2^2 \\ m_3^2 & 1 & b_3^2 \end{pmatrix} \geq 2$, and $\begin{vmatrix} m_1^2 & 1 & b_1^2 \\ m_1^2 & 1 & b_2^2 \\ m_3^2 & 1 & b_3^2 \end{vmatrix} = (m_3^2 - m_1^2)(b_2^2 - b_1^2) = 0$ if and only if $b_1^2 = b_2^2$, (55) has a solution if and only if $b_1^2 = b_2^2$. The unique solution is

$$u = S_{23} = S_{13} \tag{56}$$

and

$$v = T_{23} = T_{13} \tag{57}$$

If (56) and (57) are positive, then we let $a^2 = u$ and $b^2 = v$ to obtain the equation of the required ellipse. The condition $b_1^2 = b_2^2$ implies $(m_1h+c_1-k)^2 = (m_2h+c_2-k)^2$, which implies(using $m_1^2 = m_2^2$) that

$$2(m_2 - m_1)hk + 2(m_1c_1 - m_2c_2)h + 2(c_2 - c_1)k + c_1^2 - c_2^2 = 0 \tag{58}$$

Our next theorem covers the case when $m_2 = -m_1$.

Theorem 14. Let L_1, L_2, L_3 be distinct, non-concurrent, and non–vertical lines with equations $y = L_j(x) = m_jx + c_j$, $j = 1, 2, 3$. Assume that $m_2 = -m_1 \neq 0$ and that $m_1^2 \neq m_3^2$. Let $Q(h) = (h-x_1)(h-x_2)$, $L_1(k) = -2\dfrac{x_3 - x_2}{m_3 + m_1}\left(k - \dfrac{m_3x_3 + c_3 + y_3}{2}\right)$, and $P(k) = (k - y_1)(k - y_2)$.

(a) $m_3 \neq 0$: Let $L_2(h) = -2\dfrac{m_1m_3}{m_3 + m_1}(y_3 - y_2)\left(h - \dfrac{m_3x_3 + y_3 - c_3}{2m_3}\right)$. Then $\Re = \Re_1 \cup \Re_2$, where $\Re_1 = \{(x_3, k) : L_1(k) > 0 \text{ and } P(k) > 0\}$ and $\Re_2 = \{(h, y_3) : L_2(h) > 0 \text{ and } Q(h) > 0\}$. If $(h, k) \in \Re$, there is a unique ellipse $E : \dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ which is tangent to each of the L_j, with $a^2 = \begin{cases} L_1(k) \text{ if } (h,k) \in \Re_1 \\ Q(h) \text{ if } (h,k) \in \Re_2 \end{cases}$ and $b^2 = \begin{cases} P(k) \text{ if } (h,k) \in \Re_1 \\ L_2(h) \text{if } (h,k) \in \Re_2 \end{cases}$

(b) $m_3 = 0$: Then $\Re = \Re_1 \cup \Re_2$,where $\Re_1 = \{(x_3, k) : L_1(k) > 0 \text{ and } k \neq c_3\}$, $\Re_2 = \{(h, y_3) : h < \min(x_1, x_2) \text{ or } \max(x_1, x_2) < h\}$. Again E is unique, with a^2 as in (a), and $b^2 = \begin{cases} (k - c_3)^2 \text{if } (h,k) \in \Re_1 \\ (y_3 - c_3)^2 \text{if } (h,k) \in \Re_2 \end{cases}$

(c) Also, for (a) or (b) above, given any $0 \leq e_0 < 1$, there is an $(h, k) \in \Re$ such that E has eccentricity e_0.

Proof. Since $m_2 = -m_1$, $x_1 = \dfrac{c_2 - c_3}{m_1 + m_3}$, $x_2 = \dfrac{c_3 - c_1}{m_1 - m_3}$, $y_1 = \dfrac{m_1c_3 + m_3c_2}{m_3 + m_1}$, $y_2 = \dfrac{m_1c_3 - m_3c_1}{m_1 - m_3}$, $x_3 = \dfrac{c_2 - c_1}{2m_1}$, $y_3 = \dfrac{c_1 + c_2}{2}$, and (58) reduces to $-4m_1hk + 2m_1(c_1+c_2)h+2(c_2-c_1)k+c_1^2-c_2^2 = 0 \Rightarrow -(2k - c_1 - c_2)(2m_1h - c_2 + c_1) = 0 \Rightarrow (h-x_3)(k-y_3) = 0 \Rightarrow k = y_3$ or $h = x_3$. One can also obtain this equation by letting $m_2 = -m_1$ in (36). Hence the admissible centers must lie on $\gamma = \{(h,k) : h = x_3 \text{ or } k = y_3\}$. Note that $\dfrac{a_1}{D} = -y_3$, $\dfrac{a_2}{D} = -x_3$, and $h = x_3$ and $k = y_3$ are just the asymptotes of the hyperbola with equation (36). Since $L_3(x_2) = L_1(x_2)$, $(L_3(h) - L_1(h))/(m_3 - m_1) = h - x_2$. Write L_j in the form $x = \bar{L}_j(y) = \dfrac{1}{m_j}y - \dfrac{c_j}{m_j}$. Then $\bar{L}_3(y_2) = \bar{L}_1(y_2) \Rightarrow \left(\bar{L}_3(k) - \bar{L}_1(k)\right)/(1/m_3 - 1/m_1) = k - y_2$. By (51), $S_{13} = (h - x_2)(L_3(h) + L_1(h) - 2k)/(m_3 + m_1)$. The equation of $\bar{L}_j$ can also be written as $x - h = \dfrac{1}{m_j}(y - k) - \dfrac{b_j}{m_j}, m_j \neq 0$. Substituting $\dfrac{1}{m_j}$ for m_j and $-\dfrac{b_j}{m_j}$ for b_j in S_{13} yields T_{13}. Interchanging h and k then gives $T_{13} = (\bar{L}_3(k) - \bar{L}_1(k))(\bar{L}_3(k) + \bar{L}_1(k) - 2h)/(1/m_3^2 - 1/m_1^2) = (k - y_2)(\bar{L}_3(k) + \bar{L}_1(k) - 2h)/(1/m_3 + 1/m_1)$ We must determine the points on γ where S_{13} and T_{13} are **both positive** . We have two cases to consider.

Case 1: $h = x_3$

Then $S_{13} = (x_3 - x_2)(L_3(x_3) + L_1(x_3) - 2k)/(m_3 + m_1)$
$= -2(x_3 - x_2)(k - \frac{L_3(x_3)+y_3}{2}) =$

$$-2\frac{x_3 - x_2}{m_3 + m_1}\left(k - \frac{m_3 x_3 + c_3 + y_3}{2}\right) = L_1(k) \tag{59}$$

Now $\bar{L}_1(y_1)+x_1 = 2x_3$ and $\bar{L}_3(y_1) = x_1 \Rightarrow \bar{L}_3(k)+\bar{L}_1(k)-2x_3$ vanishes at $k = y_1 \Rightarrow$ $T_{13} = (k - y_2)\left(\bar{L}_3(k) + \bar{L}_1(k) - 2x_3\right)/\left(1/m_3 + 1/m_1\right) = (k - y_1)(k - y_2) = P(k)$. Note that if $m_3 = 0$, then $y_1 = y_2 = c_3$ and the above formula for $P(k)$ still holds since $T_{13} = (k - c_3)^2$. Also, $P(k) = \dfrac{q_3(k)}{q_1(k)}$ from Theorem 12 since $\dfrac{a_1}{D} = -y_3$.

Case 2: $k = y_3$

It also follows easily that $L_3(h) + L_1(h) - 2y_3$ vanishes at $h = x_1$. Then $S_{13} =$ $(h - x_2)\left(L_3(h) + L_1(h) - 2y_3\right)/(m_3 + m_1) = (h - x_1)(h - x_2) = Q(h)$. Note that $Q(h) = \dfrac{p_3(h)}{p_1(h)}$ from Theorem 12 since $\dfrac{a_2}{D} = -x_3$. Now $T_{13} = (y_3 - y_2)(\bar{L}_3(y_3) +$ $\bar{L}_1(y_3) - 2h)/\left(1/m_3 + 1/m_1\right) = (y_3 - y_2)(\bar{L}_3(y_3) + x_3 - 2h)/\left(1/m_3 + 1/m_1\right) =$ $-2\dfrac{m_1 m_3}{m_3 + m_1}(y_3 - y_2)(h - \frac{\bar{L}_3(y_3)+x_3}{2}) =$

$$-2\frac{m_1 m_3}{m_3 + m_1}(y_3 - y_2)\left(h - \frac{m_3 x_3 + y_3 - c_3}{2m_3}\right) = L_2(h), m_3 \neq 0$$

If $m_3 = 0$, and $h = x_3$, then $S_{13} = L_1(k)$ as above. $T_{13} = b_3^2 = (L_3(x_3) - k)^2 > 0$ if $k \neq L_3(x_3) = c_3$. If $m_3 = 0$, and $k = y_3$, then $S_{13} = Q(h)$ as above. $T_{13} = b_3^2 = (L_3(h) - y_3)^2 = (c_3 - y_3)^2 > 0$ since if $c_3 = y_3$, then $L_1(x_3) =$ $L_2(x_3) = L_3(x_3) = y_3$, which violates the assumption that the L_j are not concurrent. The uniqueness in each case follows from Theorem 1 since $m_1^2 \neq m_3^2$. That proves (a) and (b). To prove (c): (i) If $h = x_3$, then $\dfrac{b^2}{a^2} = \dfrac{P(k)}{L_1(k)}$. Since $m_2 = -m_1$, by (41) , $|M| = -m_3 c_2 + m_1 c_2 + m_3 c_1 - 2m_1 c_3 + m_1 c_1 \neq 0$. Some algebra shows that $L_1(y_1) = \dfrac{\left(m_1 c_2 - m_3 c_2 + c_1 m_1 + m_3 c_1 - 2m_1 c_3\right)^2}{4m_1^2\left(m_3 + m_1\right)^2}$ and $L_1(y_2) =$ $\dfrac{\left(m_1 c_2 - m_3 c_2 + c_1 m_1 + m_3 c_1 - 2m_1 c_3\right)^2}{4m_1^2\left(m_1 - m_3\right)^2}$, which implies that $L_1(y_j) > 0,\ j = 1, 2$. Let $0 < c \leq 1$ be given. Setting $\dfrac{b^2}{a^2} = c$ implies that $\dfrac{P(k)}{L_1(k)} = c$, which holds if and only if $E(k) \equiv P(k) - cL_1(k) = 0$, $L_1(k) \neq 0$. Let $y_{\max} = \max\{y_1, y_2\}$. Since $E(y_j) = -cL_1(y_j) < 0, E(y_{\max}) < 0$, and since $\lim\limits_{k\to\infty} E(k) = \infty$, $E(k_0) = 0$ for some $k_0 > y_{\max}$. Since $P(k_0) > 0$ it follows that $L_1(k_0) \neq 0$ and $\dfrac{P(k_0)}{L_1(k_0)} = c > 0$ implies that $L_1(k_0) > 0$. Thus $(x_3, k_0) \in \Re$ by (38), and implies, upon letting $b^2 = P(k_0)$ and $a^2 = L_1(k_0)$, that the equation $1 - \dfrac{b^2}{a^2} = e_0^2, 0 \leq e_0 < 1$, always has a solution. (ii) If $k = y_3$, the proof follows in a similar fashion.

Our next theorem covers the case when $m_2 = m_1$.

Theorem 15. Let L_1, L_2, L_3 be distinct non–vertical lines with equations $y =$

$L_j(x) = m_jx + c_j$, $j = 1,2,3$, and assume that $m_2 = m_1$, with $m_1^2 \neq m_3^2$. Let L be the line with equation $y = L(x) = m_1x + \frac{1}{2}(c_1 + c_2)$. Let $P(h) = \left(\frac{m_3 - m_1}{m_3 + m_1}\right)(h - x_1)(h - x_2)$ and $Q(k) = -\left(\frac{m_3 - m_1}{m_3 + m_1}\right)(k - y_1)(k - y_2)$.
(i) If $m_1 \neq 0$, then $\Re = \{(h,k) : k = L(h)$ with $P(h) > 0$ and $Q(k) > 0\}$.
(ii) If $m_1 = 0$, then $\Re = \left\{(h, \frac{1}{2}(c_1 + c_2)) : P(h) > 0\right\}$.
In either case, if $(h,k) \in \Re$, then there is a unique ellipse $E : \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ which is tangent to each of the L_j, with $a^2 = P(h)$ and $b^2 = Q(k)$.
(iii) Finally, given any $0 \leq e_0 < 1$, there is an $(h,k) \in \Re$ such that E has eccentricity e_0.
Proof. Note that $c_1 \neq c_2$ since L_1 and L_2 are distinct. Since $m_2 = m_1$, $x_1 = \frac{c_3 - c_2}{m_1 - m_3}$, $x_2 = \frac{c_3 - c_1}{m_1 - m_3}$, $y_1 = \frac{m_1c_3 - m_3c_2}{m_1 - m_3}$, $y_2 = \frac{m_1c_3 - m_3c_1}{m_1 - m_3}$, and (58) reduces to $2hm_1(c_1 - c_2) + 2k(c_2 - c_1) + c_1^2 - c_2^2 = 0 \Rightarrow$

$$2hm_1 - 2k + c_1 + c_2 = 0 \tag{60}$$

Note that by (60), an admissible center must lie on L, and L is parallel to, and lies exactly halfway between, L_1 and L_2. Solving (60) for k yields $k = m_1h + \frac{1}{2}(c_1 + c_2)$. Since $L_3(x_2) = L_1(x_2)$, and $L_3(x_2) + L_1(x_2) - 2m_1x_2 - (c_1 + c_2) = 0$, upon substituting for k it follows that $S_{13} = (L_3(h) - L_1(h))(L_3(h) + L_1(h) - 2m_1h - (c_1 + c_2))$, which vanishes at x_1 and x_2. Thus

$$S_{13} = \left(\frac{m_3 - m_1}{m_3 + m_1}\right)(h - x_1)(h - x_2) = P(h) \tag{61}$$

Solving (60) for h yields(if $m_1 \neq 0$)

$$h = \frac{2k - (c_1 + c_2)}{2m_1}$$

It also follows easily that, upon substituting for h

$$T_{13} = -\left(\frac{m_3 - m_1}{m_3 + m_1}\right)(k - y_1)(k - y_2) = Q(k) \tag{62}$$

That proves (i). If $m_1 = 0$, then (h,k) lies on the horizontal line with equation $y = \frac{1}{2}(c_1+c_2)$. Since $y_1 = c_2$ and $y_2 = c_1$, $k = \frac{1}{2}(c_1+c_2) \Rightarrow Q(k) = \frac{1}{4}(c_1-c_2)^2 > 0$. Using (61), the set of admissible centers now equals $\{(h, \frac{1}{2}(c_1 + c_2)) : P(h) > 0\}$, which proves (ii). The *uniqueness* follows for each case from Theorem 1 since $m_1^2 \neq m_3^2$. To prove (iii): If $(h,k) \in \gamma$, then $\frac{b^2}{a^2} = \frac{Q(k)}{P(h)} = \frac{Q(L(h))}{P(h)} = \frac{Q(k)}{P(g(k))}$ and $m_1^2a^2 + b^2 = b_1^2 \Rightarrow m_1^2P(h) + Q(k) = b_1^2 = (m_1h + c_1 - L(h)) = \frac{1}{4}(c_1 - c_2)^2$. Letting $k = y_j$ yields

$$P(g(y_j)) = \frac{1}{4}\frac{(c_1 - c_2)^2}{m_1^2} > 0 \tag{63}$$

and letting $h = x_j$ yields

$$Q(L(x_j)) = \frac{1}{4}(c_1 - c_2)^2 > 0 \tag{64}$$

Case 1: $\dfrac{m_3 - m_1}{m_3 + m_1} > 0$

Let $0 < c \le 1$ be given. Setting $\dfrac{a^2}{b^2} = c$ implies that $\dfrac{P(h)}{Q(k)} = \dfrac{P(h)}{Q(L(h))} = c$, which holds if and only if $E(h) \equiv P(h) - cQ(L(h)) = 0$, $Q(L(h)) \neq 0$. Let $x_{\max} = \max\{x_1, x_2\}$. Since $E(x_j) = -cQ(L(x_j)) < 0$ by (64) , $E(x_{\max}) < 0$. Since the coefficient of h^2 in $P(h) - cQ(L(h))$ is $\dfrac{m_3 - m_1}{m_3 + m_1} + c\dfrac{m_3 - m_1}{m_3 + m_1}m_1^2$, $\lim\limits_{h \to \infty} E(h) = \infty$. Thus $E(h_0) = 0$ for some $h_0 > x_{\max}$. Since $P(h_0) > 0$ it follows that $Q(L(h_0)) \neq 0$ and $\dfrac{P(h_0)}{Q(L(h_0))} = c > 0$ implies that $Q(L(h_0)) > 0$. Letting $k_0 = L(h_0)$ implies that $(h_0, k_0) \in \Re$ by (38), which implies, upon letting $a^2 = P(h_0)$ and $b^2 = Q(L(h_0))$, that the equation $1 - \dfrac{b^2}{a^2} = e_0^2, 0 \le e_0 < 1$, always has a solution.

Case 2: $\dfrac{m_3 - m_1}{m_3 + m_1} < 0$

Then consider $\dfrac{b^2}{a^2} = \dfrac{Q(k)}{P(h)} = \dfrac{Q(k)}{P(g(k))}$ and the rest of the proof follows in a fashion similar to case 1.

§4.2 All three of the tangents have slopes equal in absolute value

Clearly, if $m_1 = m_2 = m_3$, then there is no ellipse tangent to all of the L_j. Hence we can assume, without loss of generality, that $m_1 = m_2 = -m_3$. In this case the set of admissible centers consists of two points. For each such center, there are infinitely many ellipses tangent to the L_j.

Theorem 16. Let L_j, $j = 1, 2, 3$ be three non–vertical lines with equations $y = m_j x + c_j$, and assume that $m_1 = m_2 = -m_3$. Then $\Re = \{(x_2, y_1), (x_1, y_2)\} = \left\{\left(\dfrac{c_3 - c_2}{2m_1}, \dfrac{c_1 + c_3}{2}\right), \left(\dfrac{c_3 - c_1}{2m_1}, \dfrac{c_2 + c_3}{2}\right)\right\}$. If $(h, k) \in \Re$, the ellipse $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ is tangent to each of the L_j, where a^2 and b^2 are any positive solutions of $a^2 m_1^2 + b^2 = \dfrac{1}{4}(c_1 - c_2)^2$. Finally, given any $0 \le e_0 < 1$ and any $(h, k) \in \Re$, one can choose a^2 and b^2 such that E has eccentricity e_0.

Proof. Given $C = (h, k)$, write L_j in the form $y - k = m_j(x - h) + b_j$, $j = 1, 2, 3$. Since $m_1^2 = m_2^2 = m_3^2$, (42) has a solution if and only if $b_1^2 = b_2^2 = b_3^2$. The solutions satisfy the one equation

$$m_1^2 u + v = b_1^2 \tag{65}$$

If $b_1 \neq 0$, then (65) has infinitely many positive solutions (u, v) and the ellipse $E : \dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ is tangent to L_1, L_2, and L_3, where $a^2 = u$ and $b^2 = v$. By (8), $b_1^2 = b_2^2$ and $b_1^2 = b_3^2$ imply that $2(m_j - m_1)hk + 2(m_1 c_1 -$

$m_jc_j)h + 2(c_j - c_1)k + c_1^2 - c_j^2 = 0$ for $j = 2, 3$, which yields the two simultaneous equations

$$2hm_1 - 2k + c_1 + c_2 = 0,\ (2k - c_1 - c_3)(2m_1h - c_3 + c_1) = 0$$

with solutions $(h, k) = \left(\frac{c_3 - c_2}{2m_1}, \frac{c_1 + c_3}{2}\right) = (x_1, y_2)$ and $\left(\frac{c_3 - c_1}{2m_1}, \frac{c_2 + c_3}{2}\right) = (x_2, y_1)$. Note that those points are distinct since $c_1 \neq c_2$(else $L_1 = L_2$). It is not hard to show that $(h, k) = \left(\frac{c_3 - c_2}{2m_1}, \frac{c_1 + c_3}{2}\right)$ if and only if $b_1 = \frac{1}{2}(c_1 - c_2)$, $b_2 = -b_1$, and $b_3 = -b_1$, and $(h, k) = \left(\frac{c_3 - c_1}{2m_1}, \frac{c_2 + c_3}{2}\right)$ if and only if $b_1 = \frac{1}{2}(c_1 - c_2)$, $b_2 = -b_1$, and $b_3 = b_1$. For each choice of (h, k) above, $b_1 \neq 0$ with $b_1^2 = b_2^2 = b_3^2$. As a^2 varies between 0 and $\frac{b_1^2}{m_1^2}$, one obtains infinitely many positive solutions a^2, b^2 of the equations in (65), and $\frac{a^2}{b^2}$ varies from 0 to ∞. Thus one can also find an ellipse with any preassigned eccentricity.

Remark. For three given tangents(no two of which are parallel) there is a very nice result about ellipses inscribed in the triangle, T, formed by the tangents. Let T_M denote the midpoint triangle, that is the triangle whose vertices are the midpoints of the sides of T. Chakerian [1] proved that $\cup_{-\frac{\pi}{2}<\alpha<\frac{\pi}{2}} \Re_\alpha = int(T_M)$. Chakerian uses orthogonal projection to prove this result, methods much different than the ones used in this paper. One can also prove this result using methods similar to the proofs given for four given lines, though the proof is longer and not as concise. However, using our methods one can also show that the union, over α, of the α admissible centers for **hyperbolas** equals $int(T) - \{int(T_M) \cup T_M\}$.

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Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 1, July 2002, pp. 33–38.
Submitted: November 13, 2001. Published: July 1 2002.

A RELIABLE METHOD TO DETERMINE THE ILL-CONDITION FUNCTIONS USING STOCHASTIC ARITHMETIC

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Abstract. One of the important matters in numerical methods is to find ill-condition functions which have instability in some points. In this paper it has been explained a method to estimate the number of common significant digits between the values of $f(x)$ and $f(x+\delta)$ where $|\delta|$ is small enough. Then we develop this idea for $f(x,y)$ and $f(x+\delta, y+\epsilon)$ where $|\delta|$ and $|\epsilon|$ are small enough. It has been proved two theorems which express the relationship between these values and the condition number of the function f. It has been defined in [4], the condition number of one dimensional function $f(x)$ and in [3] the number of common significant digits between two real numbers . It has been developed these definitions for two dimensional function $f(x,y)$. It has been explained in [9,10,11], one can use the CESTAC[1] method which is a method based on stochastic arithmetic in order to find $f(x)$ numerically. It is used this method to rely and validate the results and implemented the numerical examples. In the numerical examples it has been considered the Rump's function and has been shown it is ill-condition based on the CESTAC method and a kind of perturbation method which has been explained in [1].

1991 A.M.S. (MOS) Subject Classification Codes. 65C20, 65Y20, 68W20.

Key Words and Phrases. keywords Ill-posed function,Stochastic arithmetic, CESTAC method, Condition number, Common significant digits.

1. Introduction. It has mentioned in [3], the number of significant digits common between two distinct real numbers a and b , denoted by $C_{a,b}$, can be defined by

$$C_{a,b} = \log_{10} \left| \frac{a+b}{2(a-b)} \right| = \log_{10} \left| \frac{a}{a-b} - \frac{1}{2} \right|, \tag{1}$$

[1]Control et Estimation Stochastically den Almonds die Calculus

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if $a = b$ then $C_{a,a} = +\infty$. Also if $\mid a - b \mid$ is small enough, one can take

$$C_{a,b} \simeq \log_{10} \mid \frac{a}{a-b} \mid . \tag{2}$$

One can use this definition in order to find the number of common significant digits between two ordered pairs (x, y) and (x', y').It will explain in the next section.

It has mentioned in [4], the condition number of the function f at x can be defined by

$$Cond(f(x)) = max\{|\frac{f(x) - f(x')}{f(x)}|/|\frac{x - x'}{x}| : |x - x'| \ is \ \ small\} \simeq |\frac{xf'(x)}{f(x)}|. \tag{3}$$

If $cond(f(x)) >> 1$ the function f is ill-condition at x. One can use this definition in order to find the condition number of two dimensional function f at (x, y).

2. Numerical accuracy of one variable functions.

Theorem 1. *If it is perturbed x to $x + \delta$ which $|\delta| << 1$ then the number of common significant digits between $f(x)$ and $f(x + \delta)$ is estimated as follows*

$$C_{f(x),f(x+\delta)} \simeq C_{x,x+\delta} - \log_{10} Cond(f(x)) + O(\delta), \tag{4}$$

Proof. According to (1)

$$C_{f(x),f(x+\delta)} = \log_{10} \mid \frac{f(x) + f(x+\delta)}{2(f(x) - f(x+\delta))} \mid = \log_{10} \mid \frac{\frac{f(x)+f(x+\delta)}{\delta}}{\frac{2(f(x)-f(x+\delta))}{\delta}} \mid$$

$$\simeq \log_{10} \mid \frac{f(x) + f(x+\delta)}{2\delta f'(x)} \mid = \log_{10} \mid \frac{2f(x) + O(\delta)}{2\delta f'(x)} \mid = \log_{10} \mid \frac{f(x)(1 + O(\delta))}{\delta f'(x)} \mid$$

$$= \log_{10} \mid \frac{x + O(\delta)}{\delta \frac{xf'(x)}{f(x)}} \mid = \log_{10} \mid \frac{\frac{x}{\delta} + O(1)}{\frac{xf'(x)}{f(x)}} \mid .$$

We conclude from (2) and (3)

$$C_{f(x),f(x+\delta)} \simeq \log_{10} \mid \frac{x}{\delta} \mid - \log_{10} Cond(f(x)) + O(\delta) = \log_{10} \mid \frac{x}{x + \delta - x} \mid$$

$$- \log_{10} Cond(f(x)) + O(\delta) \simeq C_{x,x+\delta} - \log_{10} Cond(f(x)) + O(\delta).$$

This result shows that if $Cond(f(x)) >> 1$ then $f(x)$ and $f(x + \delta)$ have not any common significant digits. In other words if f is ill-conditioned at x then $C_{f(x),f(x+\delta)} \leq 0$. Also if $0 \leq Cond(f(x)) \leq 1$ then $C_{f(x),f(x+\delta)} > 0$.It means that in this case f is well-conditioned. We observe that if $Cond(f(x)) \simeq 1$ then $C_{f(x),f(x+\delta)} \simeq C_{x,x+\delta}$.

3. Numerical accuracy of two variable functions.

At first we define the common significant digits between two distinct ordered pairs and the condition number of the function $f(x, y)$ as follows

Definition 1. We suppose that (x, y) and (x', y') are two distinct ordered pairs. Then

$$C_{(x,y),(x',y')} = \log_{10}\left(\frac{\| (x + x', y + y') \|_2}{2 \| (x - x', y - y') \|_2}\right), \tag{5}$$

if $(x, y) \simeq (x', y')$ then

$$C_{(x,y),(x',y')} \simeq \log_{10}\left(\frac{\| (x, y) \|_2}{\| (x - x', y - y') \|_2}\right). \tag{6}$$

Definition 2. The condition number of function $f : \Re^2 \to \Re$ at (x, y) is

$$Cond(f(x, y)) = max\{\mid \frac{f(x, y) - f(x', y')}{f(x, y)} \mid / \frac{\| (x - x', y - y') \|_2}{\| (x, y) \|_2} : |x - x'|, \mid y - y' \mid \ \ are \ \ small\},$$

(7)

$$= max\{\frac{\mid f(x, y) - f(x', y') \mid}{\| (x - x', y - y') \|_2} \| (x, y) \|_2 \ / \mid f(x, y) \mid : |x - x'|, \mid y - y' \mid \ \ are \ \ small\}.$$

We know that $f(x, y) - f(x', y') \simeq (x - x')\frac{\partial f}{\partial x} + (y - y')\frac{\partial f}{\partial y}$, therefore

$$Cond(f(x, y)) \simeq \frac{\sqrt{x^2 + y^2}}{\mid f(x, y) \mid}\left(\mid \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \mid\right). \tag{8}$$

Theorem 2. *If it is perturbed (x, y) to $(x + \delta, y + \epsilon)$ which $|\delta| << 1$ and $|\epsilon| << 1$ then the number of common significant digits between $f(x, y)$ and $f(x + \delta, y + \epsilon)$ is estimated as follows*

$$C_{f(x,y),f(x+\delta,y+\epsilon)} \simeq C_{(x,y),(x+\delta,y+\epsilon)} - \log_{10} Cond(f(x, y)) + O(\delta, \epsilon), \tag{9}$$

Proof. Let $max\{\mid \delta \mid, \mid \epsilon \mid\} = \mid \eta \mid$. According to (5)

$$C_{f(x,y),f(x+\delta,y+\epsilon)} = \log_{10} \mid \frac{f(x, y) + f(x + \delta, y + \epsilon)}{2(f(x, y) - f(x + \delta, y + \epsilon))} \mid = \log_{10} \mid \frac{f(x, y) + f(x, y) + O(\eta)}{2(\delta\frac{\partial f}{\partial x} + \epsilon\frac{\partial f}{\partial y} + O(\eta^2))} \mid$$

$$= \log_{10} \mid \frac{f(x, y)(1 + O(\eta))}{\delta\frac{\partial f}{\partial x} + \epsilon\frac{\partial f}{\partial y} + O(\eta^2)} \mid = \log_{10} \mid \frac{\frac{1+O(\eta)}{|\delta|+|\epsilon|}}{\frac{\delta\frac{\partial f}{\partial x}+\epsilon\frac{\partial f}{\partial y}+O(\eta^2)}{f(x,y)(|\delta|+|\epsilon|)}} \mid \simeq \log_{10} \mid \frac{\frac{\sqrt{x^2+y^2}}{|\delta|+|\epsilon|} + O(1)}{\frac{\sqrt{x^2+y^2}}{f(x,y)}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + O(\eta^2)\right)} \mid$$

$$= \log_{10} \frac{\sqrt{x^2+y^2}}{\mid \delta \mid + \mid \epsilon \mid} - \log_{10} \mid \frac{\sqrt{x^2+y^2}}{f(x,y)} (\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}) \mid + O(\eta)$$

Since δ and ϵ are small hence

$$C_{(x,y),(x+\delta,y+\epsilon)} \simeq \log_{10} \mid \frac{\| (x,y) \|_2}{\| (\delta,\epsilon) \|_2} \mid \simeq \log_{10} \frac{\sqrt{x^2+y^2}}{\sqrt{\delta^2+\epsilon^2}} \simeq \log_{10} \frac{\sqrt{x^2+y^2}}{\mid \delta \mid + \mid \epsilon \mid}$$

We conclude from (6) and (8)

$$C_{f(x,y),f(x+\delta,y+\epsilon)} \simeq C_{(x,y),(x+\delta,y+\epsilon)} - \log_{10} Cond(f(x,y)) + O(\eta),$$

This result shows that if $Cond(f(x,y)) >> 1$ then $f(x,y)$ and $f(x+\delta, y+\epsilon)$ have not any common significant digits. In other words if f is ill-conditioned at (x,y) then $C_{f(x,y),f(x+\delta,y+\epsilon)} \leq 0$. Also if $0 \leq Cond(f(x,y)) \leq 1$ then $C_{f(x,y),f(x+\delta,y+\epsilon)} > 0$. It means that in this case f is well-conditioned.

In the next section we remind the main idea of the CESTAC method and stochastic arithmetic [9,10,11].

4. Stochastic Arithmetic-CESTAC Method.

Let F be the set of all the values representable in the computer. Thus any value $r \in \mathbb{R}$ is represented in the form of $R \in F$ in the computer. It has been mentioned in [11] that in a binary floating-point arithmetic with P mantissa bits, the rounding error stems from assignment operator is

$$R = r - \epsilon 2^{E-p} \alpha. \tag{10}$$

In this relation ϵ is the sign of r and $2^{-p}\alpha$ is the lost part of the mantissa due to round-off error and E is the binary exponent of the result. In single precision case, $P = 24$ and in double precision case, $P = 53$. Also if the floating-point arithmetic is as rounding to $+\infty$ or $-\infty$ then $-1 \leq \alpha \leq 1$.

According to (10) , if you want to perturb the last mantissa bit of the value r, it is sufficient that you change α in the interval $[-1,1]$. In CESTAC method if the arithmetic is considered as rounding to $+\infty$ or $-\infty$, α can be considered as a random variable uniformly distributed on $[-1,1]$. Thus R, the calculated result, is a random variable and its precision depends on its mean (μ) and its standard deviation (σ).

In practice the samples R_i are obtained by perturbation of the last mantissa bit of every result R, then the mean of random samples R_i, that is $\overline{R} = \frac{\sum_{i=1}^{N} R_i}{N}$, is considered as the result of an arithmetic operation. If $N = 3$, it has been proved in [5,11] that the number of decimal significant digits common to $\overline{R}$ and to the exact value r can be estimated by

$$C_{\overline{R},r} = \log_{10} \frac{\mid \overline{R} \mid}{\sigma} - 0.39. \tag{11}$$

In this formula σ,the standard deviation of the samples R_i, is given by

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}\left(R_i - \overline{R}\right)^2}{N-1}}.$$

In CESTAC method if $C_{\overline{R},r} \leq 0$, the informational result R is insignificant and it means a numerical instability exists in its related result.

In order to simultaneous implementation of the CESTAC method we should substitute a stochastic arithmetic in place of the floating-point arithmetic. In this way every arithmetic operation is run N times synchronously before running the next operation. The term set of stochastic numbers, denoted S, is applied to the set of Gassing random variables. An element $R \in S$ is denoted $R = (\mu, \sigma^2)$ where μ is the mean value of R and σ its standard deviation. The definitions of stochastic zero and arithmetical operations and comparative operators have been defined in [9,11].

Definition 3. $R \in S$ is a "stochastic zero", if and only if

$$C_{\overline{R},r} \leq 0 \quad or \quad \overline{R} = 0.$$

Definition 4. If $R \in S$ is a "stochastic zero" then the notation @0 is used to show the detection of this case in implementation of the CESTAC method. Hence if $C_{\overline{R},r} \leq 0$ or $\overline{R} = 0$ then we write $R = @0$.

Let $a, b, c \in F$, in order to implement the arithmetical operator $c = a\omega b$, first for any of the values a and b, it is obtained $N = 3$ random samples as it was mentioned. In this case the operation ω is in the form of $c_i = a_i \omega b_i;\ i = 1, 2, 3$. Let μ_a, μ_b and μ_c be the means and σ_a^2, σ_b^2 and σ_c^2 be the variance of the random samples a_i, b_i and c_i respectively. The result c from implementation the stochastic operation S^ω between two random variables $a, b \in S$ is an element of S. Its mean and variance can be obtained directly using the samples a_i and b_i [1,2]. At last μ_c is considered as the result of the stochastic operation S^ω. σ_c is used for estimating the number of decimal significant digits of this result using relation (11).

5. Numerical Examples.

In this section three examples have been considered. At first it is shown the instability of the functions at the given points using theorems 1 and 2, then these instabilities are obtained in the stochastic arithmetic using the CESTAC method. The programs have been provided with Fortran 90.

example 1. In this example it has been considered the function $f = x^2 + x - 1150$ [7]. It is shown for the values near the roots this is an ill-conditioned function. One of the exact roots is $\alpha \simeq 33.415336$. Let $x = 33.4153$, then $Cond(f(x)) = 895796.60000$ and therefore $\log_{10} Cond(f(x)) = 5.952209$ and $C_{x,\alpha} = 5.9424851$. Hence according to the theorem 1, $C_{f(x),f(\alpha)} < 0$. It means that f is ill-conditioned at x. Now if it is used the stochastic arithmetic and is found $f(x)$ at $x = 33.4153$, it has been detected the stochastic zero $f(x) = @0$.

example 2. In this example the Wilkinson polynomial $p(x) = \prod_{i=1}^{20}(x - i)$ [8]. One can estimate the value of $p(x)$ at points near the roots and find the instability of this polynomial at them. We know that $p'(x) = \sum_{i=1}^{20} \frac{p(x)}{x-i}$. Thus if we consider

$x = 1.000001$ then $Cond(p(x)) = 1048573.0000$ and $C_{1,x} = 6.0206001$ and therefore $\log_{10} Cond(p(x)) = 6.020599$. Hence according to the theorem 1, $C_{p(1),p(x)} \simeq 0$. In the floating-point arithmetic with single precision the value of polynomial is $p(x) = -1.160094E + 11$. Using the CESTAC method in order to find $p(x)$ at $x = 1$ it is obtained $p(x) = 4.833746e+09$ in company with the notation $p(x) = @0$. It means that the solution at this point is not valid and there is an instability in finding the exact root $x = 1$.

example 3. In this example we consider the Rump's function[10]

$$f(x,y) = 333.75y^6 + x^2(11x^2y^2 - y^6 - 121y^4 - 2) + 5.5y^8 + \frac{x}{2y}$$

We evaluate this function at $(x,y) = (77617, 33096)$. In the floating point arithmetic with double precision the value of function is $f(x,y) = -1.390612133896486E+021$. Also using the relation (8) the condition number of f is $Cond(f(x,y)) = 1.665159112522263E+016$ thus $\log_{10} Cond(f(x,y)) \simeq 16.2215$. Also if we suppose $\delta = \epsilon = 0.0001$ then $C_{(x,y),(x+\delta,y+\epsilon)} = 8.62520220154603$. According to theorem 2 we conclude $C_{f(x,y),f(x+\delta,y+\epsilon)} < 0$. It means this function is ill-conditioned at this point. Using the stochastic arithmetic we can validate this result. In this arithmetic the value f is a stochastic zero. In other words $f(x,y) = @0$. Consequently the result of the floating-point arithmetic is not reliable and this function has an instability at this point.

6. Conclusion.

There is a relationship between the condition number of a function at a given point and the significant digits common to this point and its perturbation. If it is considered an ill-conditioned function because it has a large condition number, it occurs an instability at the final result. But in many cases, the calculation of the condition number is difficult. Therefore one can use the CESTAC method to find this instability. Using the stochastic arithmetic one can validate the result and find whether it is reliable or not.

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Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISBN 1083-0464
Issue 1 December 2002, pp. 39 – 70
Submitted: June 11, 2001. Published: July 1, 2002

Boundary feedback stabilization of a hybrid PDE-ODE system

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Abstract

This paper deals with boundary feedback stabilization of a variant of the SCOLE model. Namely, a flexible beam clamped at one end and free at the other end where a rigid body is attached. Using the frequency domain method, we first prove that the considered system is uniformly stabilizable when control force and control moment are simultaneously applied at the free end of the beam. Secondly when only a control force is applied, we give a sufficient condition on the physical parameters of the system to guarantee the uniform stabilization. Finally it is shown that the presence of boundary control force is necessary to achieve uniform stabilization. To verify these theoretical developments, an asymptotic analysis and numerical study of the spectrum are conducted.

A.M.S. (MOS) Subject Classification Codes. 35B40, 35M10, 35P20, 47D03, 93C20, 93D15, 93D30
Key Words and Phrases. SCOLE model, infinite dimensional systems, boundary feedback control, uniform stabilization, semigroups theory, asymptotic spectral analysis.

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1 Introduction

The purpose of this paper is to study the boundary feedback stabilization problem of the system presented in the figure 1. It consists of an elastic beam (P) clamped at one end and linked, at the other end, to a rigid body (A). Such systems arise in the study of large scale flexible space structures and well-known as SCOLE model [11]. The system is governed by one partial

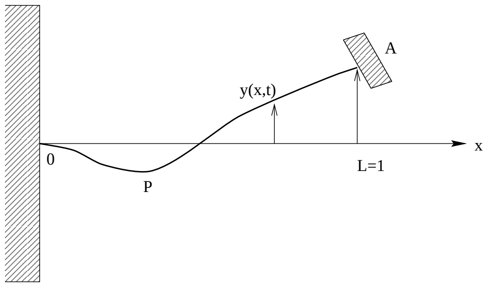

Figure 1: SCOLE model

differential equation (Euler-Bernoulli beam equation for the vibrations of the beam) coupled with two ordinary differential equations (Newton-Euler equations for the oscillations of the rigid body) and often called in literature hybrid system:

$$\begin{cases} \rho y_{tt} + EIy_{xxxx} = 0, \\ y(0,t) = y_x(0,t) = 0, \\ Jy_{xtt}(1,t) + EIy_{xx}(1,t) = \Theta_1(t), \\ my_{tt}(1,t) - EIy_{xxx}(1,t) = \Theta_2(t), \end{cases} \tag{1.1}$$

where the positive constants EI, ρ are respectively the flexual rigidity and the mass per unit length of the beam; J is the moment of inertia and m is the mass of the rigid body (A) attached at the free end of the beam. $\Theta_1(t)$ and $\Theta_2(t)$ denote respectively the linear moment control and force control acting on the free end of the beam.

The objective of the paper is to study the stabilization problem of (1.1) in the case where the **dynamical term $\boldsymbol{my_{tt}(1,t)}$ is neglected**. More precisely, the goal is to find simple feedback laws $\Theta_1(t)$ and $\Theta_2(t)$ such that the solutions of the following system

$$\begin{cases} \rho y_{tt} + EIy_{xxxx} = 0, \\ y(0,t) = y_x(0,t) = 0, \\ Jy_{xtt}(1,t) + EIy_{xx}(1,t) = \Theta_1(t), \\ -EIy_{xxx}(1,t) = \Theta_2(t), \end{cases} \tag{1.2}$$

decay uniformly to zero in some functional space.

The stabilization problem of model (1.1) has been the subject of many studies (see [11], [17], [14], [15], [5] and [10]). Indeed, it is known that the feedback law $\Theta_1(t) = -y_{xt}(1,t)$ and $\Theta_2(t) = -y_t(1,t)$ is sufficient for obtaining the strong stability of (1.1) but not sufficient for obtaining the uniform one [11]. In the case where $m = J = 1$, Slemrod [17] proved the strong stabilization with a priori bounded moment control. This result has been improved by Rao [15] in the sense that the strong stability can be obtained for any length of beam $L < 3$. Furthermore, the strong stabilization has been obtained by a priori bounded feedback force control [15]. Using a method based on compact perturbations due to Russell [16], Rao showed the lack of uniform stabilization of (1.1) with usual feedbacks $\Theta_i(t) = -\delta_{i1}y(1,t) - \delta_{i2}y_x(1,t) - \nu_{i1}y_t(1,t) - \nu_{i2}y_{xt}(1,t)$, whatever real numbers δ_{ij} and ν_{ij} are for $i,j = 1,2$ [14]. This generalizes the result of Littman-Markus [11] insofar as the feedback laws proposed in [11] is obtained by taking $\delta_{i1} = \delta_{i2} = -\nu_{ii} = 0$, for $i,j = 1,2$ and $\nu_{12} = \nu_{21} = 1$. Nevertheless, by means of higher order feedback such as $y_{xxt}(1,t)$ and $y_{xxxt}(1,t)$, uniform stabilization was obtained in [14]. By neglecting the moment of inertia J, it was shown in [5] that the system (1.1) could be exponentially stabilized by $\Theta_1(t) = 0$ and $\Theta_2(t) = -\alpha y_t(1,t) + \beta y_{xxxt}(1,t)$ (α and β are positive constants). Recently, the same result has been obtained in [10] but by using only a simple boundary control moment.

Our contribution in this article is to show that the system (1.2) is exponentially stabilizable provided that both of the two boundary controls (force and moment) are present at the free end of the beam. The proof is based on Huang's theorem (Theorem 3, p.51, [8]) which corresponds to the frequency domain method. We have tried to use the multiplier technique without much success. Moreover, if only the control force is present, we shall show uniform stability of the closed-loop system but under a condition on the physical parameters EI, J and ρ. The proof is based on compact perturbation result (see [6] and [18]). However, it is shown that the considered system can not be uniformly stabilized by only the control moment. Finally, we give asymptotic values of eigenvalues of the controlled operator. The spectral analysis permits us to prove that the condition on the physical parameters EI, J and ρ is necessary. This is illustrated by some numerical results.

Now let us briefly outline the content of this paper. In Section 2, we propose a feedback law; then we formulate the closed loop system as a standard form of evolution equation and prove some preliminary results like well-posedness and strong stability. Section 3, containing the essential part of the paper, is devoted to establishing the uniform stability of the system. Lack of uniform stabilization is also proved when the feedback laws are composed of only control moment. By using an asymptotic analysis, the real part of the spectrum is estimated in section 4 and then we prove that the condition on the physical parameters of the system is impossible to circumvent for the stability. We also give some numerical results of the spectrum.

2 Notations and preliminary results

First of all it is easy to see that, with a suitable change of physical parameters, the system (1.2) can be written as follows:

$$\begin{cases} \rho y_{tt} + EI y_{xxxx} = 0, \\ y(0,t) = y_x(0,t) = 0, \\ J y_{xtt}(1,t) + y_{xx}(1,t) = \Theta_1(t), \\ -y_{xxx}(1,t) = \Theta_2(t). \end{cases} \tag{2.1}$$

We propose the following linear feedback laws

$$\Theta_1(t) = -\alpha y_{xt}(1,t), \quad \Theta_2(t) = -\beta y_t(1,t), \tag{2.2}$$

where α and β are positive constants. For the state space of the system, we take the space

$$\mathcal{H} = H_0^2(0,1) \times L^2(0,1) \times \mathbb{R},$$

equipped with the inner product

$$\langle (f,g,\xi), (\tilde{f},\tilde{g},\tilde{\xi}) \rangle_{\mathcal{H}} = \int_0^1 \left(\rho g(x)\tilde{g}(x) + EI f_{xx}\tilde{f}_{xx} \right) dx + JEI\xi\tilde{\xi}, \tag{2.3}$$

where

$$H_0^n(0,1) = \left\{ f \in H^n(0,1);, f(0) = f_x(0) = 0 \right\}, \quad \text{for } n = 2,3,...$$

Notice that the norm induced by the scalar product (2.3) is equivalent to the usual one of the space $H^2(0,1) \times L^2(0,1) \times \mathbb{R}$ and thus the space $\mathcal{H}$ is a Hilbert space. Then, the closed loop system (2.1)-(2.2) can be written on $\mathcal{H}$ in a standard form of evolution equation

$$\dot{u}(t) = Au(t), \tag{2.4}$$

where $u(t) = \Big(y(\cdot,t), y_t(\cdot,t), y_{xt}(1,t)\Big)$ and A is an operator defined by

$$\mathcal{D}(A) = \Big\{(f,g,\xi) \in H_0^4(0,1) \times H_0^2(0,1) \times \mathbb{R}; -f_{xxx}(1) + \beta g(1) = 0, \xi = g_x(1)\Big\} \tag{2.5}$$

and

$$A(f,g,\xi) = \left(g, -\frac{EI}{\rho} f_{xxxx}, -\frac{1}{J}\Big[f_{xx}(1) + \alpha\xi\Big]\right). \tag{2.6}$$

The following lemma can be proved by using the classic semigroups theory (see [2] and [13]).

Lemma 1 . *i) The operator A defined by (2.5) and (2.6) generates a continuous semigroup of contractions $S(t)$ on $\mathcal{H}$.*
ii) For any initial data $u_0 = (y_0, z_0, \xi_0) \in \mathcal{D}(A)$, the equation (2.4) admits a unique strong solution $u(t) = (y(t), z(t), \xi(t)) = S(t)(y_0, z_0, \xi_0) \in \mathcal{D}(A)$ for all $t \geq 0$ such that $u(\cdot) \in C^1(\mathbb{R}_+; \mathcal{H}) \cap C(\mathbb{R}_+; \mathcal{D}(A))$. Moreover, the function $t \longmapsto \|Au(t)\|_{\mathcal{D}(A)}$ is decreasing.
iii) For any initial data $u_0 = (y_0, z_0, \xi_0) \in \mathcal{H}$, there exists a unique weak solution $u(t) = (y(t), z(t), \xi(t)) = S(t)(y_0, z_0, \xi_0) \in \mathcal{H}$ of (2.4) such that $u(\cdot) \in C(\mathbb{R}_+; \mathcal{H})$.

It is not difficult to show the existence of a constant $C > 0$ such that

$$\|y\|^2_{H^4(0,1)} + \|z\|^2_{H^2(0,1)} + |\xi|^2 \leq C\|(y,z,\xi)\|^2_{\mathcal{D}(A)}, \qquad \forall (y,z,\xi) \in \mathcal{D}(A).$$

Then by Sobolev embedding [1], one can establish the following elementary lemma.

Lemma 2 . *The canonical embedding $i : \mathcal{D}(A) \to \mathcal{H}$ is compact, where $\mathcal{D}(A)$ is equipped with the graph norm.*

Now, using LaSalle's invariance principle [8], we will show that the semigroup $S(t)$ is strongly stable in $\mathcal{H}$.

Lemma 3 . *For any initial data $u_0 = (y_0, z_0, \xi_0) \in \mathcal{H}$, the corresponding solution*
$u(t) = S(t)(y_0, z_0, \xi_0)$ of (2.4) tends strongly to zero in $\mathcal{H}$ as $t \longrightarrow +\infty$.

Proof of Lemma 3. Notice that by a standard argument of density of $\mathcal{D}(A)$ in $\mathcal{H}$ and the contraction of the semigroup $S(t)$, it suffices to prove Lemma 3 for any initial data $u_0 = (y_0, z_0, \xi_0) \in \mathcal{D}(A)$. Let $u(t) = (y(t), z(t), \xi(t)) = S(t)u_0$ and just take the Lyapunov function candidate

$$V(u) = \frac{1}{2}\left\{\int_0^1 \left(\rho y_t^2 + EI y_{xx}^2\right) dx + JEI y_{xt}^2(1,t)\right\}. \tag{2.7}$$

Differentiating V following the trajectories of (2.4) leads us to

$$V(u(t)) - V(u_0) = -EI \int_0^t \left(\alpha y_{x\tau}^2(1,\tau) + \beta y_\tau^2(1,\tau)\right) d\tau. \tag{2.8}$$

By lemma 1 (i), we know that the positive orbit of $S(t)$ is a graph norm bounded and thus precompact by virtue of of Lemma 2. Moreover, the operator A being the generator of semigroup of contractions $S(t)$ on $\mathcal{H}$, one can prove that the ω-limit set $\omega(u_0)$ is also in $\mathcal{D}(A)$. Applying LaSalle's principle, we deduce that $\omega(u_0)$ is non empty, compact, invariant under the semigroup $S(t)$ and in addition $S(t)u_0 \longrightarrow \omega(u_0)$ as $t \to +\infty$ [7].

In order to prove the strong stability, it is sufficient to show that $\omega(u_0)$ reduces to zero. To this end, let $\tilde{u}_0 = (\tilde{y}_0, \tilde{z}_0, \tilde{\xi}_0) \in \omega(u_0)$ and $\tilde{u}(t) = (\tilde{y}(\cdot,t), \tilde{y}_t(\cdot,t), \tilde{y}_{xt}(1,t)) = S(t)\tilde{u}_0 \in \omega(\tilde{u}_0)$ the unique strong solution of (2.4). It is well-known that $V(\tilde{u}(t))$ is constant and thus $\frac{dV}{dt}(\tilde{u}(t)) = \tilde{y}_{xt}(1,t) = \tilde{y}_t(1,t) = 0$ [8]. Hence $\tilde{y}$ is a solution of the following system

$$\begin{cases} \rho\tilde{y}_{tt} + EI\tilde{y}_{xxxx} = 0, \\ \tilde{y}(0,t) = \tilde{y}_x(0,t) = 0, \\ \tilde{y}_{xt}(1,t) = \tilde{y}_{xx}(1,t) = 0, \\ \tilde{y}_{xxx}(1,t) = \tilde{y}_t(1,t) = 0, \\ (\tilde{y}(\cdot,0), \tilde{y}_t(\cdot,0)) \in H_0^4 \times H_0^2. \end{cases}$$

It has been proved in [4] that $\tilde{y} = 0$ is the unique solution of this system and thus $\tilde{u}(t) = S(t)\tilde{u}_0 = 0$. This completes the proof of Lemma 3. $\square$

Remark 1 . *One can check that A^{-1} exists and is a compact operator on $\mathcal{H}$. Consequently, the spectrum $\sigma(A)$ of A consists of only isolated eigenvalues with finite multiplicity [12]. This, together with Lemma 3, implies that the resolvent set $\rho(A)$ of the operator A contains the set $\{i\omega;\ \omega \in \mathbb{R}\}$. Indeed, suppose the contrary to be true. Necessarily, there exists $\omega_0 \in \mathbb{R}$ such that $i\omega_0 \notin \rho(A)$ which is equivalent to saying that $i\omega_0$ is an element of the point spectrum $\sigma(A)$. This means that there exists a nontrivial element $u_0 \in \mathcal{D}(A)$ such that $(i\omega_0 I - A)u_0 = 0$. Setting $u(t) = e^{i\omega_0 t}u_0$, one can verify that $u(t)$ is the solution of (2.4) but $\|u(t)\| \to \|u_0\| \neq 0$ as $t \to +\infty$, which is inconsistent with the strong stability. We thus conclude that $\rho(A) \supset \{i\omega;\ \omega \in \mathbb{R}\}$.*

3 Main results

3.1 Uniform stabilization

Theorem 1 . *The semigroup $S(t)$ generated by A is exponentially stable in $\mathcal{H}$.*

Proof of Theorem 1. By means of Remark 1 and the contraction of $S(t)$, we only need to prove that (see Theorem 3 p. 51, [8])

$$\sup\left\{\|(i\omega - A)^{-1}\|;\ \omega \in \mathbb{R}\right\} < \infty. \tag{H}$$

For simplicity and without loss of generality, we choose $EI = \rho = 1$. Consider then the resolvent equation, i.e, given $\lambda = i\omega,\ \omega \in \mathbb{R}$ and $(f, g, \gamma) \in \mathcal{H}$, find $(y, z, \xi) \in \mathcal{D}(A)$ such that

$$(i\omega I - A)(y, z, \xi) = (f, g, \gamma). \tag{3.1}$$

In order to obtain (H), we shall first prove the existence of a positive constant M, independent of ω, such that

$$\int_0^1 \left(y_{xx}^2 + z^2\right) dx + J\xi^2 \leq M\left(\|f_{xx}\|^2 + \|g\|^2 + J\gamma^2\right), \tag{I}$$

which clearly gives the desired property (H). Furthermore, because the function $\lambda \longmapsto \|(\lambda - A)^{-1}\|$ is continuous for any $\lambda \in \rho(A)$ (see [13]), it follows that it is sufficient to show the estimation (I) for $|\omega|$ large. The proof of estimation (I), inspired by the work of Chen et al. [3], is divided into two steps.

First step- The goal is to obtain the following estimation

$$\|y_{xx}\|^2 \leq M_1 \left(\|f_{xx}\|^2 + \|g\|^2 + J\gamma^2\right). \tag{I_1}$$

To accomplish this, let $\lambda = i\omega = i\eta^2$, $\eta \in \mathbb{R}$ (the estimates for $\omega = -\eta^2$ are similar). Thus the equation (3.1) is equivalent to

$$\begin{cases} y_{xxxx} - \eta^4 y = i\eta^2 f + g, \\ y_{xx}(1) + (i\alpha\eta^2 - J\eta^4)y_x(1) = (iJ\eta^2 + \alpha)f_x(1) + \gamma J, \\ i\beta\eta^2 y(1) - y_{xxx}(1) = \beta f(1), \\ y(0) = y_x(0) = 0, \\ z = i\eta^2 y - f, \\ \xi = i\eta^2 y_x(1) - f_x(1). \end{cases} \tag{3.2}$$

As in [3], we consider the following two systems of linear differential equations:

$$\begin{cases} \hat{y}_{xxxx} - \eta^4 \hat{y} = i\eta^2 f + g, \\ \hat{y}(0) = \hat{y}_x(0) = \hat{y}_{xx}(0) = \hat{y}_{xxx}(0) = 0, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \tilde{y}_{xxxx} - \eta^4 \tilde{y} = 0, \\ \tilde{y}_{xx}(1) + \eta^2(i\alpha - J\eta^2)\tilde{y}_x(1) = l_1, \\ i\beta\eta^2 \tilde{y}(1) - \tilde{y}_{xxx}(1) = l_2, \\ \tilde{y}(0) = \tilde{y}_x(0) = 0, \end{cases} \tag{3.4}$$

where

$$\begin{cases} l_1 = -\eta^2(i\alpha - J\eta^2)\hat{y}_x(1) - \hat{y}_{xx}(1) + (iJ\eta^2 + \alpha)f_x(1) + \gamma J, \\ l_2 = -i\beta\eta^2\hat{y}(1) + \hat{y}_{xxx}(1) + \beta f(1). \end{cases} \tag{3.5}$$

Let us observe that if $\hat{y}(x)$ and $\tilde{y}(x)$ are the solutions of (3.3) and (3.4) respectively, then the element $y(x) = \hat{y}(x) + \tilde{y}(x)$ satisfies (3.2). Furthermore, it is a simple task to verify that the unique solution of (3.3) is

$$\hat{y}(x) = \frac{1}{2}\int_0^x \eta^{-3}\Big[\sinh(\eta(x-\tau)) - \sin(\eta(x-\tau))\Big]\left(i\eta^2 f(\tau) + g(\tau)\right)d\tau, \tag{3.6}$$

In order to get the unique solution of (3.4), we set the general solution of the first equation in (3.4) as

$$\tilde{y}(x) = We^{\eta x} + Xe^{i\eta x} + Ye^{-\eta x} + Ze^{-i\eta x}, \tag{3.7}$$

where, W, X, Y and Z are to be determined to satisfy the four boundary conditions of (3.4). Indeed, these boundary conditions lead us to the following linear system

$$M \begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ l_1 \\ l_2 \end{pmatrix},$$

where $M = (m_{ij})_{1 \leq i,j \leq 4}$ is a matrix whose elements are

$\mathbf{m_{11}} = 1,\ \mathbf{m_{12}} = 1,\ \mathbf{m_{13}} = 1,\ \mathbf{m_{14}} = 1,$

$\mathbf{m_{21}} = 1,\ \mathbf{m_{22}} = i,\ \mathbf{m_{23}} = -1,\ \mathbf{m_{24}} = -i,$

$\mathbf{m_{31}} = (\eta^2 + i\alpha\eta^3 - J\eta^5)e^{\eta},\ \mathbf{m_{32}} = (-\eta^2 - \alpha\eta^3 - iJ\eta^5)e^{i\eta},$

$\mathbf{m_{33}} = (\eta^2 - i\alpha\eta^3 + J\eta^5)e^{-\eta},\ \mathbf{m_{34}} = (-\eta^2 + \alpha\eta^3 + iJ\eta^5)e^{-i\eta},\ \mathbf{m_{41}} = (i\beta\eta^2 - \eta^3)e^{\eta}$

$\mathbf{m_{42}} = i(\beta\eta^2 + \eta^3)e^{i\eta},\ \mathbf{m_{43}} = (i\beta\eta^2 + \eta^3)e^{-\eta},\ \mathbf{m_{44}} = i(\beta\eta^2 - \eta^3)e^{-i\eta}.$

Also note that for η large $\det M \neq 0$ (see (3.18)) and thus after a careful computation we get

$$\begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = (i+1)\eta^2(\det M)^{-1} \begin{pmatrix} \times & \times & \mu_{13} & -\mu_{14} \\ \times & \times & -\mu_{23} & -\mu_{24} \\ \times & \times & -\mu_{33} & \mu_{34} \\ \times & \times & -\mu_{43} & -\mu_{44} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ l_1 \\ l_2 \end{pmatrix}, \quad (3.8)$$

where $\times$ denotes the unnecessary elements for subsequent calculus and

$\mu_{13} = (\beta + \eta)e^{i\eta} + i(\eta - \beta)e^{-i\eta} + [(i-1)\beta + (i+1)\eta]e^{-\eta},$

$\mu_{14} = (i + i\alpha\eta - J\eta^3)e^{i\eta} + (1 - \alpha\eta - iJ\eta^3)e^{-i\eta} + [(i+1)(1 + J\eta^3) + (1-i)\alpha\eta]e^{-\eta},$

$\mu_{23} = (\beta + i\eta)e^{\eta} + (i+1)(\eta - \beta)e^{-i\eta} + (i\beta + \eta)e^{-\eta},$

$\mu_{24} = (i - \alpha\eta - iJ\eta^3)e^{\eta} + [(i-1)(1 - \alpha\eta) + (i+1)J\eta^3]e^{-i\eta} - (1 - i\alpha\eta + J\eta^3)e^{-\eta},$

$\mu_{33} = -i(\beta + \eta)e^{i\eta} + [(i-1)\beta - (i+1)\eta]e^{\eta} + (\beta - \eta)e^{-i\eta},$

$\mu_{34} = (1 + \alpha\eta + iJ\eta^3)e^{i\eta} + [(i+1)(1 - J\eta^3) + (i-1)\alpha\eta]e^{\eta} + (i - i\alpha\eta + J\eta^3)e^{-i\eta},$

$\mu_{43} = (i+1)(\beta + \eta)e^{i\eta} + (\eta - i\beta)e^{\eta} + (i\eta - \beta)e^{-\eta},$

$\mu_{44} = [(1-i)(1 + \alpha\eta) + (i+1)J\eta^3]e^{i\eta} + (1 + i\alpha\eta - J\eta^3)e^{\eta} - (i + \alpha\eta + iJ\eta^3)e^{-\eta}.$

¿From (3.6) and integrating twice by parts, we find

$$\hat{y}_{xx}(x) = \frac{\eta^{-1}e^{\eta x}}{4}\int_0^1 e^{-\eta\tau}\left(if_{\tau\tau}(\tau) + g(\tau)\right)d\tau + \mathcal{O}\left(\eta^{-1}[\|f_{xx}\| + \|g\|]\right). \quad (3.9)$$

Similarly we have the following estimations

$$\begin{cases} \hat{y}(x) = -\dfrac{i}{\eta^2} f(x) + \dfrac{\eta^{-3}}{4} \displaystyle\int_0^1 e^{\eta(x-\tau)} \left(i f_{\tau\tau}(\tau) + g(\tau) \right) d\tau + \mathcal{O}\left(\eta^{-3} [\|f_{xx}\| + \|g\|] \right), \\ \hat{y}_x(x) = -\dfrac{i}{\eta^2} f_x(x) + \dfrac{\eta^{-2}}{4} \displaystyle\int_0^1 e^{\eta(x-\tau)} \left(i f_{\tau\tau}(\tau) + g(\tau) \right) d\tau + \mathcal{O}\left(\eta^{-2} [\|f_{xx}\| + \|g\|] \right), \\ \hat{y}_{xxx}(x) = \dfrac{1}{4} \displaystyle\int_0^1 e^{\eta(x-\tau)} \left(i f_{\tau\tau}(\tau) + g(\tau) \right) d\tau + \mathcal{O}\left(\|f_{xx}\| + \|g\| \right). \end{cases} \tag{3.10}$$

Combining (3.5) and (3.10), we obtain after a straightforward computation

$$\begin{aligned} l_1 &= \gamma J + \frac{e^{\eta}}{4} (J\eta^2 - i\alpha - \eta^{-1}) \int_0^1 e^{-\eta\tau} (i f_{\tau\tau}(\tau) + g(\tau)) d\tau \\ &+ \mathcal{O}\left(\eta^2 [\|f_{xx}\| + \|g\|] \right), \end{aligned} \tag{3.11}$$

$$l_2 = \frac{e^{\eta}}{4} (1 - i\beta\eta^{-1}) \int_0^1 e^{-\eta\tau} (i f_{\tau\tau}(\tau) + g(\tau)) d\tau + \mathcal{O}\left(\|f_{xx}\| + \|g\| \right) \tag{3.12}$$

Now let us define an element W^* by

$$W^* = (i+1)\eta^2 \left[\left(J\eta^2 - i\alpha - \eta^{-1} \right) \mu_{13} - \left(1 - i\beta\eta^{-1} \right) \mu_{14} \right]. \tag{3.13}$$

Substituting μ_{13} and μ_{14} by their expressions in the above identity, we get

$$\begin{aligned} W^* = (i+1) \Big\{ & e^{-i\eta} [2iJ\eta^5 + \beta J((1-i)\eta^4 + 2\alpha\eta^3 - \eta^2 (i+1)(1+\alpha\beta) + 2i\beta\eta] \\ & + e^{i\eta} [2J\eta^5 + \beta J((1-i)\eta^4 - 2i\alpha\eta^3 - \eta^2 (i+1)(1+\alpha\beta) - 2\beta\eta] \\ & + e^{-\eta} [2(1-i)\beta J\eta^4 + 2(\alpha\beta - 1)(i+1)\eta^2] \Big\}. \end{aligned} \tag{3.14}$$

Clearly, the dominant term of W^* is η^5 and $|W^*|$ is greater than $\mathcal{O}(\eta^4)$. Namely,

$$|W^*| = \mathcal{O}(\eta^5), \ | \ W^* \ | \geq \sqrt{2}\beta J\eta^4. \tag{3.15}$$

On the other hand, we get from the definition of the matrix M that its determinant is given by

$$\begin{aligned}\det M = &-2\eta^4 e^{(1-i)\eta}[(i-1)J\eta^4 + \beta J\eta^3 + (i+1)\alpha\eta^2 - i(1+\alpha\beta)\eta + (i-1)\beta] \\ &-2\eta^4 e^{(i-1)\eta}[(1-i)J\eta^4 + \beta J\eta^3 - (i+1)\alpha\eta^2 - i(1+\alpha\beta)\eta + (1-i)\beta] \\ &-2\eta^4 e^{(i+1)\eta}[(1+i)J\eta^4 + \beta J\eta^3 + (-i+1)\alpha\eta^2 - i(1+\alpha\beta)\eta - (i+1)\beta] \\ &-2\eta^4 e^{-(i+1)\eta}[-(1+i)J\eta^4 + \beta J\eta^3 + (i-1)\alpha\eta^2 - i(1+\alpha\beta)\eta + (i+1)\beta] \\ &+8\eta^4[\beta J\eta^3 + i(1-\alpha\beta)\eta].\end{aligned} \tag{3.16}$$

Factoring the first and the third line of (3.16) by $-\eta^3 e^\eta$ and using (3.14), we obtain

$$\begin{aligned}\det M = &-\eta^3 e^\eta \Big(W^* - e^{-\eta}[2(1-i)\beta J\eta^4 + 2(\alpha\beta - 1)(i+1)\eta^2]\Big) \\ &-2\eta^4 e^{(i-1)\eta}[(1-i)J\eta^4 + \beta J\eta^3 - (i+1)\alpha\eta^2 - i(1+\alpha\beta)\eta + (1-i)\beta] \\ &-2\eta^4 e^{-(i+1)\eta}[-(1+i)J\eta^4 + \beta J\eta^3 + (i-1)\alpha\eta^2 - i(1+\alpha\beta)\eta + (i+1)\beta] \\ &+8\eta^4[\beta J\eta^3 + i(1-\alpha\beta)\eta].\end{aligned} \tag{3.17}$$

Except the first term $-\eta^3 e^\eta W^*$ of (3.17), all the others terms are bounded by $\mathcal{O}(\eta^7)$. Then, the relation between W^* and $\det M$ is

$$\det M = -\eta^3 e^\eta W^* + \mathcal{O}(\eta^7), \tag{3.18}$$

which implies, thanks to (3.15) and (3.18) that

$$(\det M)^{-1} = -\eta^{-3} e^{-\eta} (A^*)^{-1} + \mathcal{O}(\eta^{-7} e^{-2\eta}) = \mathcal{O}(\eta^{-7} e^{-\eta}). \tag{3.19}$$

Estimation of W:

On one hand, we have from (3.8)

$$W = (i+1)\eta^2 (\det M)^{-1} (\mu_{13} l_1 - \mu_{14} l_2). \tag{3.20}$$

On the other hand, the estimations (3.11) and (3.12) yield

$$\begin{aligned}\mu_{13} l_1 - \mu_{14} l_2 = &\left(J\eta^2 - i\alpha - \eta^{-1}\right)\mu_{13} - \left(1 - i\beta\eta^{-1}\right)\mu_{14} \times \\ &\times \frac{e^\eta}{4}\int_0^1 e^{-\eta\tau}\left(if_{\tau\tau}(\tau) + g(\tau)\right) d\tau \\ &+\mu_{13}\Big(\gamma J + \mathcal{O}(\eta^2[\|f_{xx}\| + \|g\|])\Big) - \mu_{14}\mathcal{O}\left(\|f_{xx}\| + \|g\|\right).\end{aligned}$$

Because of (3.13), the above identity can be rewritten as follows

$$\mu_{13}l_1 - \mu_{14}l_2 = \frac{W^*}{(i+1)\eta^2}\frac{e^\eta}{4}\int_0^1 e^{-\eta\tau}\left(if_{\tau\tau}(\tau)+g(\tau)\right)d\tau$$
$$+\mu_{13}\Big(\gamma J + \mathcal{O}(\eta^2[\|f_{xx}\|+\|g\|])\Big) - \mu_{14}\mathcal{O}\left(\|f_{xx}\|+\|g\|\right). \tag{3.21}$$

Inserting (3.21) into (3.20) and using the first estimation of (3.19), we deduce that

$$W = \frac{-1}{4\eta^3}\int_0^1 e^{-\eta\tau}[if_{\tau\tau}(\tau)+g(\tau)]d\tau + \frac{e^\eta}{4}W^*\mathcal{O}(\eta^{-7}e^{-2\eta})\int_0^1 e^{-\eta\tau}[(if_{\tau\tau}(\tau)+g(\tau)]d\tau$$
$$+(i+1)\eta^2(\det M)^{-1}\Big[\mu_{13}\left(\gamma J + \mathcal{O}(\eta^2[\|f_{xx}\|+\|g\|])\right) - \mu_{14}\mathcal{O}\left(\|f_{xx}\|+\|g\|\right)\Big]. \tag{3.22}$$

It is not difficult to show that μ_{13} and μ_{14} (see (3.8)) are bounded by $\mathcal{O}(\eta)$ and $\mathcal{O}(\eta^3)$ respectively. Moreover, we know from (3.19) that $(\det M)^{-1} = \mathcal{O}(\eta^{-7}e^{-\eta})$ and from (3.15) that $|W^*| = \mathcal{O}(\eta^5)$. By virtue of these arguments, the estimation (3.22) becomes

$$W = -\frac{1}{4\eta^3}\int_0^1 e^{-\eta\tau}(if_{\tau\tau}(\tau)+g(\tau))d\tau + \mathcal{O}\left(\eta^{-2}e^{-\eta}[\|f_{xx}\|+\|g\|]\right) + \gamma J\mathcal{O}(\eta^{-4}e^{-\eta}). \tag{3.23}$$

Estimation of X, Y and Z :

By (3.8), we have $X = -(i+1)\eta^2(\det M)^{-1}(\mu_{23}l_1+\mu_{24}l_2)$. One can show from (3.11), (3.12) and the expressions of μ_{23}, μ_{24} that the coefficient of the dominant term $e^{2\eta}$, appearing in $\mu_{23}l_1+\mu_{24}l_2$, is zero. Indeed, we have

$$X = (\det M)^{-1}\mathcal{O}\left(\eta^5 e^\eta[\|f_{xx}\|+\|g\|]\right) + \gamma J\mathcal{O}(\eta^3 e^\eta).$$

Using this, it is easy to check from (3.19) that

$$X = \mathcal{O}\left(\eta^{-2}[\|f_{xx}\|+\|g\|]\right) + \gamma J\mathcal{O}(\eta^{-4}). \tag{3.24}$$

Similarly, we can obtain

$$Y = \mathcal{O}\left(\eta^{-2}[\|f_{xx}\|+\|g\|]\right) + \gamma J\mathcal{O}(\eta^{-4}), \tag{3.25}$$
$$Z = \mathcal{O}\left(\eta^{-2}[\|f_{xx}\|+\|g\|]\right) + \gamma J\mathcal{O}(\eta^{-4}). \tag{3.26}$$

On the other hand, it follows from (3.7) and (3.9) that

$$\begin{array}{rcl} y_{xx}(x) & = & \hat{y}_{xx}(x) + \tilde{y}_{xx}(x) \\ & = & \dfrac{1}{4\eta} e^{\eta x} \displaystyle\int_0^1 e^{-\eta\tau}(if_{\tau\tau}(\tau) + g(\tau))d\tau + \mathcal{O}\left(\eta^{-1}[\|f_{xx}\| + \|g\|]\right) \\ & + & W\eta^2 e^{\eta x} - X\eta^2 e^{i\eta x} + Y\eta^2 e^{-\eta x} - Z\eta^2 e^{-i\eta x}. \end{array}$$

This implies together with (3.23)-(3.26) that

$$\begin{array}{rcl} \|y_{xx}\| \mid & \leq & \left\{\mathcal{O}\left(e^{-\eta}[\|f_{xx}\| + \|g\|]\right) + \gamma J\mathcal{O}(\eta^{-2}e^{-\eta})\right\} \left(\displaystyle\int_0^1 e^{2\eta x}dx\right)^{1/2} \\ & + & \left\{\mathcal{O}\left(\|f_{xx}\| + \|g\|\right) + \gamma J\mathcal{O}(\eta^{-2})\right\} \left(\displaystyle\int_0^1 e^{-2\eta x}dx\right)^{1/2} \\ & + & \mathcal{O}\left(\|f_{xx}\| + \|g\|\right) + \gamma J\mathcal{O}(\eta^{-2}). \end{array} \tag{3.27}$$

Since

$$\left(\int_0^1 e^{2\eta x}dx\right)^{\frac{1}{2}} = \mathcal{O}\left(\eta^{-\frac{1}{2}}e^{\eta}\right) \text{ and } \left(\int_0^1 e^{-2\eta x}dx\right)^{\frac{1}{2}} = \mathcal{O}\left(\eta^{-\frac{1}{2}}\right),$$

the estimate (3.27) implies

$$\|y_{xx}\| = \mathcal{O}\left(\|f_{xx}\| + \|g\| + J \mid \gamma \mid\right). \tag{3.28}$$

This proves the estimation (I_1).

Final step- Now we shall prove that

$$\|z\|^2 + J\xi^2 \leq M_2\left(\|f_{xx}\|^2 + \|g\|^2 + J\gamma^2\right).$$

Since $z = i\eta^2 y - f$ and $\xi = z_x(1) = i\eta^2 y_x(1) - f_x(1)$, it is clear from (3.28) that this estimation holds if we can find a positive constant M_2 such that

$$\eta^4\left(\|y\|^2 + J \mid y_x(1) \mid^2\right) \leq M_2\left(\|f_{xx}\|^2 + \|g\|^2 + J\gamma^2\right). \tag{I_2}$$

As in [3], let us multiply (3.2) by $\bar{y}$ (conjugate of y) and integrate by parts twice. We obtain after using the boundary conditions in (3.2) that

$$\begin{array}{l} \eta^4\left(\|y\|^2 + J \mid y_x(1) \mid^2\right) = \|y_{xx}\|^2 \\ \quad +\Re e\left\{-\displaystyle\int_0^1 (g\bar{y} + i\eta^2 f\bar{y})dx - \beta f(1)\bar{y}(1) - (iJ\eta^2 + \alpha)f_x(1)\bar{y}_x(1) - \gamma J\bar{y}_x(1)\right\}. \end{array}$$

Applying Cauchy-Schwartz inequality to all terms in braces, the above equality gives

$$\frac{\eta^4}{2}\left(\|y\|^2 + J \mid y_x(1) \mid^2\right) \leq \frac{1}{2}\left(3 + J + \alpha + \beta\right)\|y_{xx}\|^2 + \frac{1}{2}(1 + J + \beta + \alpha)\|f_{xx}\|^2$$
$$+\frac{1}{2}\|g\|^2 + \frac{J}{2}\gamma^2.$$

which implies together with (3.28) that

$$\eta^4\left(\|y\|^2 + J \mid y_x(1) \mid^2\right) \leq M_2\left(\|f_{xx}\|^2 + \|g\|^2 + J\gamma^2\right),$$

with a positive constant M_2. Thus the estimation (I_2) is proved. Combining (I_1) and (I_2), we have the estimation (I). This concludes the proof of Theorem 1. □

Let A_0 be the linear operator obtained from (2.6) by taking $\alpha = 0$ in the feedback law (2.2), i.e, there is no control moment. Then it is clear that A_0 generates a continuous semigroup of contractions $S_0(t)$. We then have the following result:

Theorem 2 . *Let $\mu > 0$ be the largest constant such that*

$$\mu\varphi_{xx}^2(1) \leq \int_0^1 \varphi_{xxxxx}^2(x)dx, \qquad (*)$$

for all $\varphi \in \{H^5(0,1);\, \varphi(0) = \varphi(1) = \varphi_x(0) = \varphi_{xxx}(1) = \varphi_{xxxx}(0) = \varphi_{xxxx}(1) = 0\}$. Assume that $3\mu EIJ^2 > \rho$. Then, the semigroup $S_0(t)$ is exponentially stable in $\mathcal{H}$.

Proof of Theorem 2. First, we write the operator A defined by (2.5)-(2.6) as the sum of a linear unbounded operator A_0 with $\mathcal{D}(A_0) = \mathcal{D}(A)$ and a bounded compact operator B on $\mathcal{H}$. In other words, for all $u = (y, z, \xi) \in \mathcal{D}(A_0) = \mathcal{D}(A)$,

$$A(y, z, \xi) = (A_0 + B)(y, z, \xi),$$

where

$$A_0(y, z, \xi) = \left(z, -\frac{E}{\rho}y_{xxxx}, -\frac{1}{J}y_{xx}(1)\right),$$

and

$$B(y, z, \xi) = \left(0, 0, -\frac{\alpha}{J}\xi\right). \qquad (3.29)$$

Second, since B is a bounded compact operator on $\mathcal{H}$ and since the semigroup $S(t)$ generated by $A = A_0 + B$ is exponentially stable (see Theorem 1), then it is sufficient for us to prove the strong stability of $S_0(t)$ in $\mathcal{H}$ (see [6] and [18]). To this end, we first establish the result for smooth initial data $u_0 = (y_0, z_0, \xi_0) \in \mathcal{D}(A^2)$ and then we can extend the result to $\mathcal{H}$ by denseness. Now let $u(t) = (y(t), z(t), \xi(t))$ be the solution stemmed from $u_0 = (y_0, z_0, \xi_0)$ and consider the Lyapunov function $V(u(t))$ defined in (2.7). Since $\alpha = 0$, we obtain from (2.8) that

$$V(u(t)) - V(u_0) = -EI \int_0^t \beta y_\tau^2(1, \tau) d\tau.$$

Next, repeating the argument used in the proof of Lemma 2, we need to show a uniqueness result, i.e, the only solution of the system below stemmed from $u_0 = (y_0, z_0, \xi_0) \in \mathcal{D}(A^2)$ is the trivial one:

$$\begin{cases} \rho y_{tt} + EI y_{xxxx} = 0, \\ y(0,t) = y_x(0,t) = 0, \\ J y_{xtt}(1,t) + y_{xx}(1,t) = 0, \\ y_{xxx}(1,t) = y_t(1,t) = 0. \end{cases} \tag{3.30}$$

For this, we will use the multiplier method as developed in [15] (p. 301-303). First multiplying the equation (3.30) by $x^2(x-3)$ and integrating by parts, we obtain:

$$\begin{aligned} 6EI \int_0^T y(1,t)dt &= \rho \int_0^1 (x^3 - 3x^2) \left[y_t(x,T) - y_t(x,0) \right] dx \\ &+ 3EIJ \left[y_{xt}(1,T) - y_{xt}(1,0) \right]. \end{aligned} \tag{3.31}$$

Since $y_t(1,t) = 0$, it follows that $y(1,t)$ is constant. Then applying the Cauchy-Schwartz inequality in (3.31) and using the fact that the semigroup is a contraction, it follows that for all $T > 0$ and $u_0 \in \mathcal{D}(A^2)$, there exists a positive constant C_1 such that $|y(1,t)| \le \frac{C_1}{T} \|u_0\|_{\mathcal{H}}$. Hence $y(1,t) = 0$. Next, multiplying (3.30) by $x y_{xxx}$ and integrating by parts, we get

$$\frac{1}{2} \int_0^T \int_0^1 \left(3\rho y_{xt}^2 + EI y_{xxx}^2 \right) dx dt = \frac{\rho}{2} \int_0^T y_{xt}^2(1,t) dt + \rho \int_0^1 \left[x y_t y_{xxx} \right]_0^T dx. \tag{3.32}$$

Since $y_{xxx}(1,t)=0$, it follows that

$$\begin{aligned}
\int_0^1 xy_t y_{xxx}dx &= \int_0^1 xy_t \int_1^x y_{\xi\xi\xi\xi}d\xi dx \\
&\leq C_2 \int_0^1 \left(y_t^2 + y_{xxxx}^2\right) dx \\
&\leq C_2 \|u_0\|^2_{\mathcal{D}(A)}. \qquad (3.33)
\end{aligned}$$

Inserting (3.33) into (3.32) yields:

$$\int_0^T \int_0^1 \left(3\rho y_{xt}^2 + EIy_{xxx}^2\right) dxdt \leq \rho \int_0^T y_{xt}^2(1,t)dt + C_3\|u_0\|^2_{\mathcal{D}(A)}. \qquad (3.34)$$

Now, replacing y by y_t in (3.34) and using (3.30), we obtain:

$$\begin{aligned}
\int_0^T \int_0^1 \left(3EIy_{xxxxx}^2 + EIy_{xxxt}^2\right) dxdt &\leq \rho \int_0^T y_{xtt}^2(1,t)dt + C_3\|u_0\|^2_{\mathcal{D}(A^2)} \\
&\leq \frac{\rho}{J^2} \int_0^T y_{xx}^2(1,t)dt + C_3\|u_0\|^2_{\mathcal{D}(A^2)} \qquad (3.35)
\end{aligned}$$

Let us observe that $u_0 \in \mathcal{D}(A^2)$ implies that the corresponding solution y of (3.30) satisfies $y(0,t) = y(1,t) = y_x(0,t) = y_{xxx}(1,t) = y_{xxxx}(0,t) = y_{xxxx}(1,t)$. We deduce thus from (3.35) and the hypothesis $(*)$ of Theorem 2 that for $3\mu EIJ^2 > \rho$, we have

$$\int_0^T \int_0^1 y_{xxxt}^2 dt \leq C_3\|u_0\|^2_{\mathcal{D}(A^2)}. \qquad (3.36)$$

In addition, the conditions $y(0,t) = y_x(0,t) = y(1,t) = 0$ give us:

$$\begin{cases} y_{xt}(1,t) = \displaystyle\int_0^1 (1-x)xy_{xxxt}dx, \\ y_{xx}(1,t) = \displaystyle\int_0^1 x(2-x)xy_{xxx}dx. \end{cases}$$

Thus, by means of Cauchy-Schwartz inequality, we have

$$\begin{cases} y_{xt}^2(1,t) \leq C_4 \displaystyle\int_0^1 y_{xxxt}^2(x,t)dx, \\ y_{xx}^2(1,t) \leq C_5 \displaystyle\int_0^1 y_{xxx}^2(x,t)dx, \end{cases} \qquad (3.37)$$

which, together with (3.36), implies that

$$\int_0^T y_{xt}^2(1,t)dt \leq C_6\|u_0\|^2_{\mathcal{D}(A^2)}. \tag{3.38}$$

Combining (3.38) and (3.34), we obtain

$$\int_0^T \int_0^1 \left(3\rho y_{xt}^2 + EIy_{xxx}^2\right) dxdt \leq C_7\|u_0\|^2_{\mathcal{D}(A^2)}. \tag{3.39}$$

Because $y(0,t) = 0$, we obtain after using the Cauchy-Schwartz inequality that

$$\int_0^1 y_t^2 dx \leq \int_0^1 y_{xt}^2 dx. \tag{3.40}$$

Moreover, using Cauchy-Schwartz inequality and the second inequality of (3.37), a direct computation shows that

$$\begin{aligned} \int_0^1 y_{xx}^2(x,t)dx &= \int_0^1 \left(\int_1^x y_{\xi\xi\xi}(\xi,t)d\xi + y_{\xi\xi}(1,t)\right)^2 dx \\ &\leq 2|y_{\xi\xi}(1,t)|^2 + 2\int_0^1 y_{xxx}^2(x,t)dx \\ &\leq C_8 \int_0^1 y_{xxx}^2(x,t)dx. \end{aligned} \tag{3.41}$$

Finally, putting (3.38), (3.40) and (3.41) into (3.39), we obtain

$$\int_0^T \|u(t)\|^2_{\mathcal{H}} dt = \int_0^T \left(\int_0^1 \left(y_t^2 + y_{xx}^2\right) dx + y_{xt}^2(1,t)\right) dt \leq C_9\|u_0\|^2_{\mathcal{D}(A^2)}.$$

This means, since $\|u(t)\|^2_{\mathcal{H}}$ is constant, that

$$\|u(t)\|^2_{\mathcal{H}} \leq \frac{C_9\|u_0\|^2_{\mathcal{D}(A^2)}}{T}, \text{ for all } T > 0,$$

which gives that $y = 0$. We have proved that if $3\mu EIJ^2 > \rho$, then for any $u_0 = (u(\cdot,0), u_t(\cdot,0), u_{xt}(1,0)) \in \mathcal{D}(A_0^2)$, the corresponding solution of (3.30) is trivial. This completes the proof of Theorem 2. $\square$

Remark 2 . *It is easy to verify that the five boundary conditions of Theorem 2 yield*

$$y_{xx}(1,t) = \frac{1}{3}\int_0^1 \left(-\frac{1}{4}x^4 + x^3\right) y_{xxxxxx}(x,t)dx.$$

Then, the Cauchy-Schwartz inequality implies that

$$\begin{aligned} y_{xx}^2(1,t) &\leq \frac{1}{9}\int_0^1 x^6\left(-\frac{x}{4}+1\right)^2 dx \int_0^1 y_{xxxxxx}^2(x,t)dx \\ &\leq \frac{11}{1134}\int_0^1 y_{xxxxxx}^2(x,t)dx. \end{aligned}$$

Hence we can claim that $\mu \geq \dfrac{1134}{11}$ *in our Theorem 2.*

3.2 Lack of uniform stabilization

In order to achieve our investigation concerning stabilization problem of system (2.1), it remains to see whether the system is uniformly stabilizable by control moment. For this, assume that only the moment control is present in the feedback law (2.2), i.e, $\beta = 0$ and $\alpha > 0$. Hence the closed loop system is:

$$\begin{cases} \rho y_{tt} + EIy_{xxxx} = 0, \\ y(0,t) = y_x(0,t) = 0, \\ Jy_{xtt}(1,t) + y_{xx}(1,t) = -\alpha y_{xt}(1,t), \\ y_{xxx}(1,t) = 0, \\ y(\cdot,0) = y_0,\ y_t(\cdot,0) = z_0,\ y_t(1,0) = \xi_0, \end{cases} \tag{3.42}$$

which can be written as follows

$$\begin{cases} \dot{u}(t) = (A_1 + B)u(t), \\ u(0) = u_0, \end{cases} \tag{3.43}$$

where $u_0 = (y_0, z_0, \xi_0)$; B is given by (3.29) and A_1 is defined by:

$$\mathcal{D}(A_1) = \{(y,z,\xi) \in H_0^4 \times H_0^2 \times \mathbb{R};\ y_{xxx}(1) = 0, \xi = z_x(1)\}, \tag{3.44}$$

and for all $(y,z,\xi) \in \mathcal{D}(A_1)$,

$$A_1(y,z,\xi) = \left(z, -\frac{EI}{\rho}y_{xxxx}, -\frac{1}{J}y_{xx}(1)\right). \tag{3.45}$$

We first state a lemma which concerns the properties of the operator A_1.

Lemma 4 . *The operator A_1 defined by (3.44)-(3.45) generates an unitary group $S_1(t)$ of isometries.*

Proof of Lemma 4. Obviously, A_1 is maximal and $< A_1u, u >= 0,\ \forall u \in \mathcal{D}(A_1)$. This implies that $< A_1(u+v), (u+v) >= 0,\ \forall u,\ v \in \mathcal{D}(A_1)$, and $< A_1u, v >= - < u, A_1v >,\ \forall u,\ v \in \mathcal{D}(A_1)$. Thus the operator A_1 is antisymmetric. On the other hand, one can prove that $\mathcal{D}(A_1) = \mathcal{D}(A_1^*)$ and $A_1^* = -A_1$. Hence, A_1 is skew-adjoint. Finally, it follows that A_1 generates an unitary group $S_1(t)$ of isometries [13]. $\square$

Now, we are ready to state a result which highlights the leading role of control force.

Proposition 1 . *The closed-loop system (3.42) is not uniformly stable.*

Proof of Proposition 1. As in [14], the proof is based on the result of compact perturbations [16]. Let $S_{A_1+B}(t)$ and $S_{A_1-B}(t)$ denote the groups generated by A_1+B and A_1-B respectively. Let $u(t) = (y, z, \xi)(t)$ be the solution of (3.43) stemmed from $u_0 \in \mathcal{D}(A_1)$, i.e, $u(t) = S_{A_1+B}(t)u_0$. Define $\tilde{u}(t) = (\tilde{y}(t), \tilde{z}(t), \tilde{\xi}(t))$ where

$$\tilde{y}(x,t) = y(x,-t),\ \tilde{z}(x,t) = -z(x,-t),\ \tilde{\xi}(t) = -\xi(t).$$

It can be verified that $\tilde{u}(t) = S_{A_1-B}(t)\tilde{u}(0)$. Since B is compact, Russell's result [16] permits us to claim that there is no positive constants $0 \le \mu < 1$ and $T > 0$ such that we have simultaneously

$$||S_{A_1+B}(t)|| \le \mu \quad \text{and} \quad ||S_{A_1-B}(-t)|| \le \mu,\ \forall t \ge T. \tag{3.46}$$

Since $||\tilde{u}(-t)|| = ||\tilde{u}(t)||$, for all $t \in \mathbb{R}$, we get $||S_{A_1+B}(t)|| = ||S_{A_1-B}(-t)||$. This, together with (3.46), completes the proof of Proposition 1. $\square$

4 Asymptotic analysis of the real part of the spectrum

Let us consider the closed loop system (2.4). Using the results of Langer [9], we can estimate the real part of the spectrum with large module. The following theorem establishes explicitly a relation between the feedback gain β and the real part of the eigenvalues λ_k.

Theorem 3 . *Let λ_k be an eigenvalue of system (2.4). Then*

$$\Re e\lambda_k \longrightarrow -\beta\frac{EI}{\rho}, \qquad as\ k \longrightarrow \infty.$$

Proof of Theorem 3. First one can readily show that $\lambda \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of the operator A if and only if there exists a non trivial element $y \in H^4(0,1)$ such that

$$\left\{\begin{array}{l} \dfrac{EI}{\rho}y_{xxxx} + \lambda^2 y = 0, \\ y(0) = y_x(0) = 0, \\ y_{xx}(1) + (\alpha\lambda + J\lambda^2)y_x(1) = 0, \\ \beta\lambda y(1) - y_{xxx}(1) = 0. \end{array}\right. \tag{4.1}$$

Substituting $\lambda = i\omega^2\sqrt{\dfrac{EI}{\rho}}$, $(\omega \in \mathbb{C} \setminus \{0\})$, $\tilde{\beta} = \beta\sqrt{\dfrac{EI}{\rho}}$, $\tilde{\alpha} = \alpha\sqrt{\dfrac{EI}{\rho}}$ and $\tilde{J} = J\dfrac{EI}{\rho}$ into (4.1), we obtain

$$\left\{\begin{array}{l} y_{xxxx} - \omega^4 y = 0, \\ y(0) = y_x(0) = 0, \\ y_{xx}(1) + (i\tilde{\alpha}\omega^2 - \tilde{J}\omega^4)y_x(1) = 0, \\ i\tilde{\beta}\omega^2 y(1) - y_{xxx}(1) = 0. \end{array}\right.$$

In the same way as for (3.2), one can verify that the system below has a non trivial solution if and only if ω satisfies

$$\begin{array}{l} P(\omega) = e^{(1-i)\omega}[(i-1)\tilde{J}\omega^4 + \tilde{\beta}\tilde{J}\omega^3 + (i+1)\tilde{\alpha}\omega^2 - i(1+\tilde{\alpha}\tilde{\beta})\omega + (i-1)\tilde{\beta}] \\ +e^{(i-1)\omega}[(1-i)\tilde{J}\omega^4 + \tilde{\beta}\tilde{J}\omega^3 - (i+1)\tilde{\alpha}\omega^2 - i(1+\tilde{\alpha}\tilde{\beta})\omega + (1-i)\tilde{\beta}] \\ +e^{(i+1)\omega}[(1+i)\tilde{J}\omega^4 + \tilde{\beta}\tilde{J}\omega^3 + (-i+1)\tilde{\alpha}\omega^2 - i(1+\tilde{\alpha}\tilde{\beta})\omega - (i+1)\tilde{\beta}] \\ +e^{-(i+1)\omega}[-(1+i)\tilde{J}\omega^4 + \tilde{\beta}\tilde{J}\omega^3 + (i-1)\tilde{\alpha}\omega^2 - i(1+\tilde{\alpha}\tilde{\beta})\omega + (i+1)\tilde{\beta}] \\ -4(\tilde{\beta}\tilde{J}\omega^3 + i(1-\tilde{\alpha}\tilde{\beta})\omega) = 0. \end{array} \tag{4.2}$$

A straightforward computation shows that $P(-\omega) = -P(\omega)$ and $iP(\omega) = \overline{P(i\bar{\omega})}$. Thus the solution set of (4.2) admits the two bisectors of the complex plan as symmetrical axes and the origin as a symmetrical center. It is sufficient then to seek the solutions of (4.2) in a sector S_1 defined by $\arg(\omega) = \theta$ with $\epsilon < \theta < \pi/2 + \epsilon$ for small ϵ. Next, following the same step as in [9] , we claim that in the sector S_1, the solutions of (4.2) are asymptotically represented by those of

$$\hat{P}(\omega) = e^{\omega} A_1(\omega) + e^{-\omega} A_2(\omega) = 0, \tag{4.3}$$

where

$$\begin{aligned} A_1(\omega) &= (i-1)\tilde{J}\omega^4 + \tilde{\beta}\tilde{J}\omega^3 + (i+1)\tilde{\alpha}\omega^2 - i(1+\tilde{\alpha}\tilde{\beta})\omega + (i-1)\tilde{\beta}, \\ A_2(\omega) &= -(1+i)\tilde{J}\omega^4 + \tilde{\beta}\tilde{J}\omega^3 + (i-1)\tilde{\alpha}\omega^2 - i(1+\tilde{\alpha}\tilde{\beta})\omega + (i+1)\tilde{\beta}. \end{aligned}$$

Because of $\tilde{J} = J\dfrac{EI}{\rho} > 0$, the solutions of (4.3) are themselves asymptotic to the solutions of

$$(i-1)e^{\omega} - (i+1)e^{-\omega} = 0. \tag{4.4}$$

Obviously, the solutions of (4.4) are given by

$$\omega_k = i\pi\left(\frac{3}{4} + k\right), \quad k \in \mathbb{N}.$$

Now, we seek ω_k as follows

$$\omega_k = i\pi\left(\frac{3}{4} + k\right) + \frac{z_1}{k} + \frac{z_2}{k^2} + \frac{z_3}{k^3} + \frac{z_4}{k^4} + \mathcal{O}\left(\frac{1}{k^5}\right), \tag{4.5}$$

where $z_i,\ i = 1, 2, 3, 4$ are complex constants.
Next, by using (4.5), we calculate the development of e^{ω}, $e^{-\omega}$ and the expression of $\hat{P}(\omega)$ defined in (4.3). We obtain after a careful computation that

$$\begin{aligned} \hat{P}(\omega) \;=\; & (-1)^k \frac{\sqrt{2}}{2}\Bigg\{ 2i\tilde{J}\pi^3(\tilde{\beta} - 2\pi z_1)k^3 + \tilde{J}\pi^3\left[2(\tilde{\beta} - 6i\pi)z_1 - 4i\pi z_2 + \frac{9}{2}i\tilde{\beta}\right]k^2 \\ & + k\Big[(-6\tilde{\beta}\tilde{J} + 4\tilde{\alpha} - 16\pi + i\tilde{\beta}\tilde{J}\pi z_1)\pi^2 z_1 + 2(\tilde{\beta} - 6i\pi)\tilde{J}\pi^3 z_2 - 4i\tilde{J}\pi^4 z_3 \\ & -\frac{9}{8}i\tilde{\beta}\tilde{J}\pi^3 - \frac{27}{2}i\tilde{J}\pi^4 - 2\pi(1+\tilde{\alpha}\tilde{\beta})\Big] + Z + \mathcal{O}\left(\frac{1}{k}\right)\Bigg\}, \end{aligned}$$

where Z is a complex number depending on z_i, $i = 1, 2, 3, 4$. Finally, we choose the complex z_i such that each coefficient of k^{4-i} is zero for $i = 1, 2, 3, 4$. Next, we replace the obtained values in (4.5) to get:

$$\omega_k = i\pi\left(\frac{3}{4} + k\right) + \frac{\tilde{\beta}}{2\pi k} + \frac{\tilde{\beta}}{4i\pi}\left(\frac{\tilde{\beta}}{\pi} - \frac{3}{2}i\right)\frac{1}{k^2} + \cdots .$$

Since $\lambda_k = i\omega^2\sqrt{\frac{EI}{\rho}}$, we conclude that $\Re e\lambda_k = \Re e\left(i\omega^2\sqrt{\frac{EI}{\rho}}\right) \longrightarrow -\beta\frac{EI}{\rho}$, as $k \longrightarrow \infty$. This completes the proof of Theorem 3. $\square$

As mentioned in Remark 2, assume that the condition $(*)$ of Theorem 2 is satisfied with $\mu = \frac{1134}{11}$. Recall that if the feedback law consists of only a control force ($\alpha = 0$) then the hybrid system (2.1) is governed by the operator A_0 which generates a semigroup $S_0(t)$. An important consequence of the previous spectral analysis is the following proposition:

Proposition 2 . *Assume that $3\mu EIJ^2 \leq \rho$. Then, the semigroup $S_0(t)$ is not strongly stable in $\mathcal{H}$.*

Proof of Proposition 2. Consider the following physical parameters:

$$J = \frac{\sinh\pi}{(1 + \cosh\pi)\pi^3},\ EI = \rho = \beta = 1.$$

Then it is easy to check that $3\mu EIJ^2 < \rho$ where $\mu = \frac{1134}{11}$. Since $EI = \rho = 1$, we can claim from the proof of Theorem 3 that $\lambda_k = i\omega^2$ is an eigenvalue of A_0 of our hybrid system if and only if ω satisfies equation (4.2) with $\tilde{\alpha} = 0$ and $\tilde{\beta} = \beta$, $\tilde{J} = J$ (recall that $\tilde{\beta} = \beta\sqrt{\frac{EI}{\rho}}$, $\tilde{\alpha} = \alpha\sqrt{\frac{EI}{\rho}}$ and $\tilde{J} = J\frac{EI}{\rho}$). This gives after straightforward computation that ω is solution of the following equation:

$$\begin{aligned} &e^{\omega}\Big[\cos\omega\,(2J\omega^3 - 2) + 2\sin\omega + i\left\{\cos\omega\,(2J\omega^4 - 2\omega) + 2J\omega^4\sin\omega\right\}\Big] \\ &+e^{-\omega}\Big[\cos\omega\,(2J\omega^3 + 2) + 2\sin\omega + i\left\{\cos\omega\,(-2J\omega^4 - 2\omega) + 2J\omega^4\sin\omega\right\}\Big] \\ &-4J\omega^3 - 4i\omega = 0. \end{aligned} \tag{4.6}$$

Since $J = \frac{\sinh\pi}{(1+\cosh\pi)\pi^3}$, one can check that $\omega = \pi$ is solution of (4.6) and thus $\lambda = i\omega^2 = i\pi^2$ is an eigenvalue of A_0. We conclude that the semigroup $S_0(t)$ is not stable. $\square$

Numerical results: To see how different the eigenvalues are for the values of β and α, we shall do some computer simulations. First we know from the previous spectral study that λ is an eigenvalue if and only if ω is solution of (4.2) where λ and ω are related by $\lambda = i\omega^2\sqrt{\frac{EI}{\rho}}$. Let $\omega = x + iy$, with x and y being real numbers. Substituting ω into $P(\omega) = 0$ (see (4.2)), it follows that (x, y) is solution of the equation $R(x, y) + iI(x, y) = 0$, where R and I are two real **explicit** functions. Obviously, λ is an eigenvalue if and only if $R \equiv I \equiv 0$. Using MAPLE V, we find the points of intersection (x, y) of the graphs of $R = 0$ and $I = 0$. Once the values of $\omega = x + iy$ are obtained, we can recover the eigenvalues λ by using the relationship $\lambda = i\omega^2\sqrt{\frac{EI}{\rho}} = \sqrt{\frac{EI}{\rho}}\Big(- 2xy + i(x^2 - y^2)\Big)$. We consider three cases:

Case 1: $\boldsymbol{\alpha > 0,\ \beta > 0}$.

We observe that the real part of the spectrum approaches the vertical axis $-\beta\frac{EI}{\rho}$ (see fig. 2, fig.3,...fig.9). This confirms the result obtained in Theorem 3. It is also interesting to note that if we fix $\alpha = J = 1,\ EI = 9,\ \rho = 4$ and we vary β then we notice two facts: On one hand, the more β increases, for $\beta > 1$, the more the spectrum moves towards the left. On the other hand, the more β decreases, for $\beta < 1$, the more the spectrum moves towards the right (fig. 2,...fig.6). In return, if we fix $\beta = J = 1,\ EI = 9,\ \rho = 4$ and α vary, then we notice that the spectrum is "stable" (fig. 2, fig.7, fig.8, fig.9). This means that the influence of β is numerically dominant.

Case 2: $\boldsymbol{\alpha = 0,\ \beta > 0}$ and $\boldsymbol{3\mu EIJ^2 > \rho}$.

As mentioned in Remark 2, assume that $\mu = \frac{1134}{11}$. Then, we observe that the vertical axis $-\beta\frac{EI}{\rho}$ is still an asymptote for the real part of the spectrum (see fig. 10, fig.11, fig.12). This illustrates the result of Theorem 3.

Case 3: $\boldsymbol{\alpha = 0,\ \beta > 0}$ and $\boldsymbol{3\mu EIJ^2 < \rho}$.

As in the previous case, let $\mu = \frac{1134}{11}$. Clearly, there exists a pure imaginary eigenvalue (see fig. 13 and fig. 14) which means that the system is not even strongly stable as proved in Proposition 2.

5 Conclusion

In this article, we considered the stabilization problem of a variant of the SCOLE model. We have proposed a simple and usual stabilizing feedback law composed of control force and control moment. Then the system is uniformly stabilized by the proposed feedback law as soon as the boundary control force and control moment are present in the feedback. Nevertheless, if only the control force is applied, then the exponential stability is also obtained provided the physical parameters of the system satisfy a certain condition (see Theorem 2). We have shown that this condition is necessary to obtain uniform stabilization. An asymptotic analysis of the real part of the spectrum was also done. This spectral study establishes explicitly a relationship between the feedback gain of the control force and the eigenvalues. The theoretical results obtained in this paper have been illustrated by several numerical results.

Acknowledgements: The author expresses his Gratitude to Professor B. Rao for calling his attention to the hybrid system of the SCOLE model; and to C. Z. Xu for the helpful discussions on the numerical study of spectrum.

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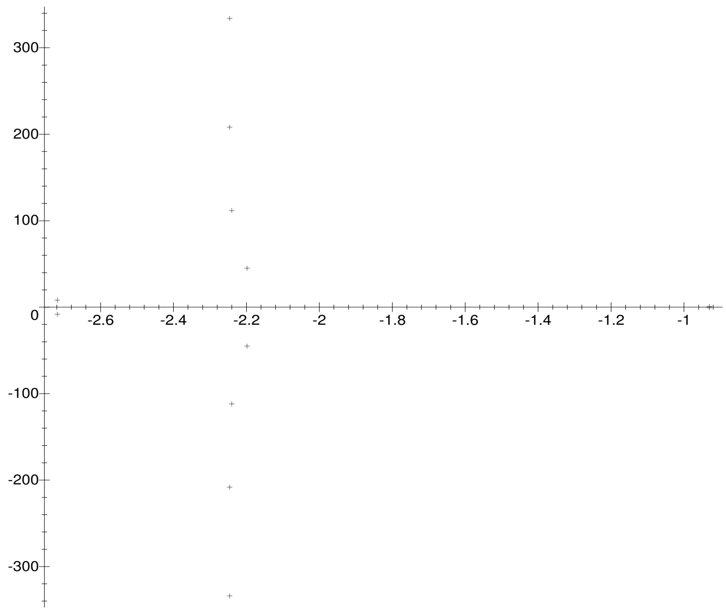

Figure 2: Spectrum for $J = 1,\ \alpha = 1,\ \beta = 1,\ EI = 9,\ \rho = 4$

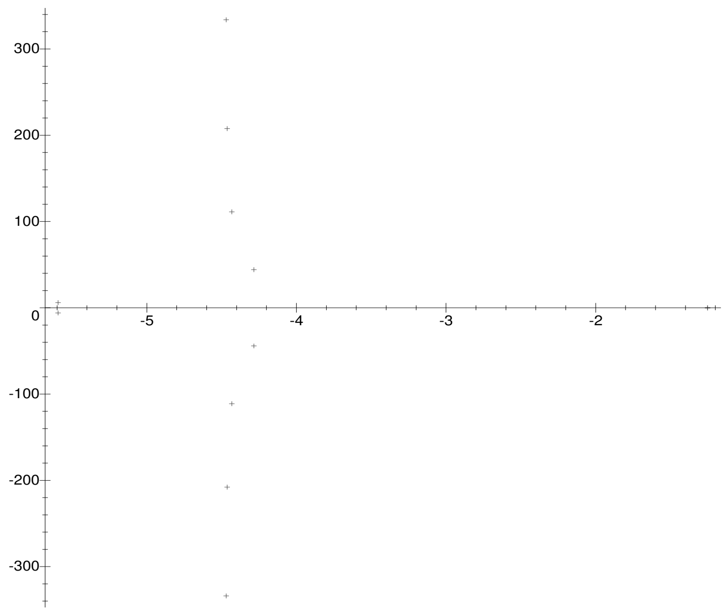

Figure 3: Spectrum for $J = 1,\ \alpha = 1,\ \beta = 2,\ EI = 9,\ \rho = 4$

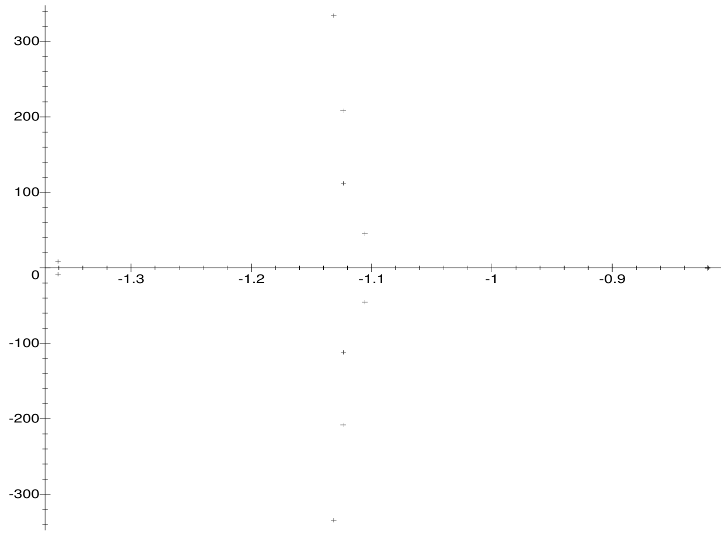

Figure 4: Spectrum for $J = 1,\ \alpha = 1,\ \beta = 0.5,\ EI = 9,\ \rho = 4$

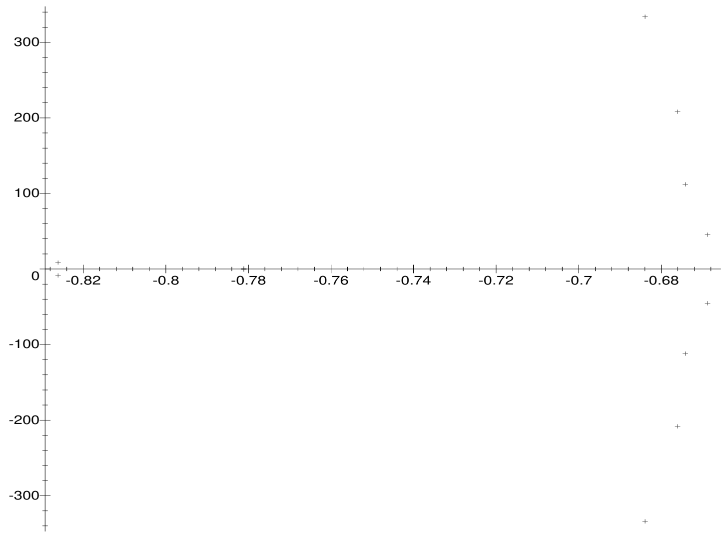

Figure 5: Spectrum for $J = 1,\ \alpha = 1,\ \beta = 0.3,\ EI = 9,\ \rho = 4$

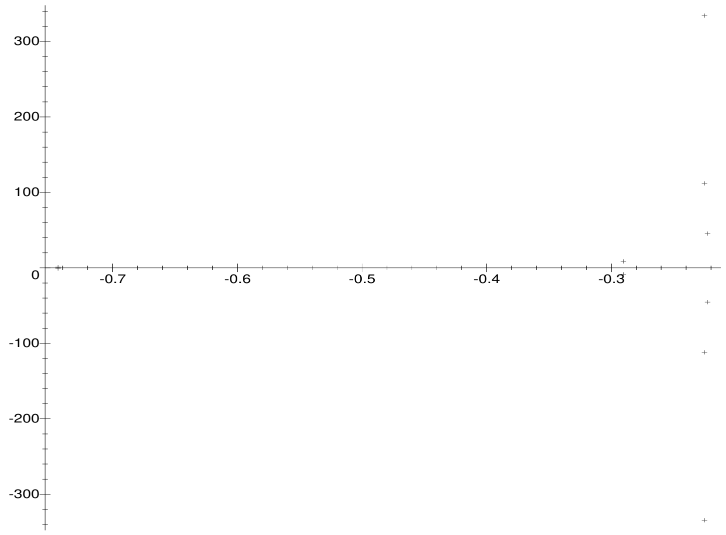

Figure 6: Spectrum for $J = 1,\ \alpha = 1,\ \beta = 0.1,\ EI = 9,\ \rho = 4$

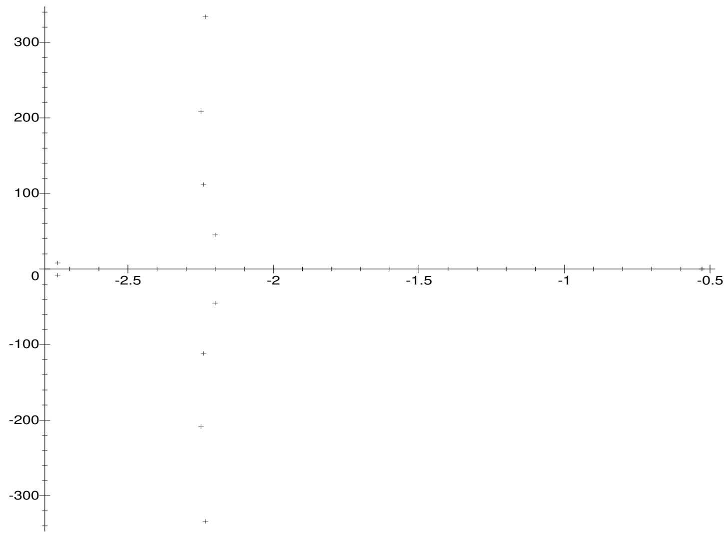

Figure 7: Spectrum for $J = 1,\ \alpha = 2,\ \beta = 1,\ EI = 9,\ \rho = 4$

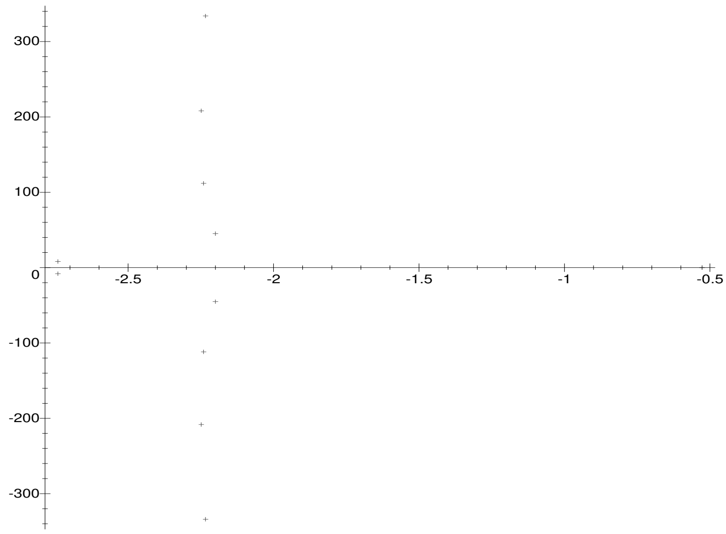

Figure 8: Spectrum for $J = 1,\ \alpha = 1.3,\ \beta = 1,\ EI = 9,\ \rho = 4$

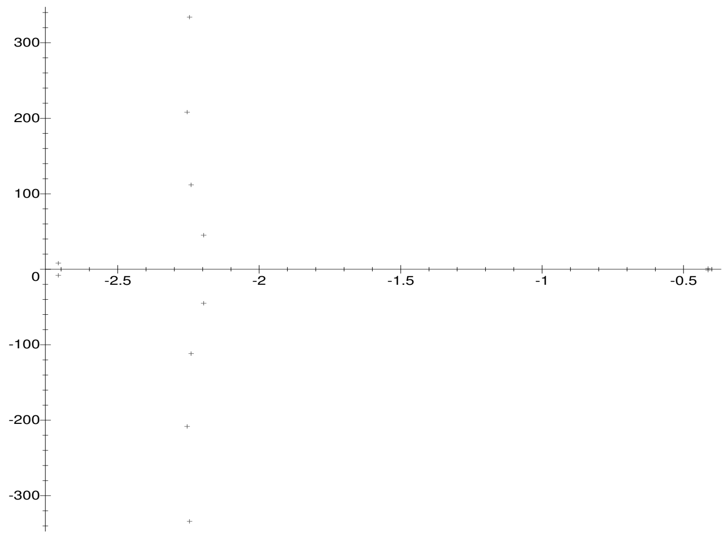

Figure 9: Spectrum for $J = 1,\ \alpha = 0.3,\ \beta = 1,\ EI = 9,\ \rho = 4$

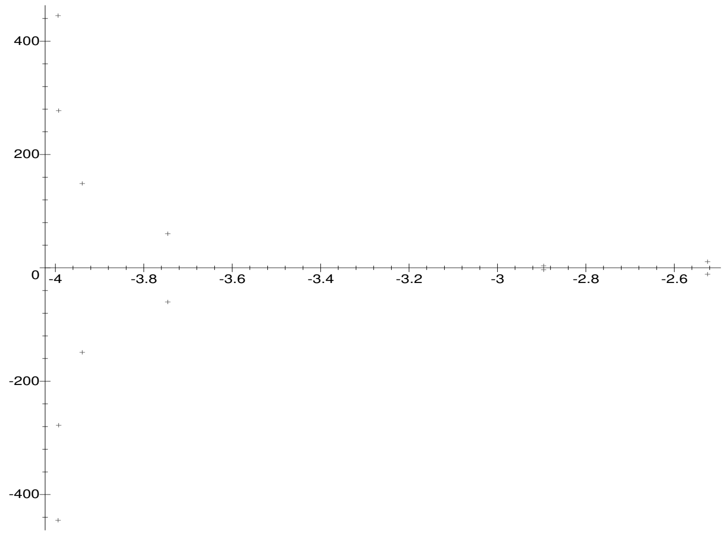

Figure 10: Spectrum for $J = 0.1,\ \alpha = 0,\ \beta = 1,\ EI = 8,\ \rho = 2$

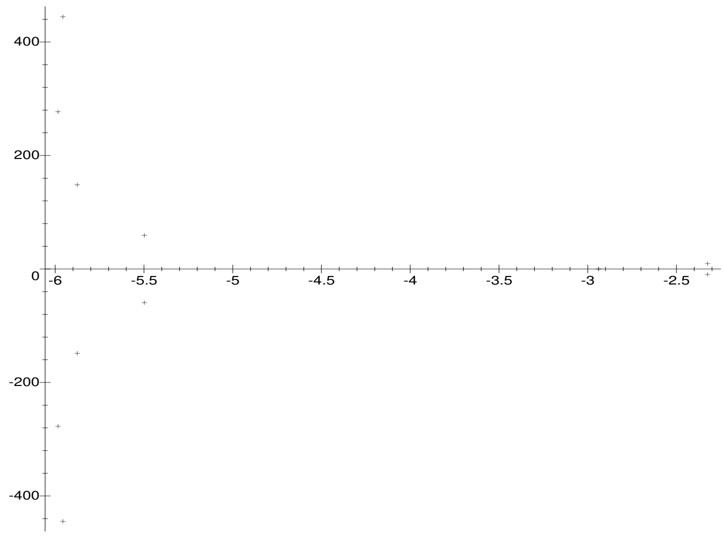

Figure 11: Spectrum for $J = 0.1,\ \alpha = 0,\ \beta = 1.5,\ EI = 8,\ \rho = 2$

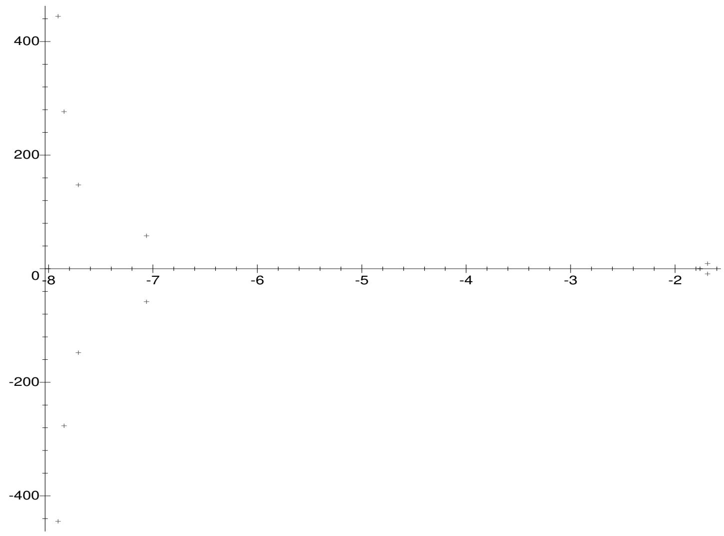

Figure 12: Spectrum for $J = 0.1,\ \alpha = 0,\ \beta = 2,\ EI = 8,\ \rho = 2$

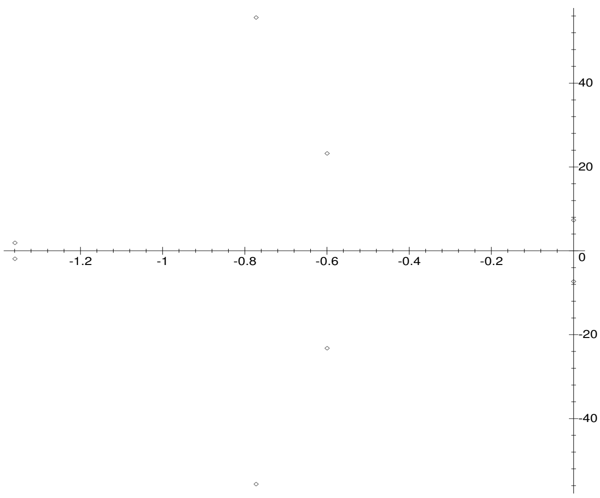

Figure 13: Spectrum for $J = 0.0399,\ \alpha = 0,\ \beta = 1.5,\ EI = 1.1,\ \rho = 2$

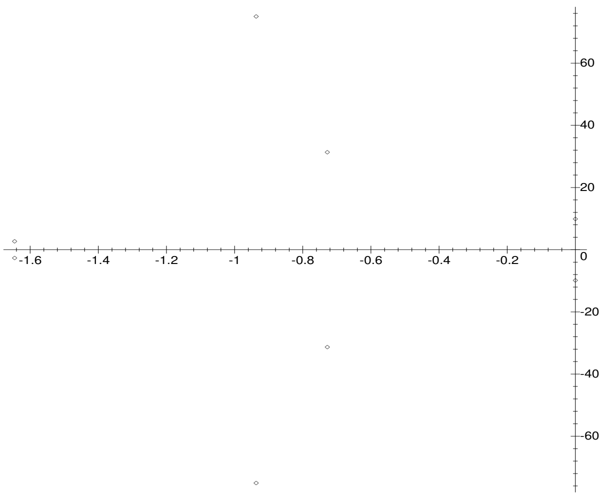

Figure 14: Spectrum for $J = \frac{\sinh \pi}{(1+\cosh \pi)\pi^3},\ \alpha = 0,\ \beta = 1,\ EI = 1,\ \rho = 1$

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 1, July 1, 2002 , pp. 71–80.
Submitted: January 15, 2002. Published: July 1, 2002.

ON FOURIER COEFFICIENTS AND FACTORABLE OPERATORS

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Abstract. In this paper we combine the new interpolation method with the well-known results of convolution operator and we give a good estimate for the Fourier coefficients of a periodic fuctions f belonging to $L_\Phi(2\pi), L_{\Phi q}(2\pi),\ W_\Phi^q(2\pi)$ and $B_{\Phi q}^s(2\pi)$. Further,we show that the Fourier coefficients belong to some Lorentz-Orlicz space $l_{\Psi,q}$.

It is one of the aims of this paper to give a discription of the Fourier coefficients with some applications in the convolution operators,we refer to the books written by A.Pietsch [15] and H.König [10]. Also, we give estimates of entropy numbers of diagonal operators acting between Lorentz-Orlicz spaces which is a consequence of the paper by B.Carl [2]. Applying these results to foctorable operators acting between arbitrary Banach spaces, we get information about the distribution of eigenvalues.

1991 A.M.S. (MOS) Subject Classification Codes. 46,47.

Key Words and Phrases. Convolution operator, Fourier coefficients, Banach space, Hausdorff-Young inequality, Besov spaces.

1. Fourier coefficients.

In this section, we introduce the necessary definitions and we collect a number of basic results. In many cases,the optimality of the eigenvalues estimates follows considering convolution operator for which the Fourier coefficients are eigenvalues. The fact that the fourier coefficients of a periodic function f coincid up to the factor 2π with the eigenvalues of the associated convolution operator was realized rather late.The first remark concerning due to Carleman [2].

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Remark 1.1. *Every function $f \in L_1(2\pi)$ induces a convolution operator C_f defined by*

$$C_f g(x) \;=\; \int_0^{2\pi} f(y-x)g(x)dx \tag{1}$$

on $C(2\pi)$ and $L_p(2\pi)$ with $1 \le p \le \infty$ (see [7] and [17]).

Proposition 1.2. *Let $f \in L_{r,w}(2\pi)$. Then the associated convolution operator C_f acts from $L_{r',w'}(2\pi)$ into $L_\infty(2\pi)$.*

Proof.

Given $g \in L_{r',w'}(2\pi)$, by Hölder's inequality,for almost all $x \in \mathbb{R}$ we have

$$|\int_0^{2\pi} f(x-y)g(y)dy| \le |\int_0^{2\pi} f^*(t)g^*(t)dt| \le |\int_0^{2\pi} t^{\frac{1}{r}-\frac{1}{w}} f^*(t) t^{\frac{1}{r'}-\frac{1}{w'}} g^*(t)dt|$$

$$\le \{\int_0^{2\pi} [t^{\frac{1}{r}-\frac{1}{w}} f^*(t)]^w dt\}^{\frac{1}{w}} \{\int_0^{2\pi} [t^{\frac{1}{r'}-\frac{1}{w'}} g^*(t)]^{w'} dt\}^{\frac{1}{w'}}.$$

This prove that $\|C_f g\|_{L_\infty} \le \|f\|_{L_{r,w}} \|g\|_{L_{r',w'}}$.

Definition 1.3. *For every function $f \in L_1(2\pi)$ and $k \in \mathbb{Z}$ the k-th Fourier coefficient is defined by*

$$\nu_k(f) \;:=\; \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx \quad . \tag{2}$$

Proposition 1.4. *(T.Carleman 1918) Let $f \in L_1(2\pi)$.The sequence of eigenvalues of C_f can be obtained as the non-increasing rearrangement of $(2\pi\nu_k(f))$.*

Proof.

Define $e_k(x) := e^{ikx}$ for all $x \in \mathbb{Z}$. Then it follows from

$$\int_0^{2\pi} f(x-y)e_k(y)dy \;=\; \int_0^{2\pi} f(y)e^{ik(x-y)}dy \;=\; 2\pi\nu_k(f)e_k(x)$$

that $2\pi\nu_k(f)$ is an eigenvalue of C_f with the associated eigenfuction e_k. Next,if λ denotes an arbitrary eigenvalue of C_f then for any associated eigenfunction $g \in L_1(2\pi)$ we have

$$\begin{aligned}\lambda(g,e_k) \;=\; (C_f g, e_k) \;&=\; \int_0^{2\pi}\int_0^{2\pi} f(x-y)g(y)e^{-ikx}dydx \\ &=\; \int_0^{2\pi}\int_0^{2\pi} f(x)g(y)e^{-ik(x+y)}dxdy \\ &=\; (f,e_k)(g,e_k) \;=\; 2\pi\nu_k(f)(g,e_k).\end{aligned}$$

Hence $\lambda \ne 2\pi\nu_k(f)$ implies $(g,e_k) = 0$.Since $g \ne 0$,the eigenvalue must coincide with at least one of the coefficients $2\pi\nu_k(f)$.

Theorem 1.5. *(Hausdorff-Young inequality)[1],[17]. Let $2 \leq p < \infty$ and $f \in L_{p'}(2\pi)$. Then the Fourier coefficients*

$$(\nu_n(f))_{n\in\mathbb{Z}} \in l_p(\mathbb{Z}) \quad with \quad |(\nu_n(f))\|_{l_p} \leq \|f\|_{L_{p'}}.$$

Remark 1.6. *[10] The proof of the Haosdorff-Young inequality is usually accomplished by using the complex interpolation method (cf. [17]).The proof in König (cf.3a.4 [10]) does not use any interpolation theory.Using real interpolation theory one can slightly improve the following corollary:*

Let $2 < p < \infty$ and $f \in L_{p'}(2\pi)$. Then the Fourier coefficients $(\nu_n(f))_{n\in\mathbb{Z}} \in l_{p,p'}(\mathbb{Z})$ and with some constant $c_p > 0$ depending only on p we have

$$\|(\nu_n(f))\|_{l_{p,p'}} \leq c_p\|f\|_{L_{p'}}.$$

Note that $l_{p,p'}$ is a slightly smaller Lorentz sequence space than $l_p = l_{p,p}$ since $p' < 2 < p$.

Theorem 1.7. *(W.H.Young 1912 ; F.Hausdorff 1923 ; G.Hardy , J.E. Littlewood 1931) Let $1 < r < 2$ and $1 \leq w \leq \infty$. Then*

$$f \in L_{r,w}(2\pi) \quad implies \quad (\nu_n(f)) \in l_{r',w}(\mathbb{Z}). \tag{3}$$

Theorem 1.8. *(A.Pietsch 1980) Let $\frac{1}{s} = \lambda + \frac{1}{r^+}$ where $\frac{1}{r^+} = \frac{1}{\max(r',2)}$. Then*

$$f \in B^{\lambda}_{rw}(2\pi) \quad implies \quad (\nu_n(f)) \in l_{s,w}(\mathbb{Z}). \tag{4}$$

Theorem 1.9. *[16] Let $1 < r < \infty, \frac{1}{s} = l + \frac{1}{r^+}$ and $w := \min(2,r)$. Then*

$$f \in W^l_r(2\pi) \quad implies \quad (\nu_n(f)) \in l_{s,w}(\mathbb{Z}). \tag{5}$$

2. Entropy numbers of diagonal operators .

In the last year several results on the beheviour of entropy numbers of diagonal operators acting between l_p-spaces as well as embedding maps between Besov spaces have been derived. In the following ,we deal with diagonal operators $D \in L(l_p, l_q)$ defined by $D(\xi_i) := (\sigma_i\xi_i)$ and generated by a sequence (σ_i) belongs to Lorentz space $l_{r,t}$.

Definition 2.1. *Let S be a bounded linear operator acting from E into F (i.e. $S \in L(E,F)$. Then the n-th entropy number $e_n(S)$ is defined to be the infimum of all $\varepsilon \geq 0$ such that there are $y_1, y_2, \dots, y_{2^{n-1}} \in F$ for which*

$$S(U_E) \subseteq \bigcup_{i=1}^{2^{n-1}} \{y_i + \varepsilon U_F\} \qquad holds.$$

The sets U_E and U_F are the closed unit balls of E and F,respectively. Put

$$L^{(e)}_{s,t} := \{S \in L : (e_n(S)) \in l_{s,t}\} \text{ and } \|S\|_{L^{(e)}_{s,t}} := k_{s,t}\|(e_n(S))\|_{l_{s,t}}, \tag{6}$$

where $k_{s,t}$ is some norming constant.

In the following ,we deal with diagonal operators $D \in L(l_p, l_q)$ defined by $D(\xi_i) := (\sigma_i\xi_i)$ and generated by a sequence (σ_i) belongs to Lorentz space $l_{r,t}$.

Theorem 2.2. *[2] Let* $1 \leq p,q \leq \infty$ $\frac{1}{r} > \max(\frac{1}{q}-\frac{1}{p},0)$ $,0 < t \leq \infty$ *and* $\frac{1}{s}=\frac{1}{r}+\frac{1}{p}-\frac{1}{q}$. *Then*

$$D \in L^{(e)}_{s,t}(l_p,l_q) \qquad if and only if \qquad (\sigma_i) \in l_{r,t}. \tag{7}$$

Remark 2.3.

(i) The statement of the above theorem extend those of Marcus [13] and Oloff [14] to the case $1 \leq p,q \leq \infty$ *which possesses interesting applications to eigenvalue problems of operators acting on Banach spaces.*

(ii) In the sequel we extend the statement of the preceding theotem to diagonal operators acting from $l_{p,u}$ *into* $l_{q,v}, 1 < p,q < \infty, 1 \leq u,v \leq \infty$. *It turns that the characterization of diagonal operators belonging to the entropy ideals* $L^{(e)}_{s,t}(l_{p,u},l_{q,v})$ *does not depend on the indices* u *and* v *of the underlying Lorents sequence spaces.*

Theorem 2.4. *[2] Let* $1 < p,q < \infty$, $1 \leq u,v \leq \infty$, $\frac{1}{r} > \max(\frac{1}{q}-\frac{1}{p},0)$, $0 < t \leq \infty$ *and* $\frac{1}{s}=\frac{1}{r}+\frac{1}{p}-\frac{1}{q}$. *Then*

$$D \in L^{(e)}_{s,t}(l_{p,u},l_{q,v}) \qquad if and only if \qquad (\sigma_i) \in l_{r,t}. \tag{8}$$

Definition 2.5. *An operator* $S \in L(E,F)$ *is called an* $(r,t;p,q)$*-factorable operator,*$0 < r < \infty, 0 < t \leq \infty, 1 < p,q < \infty$, *iff there exists a commutative diagram*

$$S : E \underrightarrow{A} l_{p,\infty} \underrightarrow{D} l_{q,1} \underrightarrow{Y} F$$

such that $D \in L(l_{p,\infty}, l_q,1)$ *is a diagonal operator of the form* $D(\xi_i) = (\sigma_i\xi_i)$ *with* $(\sigma_i) \in l_{r,t}$, $\frac{1}{r} > \max(\frac{1}{q}-\frac{1}{p},0)$, $A \in (E,l_{p,\infty})$ *and* $Y \in (l_{q,1},F)$.

The class of those operators is denoted by $\mathbb{Z}_{(r,t;p,q)}$.

Applying the preceding results to eigenvalue problems of factorable operators acting on arbitrary Banach spaces.For $1 < p \leq q < \infty$

B.Carl [2] proved that the sequence of eigenvalues $(\lambda_n(S))$ of operators $S \in \mathbb{Z}_{(r,t;p,q)}(E,E)$ belongs to certain Lorentz sequence spaces $l_{s,t}$. The following results complement those obtained in [3,11,12].For nuclear operators (cf also [4,6]).

Theorem 2.6. *[2] Let* $1 < p \leq q < \infty$, $0 < r < \infty$, $0 < t \leq \infty$ *and* $\frac{1}{s}=\frac{1}{r}+\frac{1}{p}-\frac{1}{q}$. *Then,for any Banach space E*

$$(\lambda_n(S)) \in l_{s,t} \qquad for any \qquad S \in \mathbb{Z}_{(r,t;p,q)}(E,E) \quad .$$

3. Interpolation spaces.

The following theorem shows that reflexive Orlicz spaces can be characterized as interpolation spaces between L_p-spaces. This is a converse of Gustavsson-Peetre's theorem ([18] theorem (7.3)).

By an Orlicz function we mean a convex and continuous function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$.

Theorem 3.1. *[9] Let Φ be an Orlicz function and let $L_\Phi(E)$ be the corresponding E-valued Orlicz space on a σ-finite measure space, where E is a U.M.D. Banach space. Suppose that Φ and the conjugate Φ^* satisfy the Δ_2-condition. Then there exist p_0 and p_1, with $1 < p_1 < p_0 < \infty$,*

$$\langle L_{p_0}(E), L_{p_1}(E)\rangle_{\rho,p} = L_\Phi(E) \quad , where \tag{9}$$

$$\rho(t) = \begin{cases} t^{\frac{p_1}{p_1-p_0}} \quad \Phi^{-1}(t^{\frac{p_0p_1}{p_0-p_1}}) & t > 0, \\ 0 & t = 0 \end{cases}. \tag{10}$$

and $1 < p < \infty$.

Definition 3.2. *Let Φ be an Orlicz function as before. We define for $1 \le q < \infty$, the Lorentz-Orlicz spaces $l_{\Phi q}$ as follows*

$$l_{\Phi q} = \left\{ x = (x_n)_n : \|x\|_{l_{\Phi q}} := \inf\left(c : (\sum_{n=1}^{\infty} \Phi(\frac{|x_n|}{c})^q)^{1/q} \le 1 \right) < \infty \right\}, \tag{11}$$

and,

$$l_{\Phi\infty} = \left\{ x = (x_n)_n \ : \|x\|_{l_{\Phi\infty}} := \inf\left(c : \sup_n \Phi(\frac{|x_n|}{c}) \le 1 \right) < \infty \right\}. \tag{12}$$

Definition 3.3. *Let Φ be an Orlicz function . We define for $1 \le q \le \infty$, the Schatten-Orlicz classes $S^s_{\Phi q}$ as follows*

$$S^s_{\Phi q}(X) := \{T \in L(X) \quad : (s_n(T))_n \in l_{\Phi q}\}. \tag{13}$$

Here, $s_n(T)$ be the n-th s-number of T.

Theorem 3.4. *[8] Let $1 < p_0, p_1 < \infty$. Then we have*

$$\langle S^s_{p_0 q}(X), S^s_{p_1 q}(X)\rangle_{\rho,p} \subset S^s_{\Phi q}(X), \tag{14}$$

where ρ is given by (10).

Definition 3.5. *Let Φ be an Orlicz function . We define for $1 \le t \le \infty$, the classes $L^e_{\Phi t}$ as follows*

$$L^e_{\Phi t} := \{T \in L \quad : (e_n(T))_n \in l_{\Phi t}\}. \tag{15}$$

Corollary 3.6. *Let $1 < p_0, p_1 < \infty$. Then we have*

$$\langle L^e_{p_0 t}, L^e_{p_1 t}\rangle_{\rho,p} \subset L^e_{\Phi t}, \tag{16}$$

where ρ is given by (10).

Definition 3.7. *Suppose $s \in \mathbb{Z}$ and let (ϕ_n) be a system of functions (see [19] p. 171). Let X be a Banach space and Φ be an Orlicz function satisfy the Δ_2-condition,we define for $1 \leq q \leq \infty$, the Besov-Orlicz spaces*

$$B^s_{\Phi q}(\mathbb{Z}^N; X) = \{f \in S' \ : (\phi_n * f)_n \in l^s_q(L_\Phi(\mathbb{Z}^N; X))\} \tag{17}$$

We equip the spaces $B^s_{\Phi q}(\mathbb{Z}^N; X)$ with the norm

$$\|f\|_{B^s_{\Phi q}} = \|(\phi_n * f)_n\|_{l^s_q(L_\Phi(\mathbb{Z}^N;X))} = \|(\|\phi_n * f\|_{L_\Phi(X)})_n\|_{l^s_q}, \tag{18}$$

Theorem 3.8. *[8] Let Φ be an Orlicz function which satisfies, together with its conjugate Φ^*, the Δ_2-condition. Then there exist p_0 and p_1, with $1 < p_0, p_1 < \infty$, such that*

$$\langle B^s_{p_0 q}(\mathbb{Z}^N; X), B^s_{p_1 q}(\mathbb{Z}^N; X)\rangle_{\rho,p} \ = \ B^s_{\Phi q}(\mathbb{Z}^N; X), \tag{19}$$

where ρ is given by (10) .

4. Some applications of the interpolation theory.

Now, we derive our main results using the well-known results of Fourier coefficients and the new interpolation results (interpolation of S^s_{pq}-spaces, and B^s_{pq}-spaces). We will prove that the Fourier coefficients of a periodic function $f \in B^s_{\Phi q}(2\pi)$ belong to some Lorentz-Orlicz space $l_{\Psi q}$. Finally,we give some statements which are obtained using results recently proved and developed by the author in [8] and the Carl's results

4.1 Fourier coeficients of some periodic functions.

Theorem 4.1. *Let Φ and Ψ be two Orlicz functions satisfying the Δ_2 condition together with their conjugate functions Φ^* and Ψ^* , let $1 \leq w \leq \infty$. Then the convolution operator C_f of any function $f \in L_{\Phi w}(2\pi)$ acts from $L_{\Psi w'}(2\pi)$ into $L_\infty(2\pi)$ with*

$$\Phi^{-1}(\tau) = \tau \Psi^{-1}(\tau^{-1}).$$

Proof.

By using proposition (1.2) for different values of r ,we get

$$f \in L_{r_i w}(2\pi) \qquad \text{implies} \qquad C_f : L_{r_i' w'}(2\pi) \to L_\infty(2\pi) \qquad , i = 0, 1.$$

Therefore,by the interpolation method we have

$$f \in L_{\Phi w}(2\pi) \qquad \text{implies} \qquad C_f : L_{\Psi w'}(2\pi) \to L_\infty(2\pi) \qquad \text{where}$$

$$\begin{aligned} \rho(t) \ &= \ t^{\frac{r_1}{r_1 - r_0}} \Phi^{-1}(t^{\frac{r_0 r_1}{r_0 - r_1}}) \ = \ t^{\frac{r_1'}{r_1' - r_0'}} \Psi^{-1}(t^{\frac{r_0' r_1'}{r_0' - r_1'}}) \quad \text{i.e.} \\ t^{\frac{-\alpha}{r_0}} \Phi^{-1}(t^\alpha) \ &= \ t^{\frac{\alpha}{r_0'}} \Psi^{-1}(t^{-\alpha}) \qquad , \alpha := \frac{r_0 r_1}{r_0 - r_1} = -\frac{r_0' r_1'}{r_0' - r_1'} \end{aligned}$$

Hence,taking $t^\alpha = \tau$,we have

$$\Phi^{-1}(\tau) = \tau \Psi^{-1}(\tau^{-1}).$$

Theorem 4.2. *Let Φ and Ψ be two Orlicz functions satisfying the Δ_2 condition together with their conjugate functions Φ^* and Ψ^* . Then the Fourier coefficients of any function $f \in L_\Phi(2\pi)$ belong to the Lorentz-Orlicz space $l_\Psi(\mathbb{Z})$ with*

$$\Phi^{-1}(\tau) = \tau\Psi^{-1}(\tau^{-1}) \qquad i.e.$$

$$f \in L_\Phi(2\pi) \qquad \textit{implies} \qquad (\nu_n(f))_{n\in\mathbb{Z}} \in l_\Psi(\mathbb{Z}). \tag{20}$$

Proof.

By using Hausdorff-Young theorem (1.5) ,we have

$$f \in L_{p_i}(2\pi) \qquad \text{implies} \qquad (\nu_n(f)) \in l_{p_i'}(\mathbb{Z}) \qquad ,i = 0,1.$$

Therefore, by the interpolation theorem (3.1), we have

$$f \in L_\Phi(2\pi) \qquad \text{implies} \qquad (\nu_n(f))_{n\in\mathbb{Z}} \in l_\Psi(\mathbb{Z}). \qquad \text{where}$$

$$\rho(t) \;=\; t^{\frac{p_1}{p_1-p_0}}\Phi^{-1}(t^{\frac{p_0p_1}{p_0-p_1}}) \;=\; t^{\frac{p_1'}{p_1'-p_0'}}\Psi^{-1}(t^{\frac{p_0'p_1'}{p_0'-p_1'}}) \quad \text{i.e.}$$

$$t^{\frac{-\alpha}{p_0}}\Phi^{-1}(t^\alpha) \;=\; t^{\frac{\alpha}{p_0'}}\Psi^{-1}(t^{-\alpha}) \qquad ,\alpha := \frac{p_0p_1}{p_0-p_1} = -\frac{p_0'p_1'}{p_0'-p_1'}$$

Hence,taking $t^\alpha = \tau$,we have

$$\Phi^{-1}(\tau) = \tau\Psi^{-1}(\tau^{-1}).$$

The following main results are aconsequence of theorems (1.7),(1.8) and (1.9) and the interpolation theorems (3.4) and (3.8).

Theorem 4.3. *Let Φ and Ψ be two Orlicz functions satisfying the Δ_2 condition together with their conjugate functions Φ^* and Ψ^* , let $1 \le w \le \infty$. Then the Fourier coefficients of any function $f \in L_{\Phi w}(2\pi)$ belong to the Lorentz-Orlicz space $l_{\Psi w}(\mathbb{Z})$ with*

$$\Phi^{-1}(\tau) = \tau\Psi^{-1}(\tau^{-1}) \qquad i.e.$$

$$f \in L_{\Phi w}(2\pi) \qquad \textit{implies} \qquad (\nu_n(f))_{n\in\mathbb{Z}} \in l_{\Psi w'}(\mathbb{Z}). \tag{21}$$

Theorem 4.4. *Let Φ and Ψ be two Orlicz functions satisfying the Δ_2 condition together with their conjugate functions Φ^* and Ψ^* , let $1 \le w \le \infty$. Then the Fourier coefficients of any function $f \in B^s_{\Phi q}(2\pi)$ belong to the Lorentz-Orlicz space $l_{\Psi q}(\mathbb{Z})$ with*

$$\Phi^{-1}(\tau) = \tau^{s+1}\Psi^{-1}(\tau) \qquad i.e.$$

$$f \in B^s_{\Phi q}(2\pi) \qquad \textit{implies} \qquad (\nu_n(f))_{n\in\mathbb{Z}} \in l_{\Psi q}(\mathbb{Z}). \tag{22}$$

Theorem 4.5. *Let Φ and Ψ be two Orlicz functions satisfying the Δ_2 condition together with their conjugate functions Φ^* and Ψ^* , let $1 \leq w \leq \infty$. Then the Fourier coefficients of any function $f \in W^l_\Phi(2\pi)$ belong to the Lorentz-Orlicz space $l_\Psi(\mathbb{Z})$ with*

$$\Phi^{-1}(\tau) = \tau^l \Psi^{-1}(\tau) \qquad i.e.$$

$$f \in W^l_\Phi(2\pi) \qquad \text{implies} \qquad (\nu_n(f))_{n\in\mathbb{Z}} \in l_\Psi(\mathbb{Z}). \tag{23}$$

4.2 Eigenvalue distribution of factorable operators .

The following statements are obtained using results recently proved and developed by the author in [8] and the Carl's results in the second section.

Theorem 4.6. *Let Φ and Ψ be Orlicz functions satisfying the Δ_2-condition together with their conjugate functions ,$1 \leq p, q \leq \infty$,$0 < t \leq \infty$. Then*

$$D \in L^{(e)}_{\Phi,t}(l_p, l_q) \qquad \text{if and only if} \qquad (\sigma_i) \in l_{\Psi,t} \tag{24}$$

with

$$\Phi^{-1}(\tau) \; = \; \tau^{\frac{1}{p}-\frac{1}{q}} \;\; \Psi^{-1}(\tau).$$

Proof.

Using theorem (2.4), we have

$$D \in \mathrm{L}^e_{s_i,t}(l_p, l_q) \qquad \text{iff} \qquad (\sigma_i) \in l_{r_i,t} \quad \text{with} \;\; \frac{1}{s_i} := \frac{1}{r_i} + \frac{1}{p} - \frac{1}{q}. \tag{25}$$

By corollary (3.6), we have

$$\langle L^e_{s_0,t}(l_p, l_q), L^e_{s_1,t}(l_p, l_q)\rangle_{\rho,q} = L^e_{\Phi,t}(l_p, l_q), \tag{26}$$

with,

$$\rho(\tau) = \tau^{\frac{s_1}{s_1-s_0}} \quad \Phi^{-1}(\tau^{\frac{s_0 s_1}{s_0-s_1}}). \tag{27}$$

Using theorem (3.1),we have,

$$\langle l_{r_0 t}, l_{r_1 t}\rangle_{\rho,q} = l_{\Psi t}, \tag{28}$$

with,

$$\rho(\tau) = \tau^{\frac{r_1}{r_1-r_0}} \quad \Psi^{-1}(\tau^{\frac{r_0 r_1}{r_0-r_1}}). \tag{29}$$

Let $\beta = \frac{s_0 s_1}{s_0-s_1} \Rightarrow \quad \frac{1}{\beta} = \frac{1}{s_1} - \frac{1}{s_0} = \frac{1}{r_1} - \frac{1}{r_0} \Rightarrow \quad \beta = \frac{r_0 r_1}{r_0-r_1}$, Consequently, we have

$$\rho(\tau) = \tau^{\frac{-\beta}{s_0}} \; \Phi^{-1}(\tau^\beta) = \tau^{\frac{-\beta}{r_0}} \; \Psi^{-1}(\tau^\beta). \tag{30}$$

Take $\tau^\beta = \xi$, we find

$$\Phi^{-1}(\xi) = \xi^{\frac{1}{s_0}-\frac{1}{r_0}} \quad \Psi^{-1}(\xi) = \xi^{\frac{1}{p}-\frac{1}{q}} \quad \Psi^{-1}(\xi). \tag{31}$$

Then

$$D \in L^{(e)}_{\Phi,t}(l_p, l_q) \qquad \text{if and only if} \qquad (\sigma_i) \in l_{\Psi,t}$$

with

$$\Phi^{-1}(\tau) \; = \; \tau^{\frac{1}{p}-\frac{1}{q}} \;\; \Psi^{-1}(\tau).$$

Theorem 4.7. *Let Φ and Ψ be Orlicz functions satisfying the Δ_2-condition together with their conjugate functions ,$1 \le p, q \le \infty$,$0 < t \le \infty$, $1 \le u, v \le \infty$. Then*

$$D \in L^{(e)}_{\Phi,t}(l_{p,u}, l_{q,v}) \quad \text{if and only if} \quad (\sigma_i) \in l_{\Psi,t} \tag{32}$$

with

$$\Phi^{-1}(\tau) \;=\; \tau^{\frac{1}{p}-\frac{1}{q}} \;\; \Psi^{-1}(\tau).$$

Proof.

Replacing $L^e_{s,t}(l_p, l_q)$ by $L^e_{s,t}(l_{p,u}, l_{q,v})$ in the preceding theorem and using the same method, we get directly the desired results.

Theorem 4.8. *Let Φ and Ψ be Orlicz functions satisfying the Δ_2-condition together with their conjugate functions ,$1 \le p, q \le \infty$,$0 < t \le \infty$. Then ,for any Banach space E*

$$(\lambda_n(S)) \in l_{\Phi,t} \quad \text{for any} \quad S \in \mathbb{Z}_{(\Psi,t;p,q)}(E, E)$$

with

$$\Phi^{-1}(\tau) \;=\; \tau^{\frac{1}{p}-\frac{1}{q}} \;\; \Psi^{-1}(\tau).$$

Proof.

Using theorem (2.6),(4.7) and the interpolation method ,we get directly the desired results.

The following results are obtained using the same techniques recently developed above.

Theorem 4.9. *Let Φ and Ψ be Orlicz functions satisfying the Δ_2-condition together with their conjugate functions ,$1 \le q \le \infty$,$0 < t \le \infty$. Then*

$$D \in L^{(e)}_{s,t}(l_\Phi, l_q) \quad \text{if and only if} \quad (\sigma_i) \in l_{\Psi,t} \tag{33}$$

with

$$\Phi^{-1}(\tau) \;=\; \tau^{\frac{1}{s}+\frac{1}{q}} \;\; \Psi^{-1}(\tau^{-1}).$$

Theorem 4.10. *Let Φ and Ψ be Orlicz functions satisfying the Δ_2-condition together with their conjugate functions ,$1 \le q \le \infty$,$0 < t \le \infty$,$1 \le u, v \le \infty$. Then*

$$D \in L^{(e)}_{s,t}(l_{\Phi,u}, l_{q,v}) \quad \text{if and only if} \quad (\sigma_i) \in l_{\Psi,t}. \tag{34}$$

with

$$\Phi^{-1}(\tau) \;=\; \tau^{\frac{1}{s}+\frac{1}{q}} \;\; \Psi^{-1}(\tau^{-1}).$$

Theorem 4.11. *Let Φ and Ψ be Orlicz functions satisfying the Δ_2-condition together with their conjugate functions ,$1 \le q \le \infty$,$0 < t \le \infty$. Then ,for any Banach space E*

$$(\lambda_n(S)) \in l_{s,t} \quad \text{for any} \quad S \in \mathbb{Z}_{(\Psi,t;\Phi,q)}(E, E).$$

with

$$\Phi^{-1}(\tau) \;=\; \tau^{\frac{1}{s}+\frac{1}{q}} \;\; \Psi^{-1}(\tau^{-1}).$$

Example 4.12. *Take $\Phi(\tau) = \tau^s$ in theorem (4.6),then we have*

$$\Phi^{-1}(\tau) \;=\; \tau^s \;=\; \tau^{\frac{1}{p}-\frac{1}{q}} \;\; \Psi^{-1}(\tau).$$

Therefore,

$$\Psi^{-1}(\tau) \;=\; \tau^{\frac{1}{s}-\frac{1}{p}+\frac{1}{q}} \;=\; \tau^{\frac{1}{r}} \qquad : \frac{1}{s} = \frac{1}{r} + \frac{1}{p} - \frac{1}{q}.$$

Hence,

$$D \in L^{(e)}_{s,t}(l_p, l_q) \qquad \textit{if and only if} \qquad (\sigma_i) \in l_{r,t} \qquad : \frac{1}{s} = \frac{1}{r} + \frac{1}{p} - \frac{1}{q}.$$

This means that the Carl's theorem (2.2) is a special case of theorem (4.6)

Example 4.13. *Taking $\Phi(\tau) = \tau^p(\log\tau)$ in theorem (4.9) ,we have*

$$D \in L^{(e)}_{s,t}(l_p(\log l), l_q) \qquad \textit{if and only if} \qquad (\sigma_i) \in l_{r,t}(\log 1/l) \quad \textit{where} \qquad \frac{1}{s} = \frac{1}{r}+\frac{1}{p}-\frac{1}{q}.$$

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Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISBN 1083-0464
Issue 1 July 2002, pp. 81 – 110
Submitted: November 25, 2001. Published: July 1, 2002

A Probabilistic Description of the Three Theorems of Hardy

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Abstract

The purpose of this paper will be to transpose three of Hardy's theorems to the context of unimodality, large deviation theory, and local limits theorems. In his own work, these three theorems were strategic and were used to prove among other things, certain Tauberian theorems for the Euler-Borel class. But where Hardy's results are for particular sums. Our generalizations apply to various general methods of summability; including for example, random-walk methods, and the convolution method.

A.M.S. (MOS) Subject Classification Codes. 30, 40, 60, 62
Key Words and Phrases. Hardy's Theorems, Summability, Random-walk methods, Convolution Summability Method, Large Deviation, Local Limit Theorems, Central Limit Theorem

I. INTRODUCTION

Let $X_1, X_2, \cdots$ be a sequence of random variables either dependent or independent, and let $\frac{S_n}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i$. The main object of study in probability theory has been the quentity $\frac{S_n}{n}$. The three main branches of classical probability deal with (i). the almost sure behavior of $\frac{S_n}{n}$, (ii). the weak convergence of the appropriate normalized form of $\frac{S_n}{n}$ and (iii). the large deviation results.

From the summability point of view, $\frac{S_n}{n}$ is simply the Cesáro transform of $X_1, X_2, \cdots$. It is natural to ask analogous questions concerning other well known summability transforms of $X_1, X_2, \cdots$. For example, various authors have considered Borel, Euler and Abel summability transforms [2], [9] and [3] for this purpose. It seems that most of the convergence results hold for these summability analogs, under appropriate moment conditions.

The results presented in this paper provide some new results in the summability theory using probabilistic tools.

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II. SOME SUMMABILITY METHODS

In this section, we collect relevant summability methods that are used in this paper. A natural extension of the Euler method of summability is the random-walk family defined as follows.

Definition 1 Random-walk method of summability
Let $\{X_i\}_{i\geq 1}$ be an independent sequence of random variables taking values over nonnegative integers and let

$$S_n = \sum_{k=1}^{n} X_k, \text{ with } S_0 = 0.$$

Then the matrix entries associated with $a_{n,k} = P(S_n = k)$, such that $A = \{a_{n,k}\}_{n,k\geq 0}$ is called the random-walk method.

Let $f(k)$ be a sequence so that $\sum_{k=0}^{\infty} f(k)P(S_n = k)$ exists for each n. If

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} f(k)P(S_n = k) = s$$

then we say that $f(n)$ converges to s in the sense of the random-walk method and write $f(n) \to s$ (P). However, one should note that if $E|f(S_n)| < \infty$ for each n, then the above requirement is equivalent to asking that $E(f(S_n)) \to s$ to conclude that $f(n) \to s$ (P).

Even though the above family of random-walk methods contains the Euler and Borel matrix families, it does not include the Taylor family. The following generalization contains Taylor and other similar methods of summability [8].

Definition 2 Convolution method of summability Let $(p_k)_{k\geq 0}$ and $(q_k)_{k\geq 0}$ be two sequences of nonnegative numbers with $\sum_{k=0}^{\infty} p_k = 1$ and $\sum_{k=0}^{\infty} q_k = 1$. Define a summability matrix $C = \{C_{n,k}\}$, the entries of which are given as follows. Take $C_{0,k} = q_k$ and the $(n+1)$th row, $n \geq 0$ is defined as

$$C_{n+1,k} := (C_{n,\cdot} * p)_k = \sum_{j=0}^{k} p_j C_{n,k-j};\ k = 0, 1, 2, \cdots.$$

We will call C to be a convolution summability method generated by (p, q) and denote it by $C(p, q)$.

Remark 1 Khan [8] showed that $C(p, q)$ is regular and takes l^1 sequences into l^1 sequences if and only if $p_0 < 1$.

The probability description for this matrix summability method is as follows.

Let $Y, X_1, X_2, \cdots$ be a sequence of independent random variables taking nonnegative integer values. Define $S_n = Y + X_1 + X_2 + \cdots + X_n$ for $n \geq 1$ and for $n = 0$ take $S_0 = Y$. Assume that the random variables $X_1, X_2, \cdots$ are identically distributed with probability distribution p and Y has the probability distribution q. The nth row of the matrix $C(p, q)$ is the probability distribution of the random variable S_n. We now consider the conditions on p and q that lead the convolution summability method to Euler, Borel, Taylor and Meyer-König Methods.

Example 1 In the following examples, we assume $0 < r \le 1$ and $\lambda > 0$.

1. Euler method: Take $p = (1-r, r, 0, 0, \cdots)$, $q = (1, 0, 0, 0, \cdots)$. It is seen that this choice of p and q generates the Euler summability method E_r.

2. Borel matrix method: Take $p = q = (e^{-\lambda}, e^{-\lambda}\lambda, e^{-\lambda}\lambda^2/2!, e^{-\lambda}\lambda^3/3!, \cdots)$. This generates Borel matrix summability method B_λ.

3. Taylor method: Take $p = (0, 1-r, (1-r)r, (1-r)r^2, \cdots)$, $q = (1-r, (1-r)r, (1-r)r^2, \cdots)$. That is, Y is the geometric random variable with parameter $(1-r)$ and X_1 is the shifted (or Pascal) random variable with parameter $(1-r)$. This gives rise to the Taylor summability method T_r.

4. Meyer-König method: Take $p = q = (1-r, (1-r)r, (1-r)r^2, \cdots)$. This gives rise to Meyer-König method of summability denoted by MK_r.

We now discuss two well-known methods of summability called Valiron and circle methods of summability [1].

Definition 3 Valiron and Circle methods of summability
For $a > 0$, the sequence $\{s_n\}_{n\ge 0}$ is said to converge to s in the sense of the Valiron method of summability with parameter a if

$$\sqrt{\frac{a}{2\pi n}} \sum_{i=0}^{\infty} s_i \exp\{-\frac{1}{2}a(i-n)^2/n\} \to s \text{ as } n \to \infty;$$

and we write $s_n \to s\ (V_a)$.

On the other hand the circle family of summability methods contains the Valiron method and the following.

1. Borel exponential method B, for which $s_n \to s\ (B)$ means

$$\sum_{i=0}^{\infty} \frac{e^{-x} x^i}{i!} s_i \to s \text{ as } n \to \infty.$$

2. Euler method E_r, $r \in (0,1)$, for which $s_n \to s\ (E_r)$ means

$$\sum_{i=0}^{n} \binom{n}{i} r^i (1-r)^{n-i} s_i \to s \text{ as } n \to \infty.$$

3. Meyer-König method S_r (or MK_r), $r \in (0,1)$, for which $s_n \to s\ (S_r)$ means

$$(1-r)^{n+1} \sum_{i=0}^{\infty} \binom{n+i}{i} r^i s_i \to s \text{ as } n \to \infty.$$

4. Taylor method T_r, $r \in (0,1)$, for which $s_n \to s\ (T_r)$ means

$$(1-r)^{n+1} \sum_{i=0}^{\infty} \binom{n+i}{i} r^i s_{n+i} \to s \text{ as } n \to \infty.$$

Note that the last three members of the circle family are also members of the convolution summability method.

III. THREE THEOREMS OF HARDY

Much of the second half of Hardy's text [6] uses three important theorems, namely, Theorems 137, 138, and 139, pp. 200 - 201, [6]. These theorems have direct probabilistic analogs. These theorems will be extended in the next few sections to the random-walk method and, when appropriate, to the convolution summability method. Here we provide the original theorems of Hardy for the sake of easy referencing. The proposed generalization serves a variety of purposes. The first purpose is to view the results from the probability point of view. The second purpose is to extend these results for the convolution summability method. It is anticipated that the resulting extensions may be useful for extending Hardy's Tauberian theorems. The first of them deals with the Borel exponential method of summability.

Theorem 1 *(Theorem 137, p. 200, [6]) Suppose that $x > 0$ and*

$$u_m = u_m(x) = e^{-x}\frac{x^m}{m!} \quad (m = 0, 1, 2, \cdots),$$

so that $\sum u_m = 1$. Then

(1). the largest u_m is u_M, where $M = [x]$, two terms, u_{M-1} and u_M, being equal when x is an integer;

(2). if $m = M + h$ and $0 < \delta < 1$, then

$$\sum_{|h|>\delta x} u_m = O\Big(e^{-\gamma x}\Big), \quad \textit{where } \gamma = \frac{1}{3}\delta^2;$$

(3). if $\frac{1}{2} < \xi < \frac{2}{3}$, then

$$\sum_{|h|>x^{\xi}} u_m = O\Big(e^{-x^{\eta}}\Big),$$

where η is any number less than $2\xi - 1$;

(4). if $\lambda > 0$, then

$$\sum_{|h|>\lambda x^{\frac{1}{2}}} u_m < \epsilon,$$

for $x > x_0(\epsilon)$, $\lambda > \lambda_0(\epsilon)$;

(5). if $|h| \le x^{\xi}$, then

$$u_m = \sqrt{\Big(\frac{c}{\pi M}\Big)}e^{-ch^2/M}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\},$$

where $c = \frac{1}{2}$;

(6). the estimates (2) and (3) remain valid if u_m is multiplied by any fixed power of m.

The next result is about the Euler method.

Theorem 2 *(Theorem 138, p. 201, [6]) Suppose that* $q > 0$ *and*

$$u_m = u_m(n) = (q+1)^{-n} \binom{n}{m} q^{n-m} \quad (0 \le m \le n),$$

so that $\sum u_m = (q+1)^{-n} \sum \binom{n}{m} q^{n-m} = 1$*. Then*

(1). the largest u_m *is* u_M*, where* $M = \left[\frac{n+1}{q+1}\right]$*, two terms,* u_{M-1} *and* u_M*, being equal when* $(n+1)/(q+1)$ *is an integer;*

(2). clauses (2) - (6) of Theorem 1 hold with $c = \frac{q+1}{2q}$*, some positive* $\gamma = \gamma(q, \delta)$ *in clause (2)*[1] *and* n *in the place of* x*.*

The next theorem is the Taylor method.

Theorem 3 *(Theorem 139, p. 201, [6]) Suppose that* $0 < k < 1$ *and*

$$u_m = u_m(n) = k^{n+1} \binom{m}{n} (1-k)^{m-n} \quad (m \le n),$$

so that $\sum u_m = k^{n+1}\Big\{1 + (n+1)(1-k) + \frac{(n+1)(n+2)}{2!}(1-k)^2 + \cdots\Big\} = 1$*. Then*

(1). the largest u_m *is* u_M*, where* $M = [n/k]$*, two terms,* u_{M-1} *and* u_M*, being equal when* n/k *is an integer;*

(2). clauses (2) - (6) of Theorem 1 hold with $c = \frac{k}{2(1-k)}$*, some positive* $\gamma = \gamma(k, \delta)$ *in clause (2)*[2] *and* n *in the place of* x*.*

As we shall see later that in the extensions of the results for other summability methods, several restrictions on the parameters are relaxed. In proving our results, we draw attention to three important topics in probability theory; (i) unimodality of a discrete distribution, (ii) large deviation theory, and (ii) local limit theorems. Among them, the theory of large deviations (LDT) is one of the most active areas in probability today, having applications to many areas including statistics, communication networks, information theory, statistical mechanics, risk sensitive control, and queuing systems [13]. The richness of the theory of large deviations derives, in part, from the fact that it is nonlinear, and this in turn contributes to much of the difficulty of large deviation analysis.

IV. LARGE DEVIATION THEORY (LDT)

We will discuss the relatively simple case of the theorem of Cramer in order to introduce the main ideas of the approach. In the context of a summability method, we will explore connections between the representation formulas and computations. This is an aspect of LDT that has not been extensively studied in summability context. Although the general theory of large deviations is well developed, in the summability context there is still a great deal of work to be done. In the end, we will discuss the use of the representation formula to obtain some results.

[1] a definite value to γ is not assigned in this case.

[2] a definite value to γ is not assigned in this case.

The weak laws of large numbers (WLLN) for coin tossing says:

$$P\Big\{\Big|\frac{S_n}{n}-\frac{1}{2}\Big|\geq\epsilon\Big\}\to 0 \text{ as } n\to\infty$$

for all $\epsilon>0$, where S_n represents the number of heads out of n tossings of a fair coin. Of course, the rate at which it tends to 0 and how it depends on $\epsilon>0$ is given by the Chebychev theorem. Using the second moment, we get

$$P\Big\{\Big|\frac{S_n}{n}-\frac{1}{2}\Big|\geq\epsilon\Big\}\leq\frac{1}{4n\epsilon^2}.$$

Using the fourth moments, we obtain

$$P\Big\{\Big|\frac{S_n}{n}-\frac{1}{2}\Big|\geq\epsilon\Big\}\leq\frac{C}{n^2\epsilon^4},$$

etc.. Using higher moments, one can show that the rate of decrease of the left side is even faster. The key result of the LDT is the following theorem of Cramer [13], which shows that the rate of decrease is in fact exponential.

Theorem 4 *(Cramer [13]) Assume $X_1,X_2,\cdots,X_n$ are i. i. d. random variables with common distribution function $F(x)$ with the following properties:*

1. *$\int e^{\theta x}dF(x)=m(\theta)<\infty,$ for all θ;*
2. *$0<F(x)<1,$ for all $-\infty<x<\infty.$*

Then there exists a function $h(x)$ on $\mathbb{R}$ such that

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log P\Big[\frac{S_n}{n}\in A\Big] &\leq -\inf_{x\in\bar{A}}h(x) \quad\text{and}\\ \liminf_{n\to\infty}\frac{1}{n}\log P\Big[\frac{S_n}{n}\in A\Big] &\geq -\inf_{x\in A^0}h(x),\end{aligned}$$

where $h(x)=\sup_\theta\{\theta x-\log m(\theta)\}$.

However, this theorem is not applicable to our example, since the coin tossing example is precluded by the second condition above. Nevertheless, the result still holds due to the following version of Cramer's theorem [4].

Theorem 5 *(Cramer [4]) Let $\{X_i\}_{i\geq1}$ be a sequence of independent and identically distributed random variables and let $S_n=\sum_{i=1}^n X_i$. Suppose in addition to*

1. *$\varphi(\theta)=E\exp(\theta X_1)<\infty$ for some $\theta>0$ and*
2. *the distribution F is not a point mass at μ,*

that for a given $a>\mu=EX_1$, there is a $\theta_a\in(0,\theta_+)$ so that $a=\varphi'(\theta_a)/\varphi(\theta_a)$ and $\theta_+=\sup\{\theta:\varphi(\theta)<\infty\}$. Then as $n\to\infty$,

$$n^{-1}\log P\{S_n\geq na\}\to -a\theta_a+\log\varphi(\theta_a).$$

The purpose of this paper will be to transpose three of Hardy's theorems [6, pp. 200 - 201] to the context of unimodality, large deviation theory, and local limits theorems. In his own work, these three theorems were strategic and were used to prove among other things, certain Tauberian theorems for the Euler-Borel class. But where Hardy's results are for particular sums. Our generalizations apply to various general methods of summability; including for example, random-walk methods, and the convolution method. Corresponding generalizations of Tauberian theorems will be the subject of later work.

Regarding notation, since the general summability methods do not depend as such on [x] (the integer part of x), we simply use x in our formulation.

V. UNIMODALITY FOR RANDOM-WALK METHOD

The following discussion will shed some light on the theorem in extending the first assertion of Hardy's theorem for random-work method. This has to do with the unimodality of the entries of the nth row of the three summability methods (Borel matrix, Euler and Taylor). A probability distribution $\{p_j\}_{j\geq 0}$ is called unimodal if the sequence $\{p_{j+1}-p_j\}_{j=-\infty}^{\infty}$ has exactly one change of sign. The unimodality of the distribution function has been of substantial interest, especially in connection with the question of whether distribution functions of the $n-$fold convolution are unimodal [10].

It was conjectured [**?**] that something stronger ought to hold for a discrete distribution $\{p_j\}$ on the integers, namely that for each such distribution there ought to be an integer n_0 such that the $n-$fold convolution $\{p_j\}^{\star n}$ is unimodal for all $n \geq n_0$. There are counter examples to show that it is hard to guarantee unimodality even for high convolution of a discrete distribution when the distribution has integer support.

A more restrictive concept than that of unimodality is that of strong unimodality; a discrete distribution $\{p_j\}$ is strongly unimodal if $\{p_j\}\star\{q_j\}$ is unimodal for any unimodal discrete distribution $\{q_j\}$. A strongly unimodal distribution is unimodal, but not necessarily conversely. A discrete distribution $\{p_j\}$ is strongly unimodal if and only if it is log concave; $p_j^2 \geq p_{j-1}p_{j+1}$ for all integer values of j [7].

In view of these results, the conclusions of Hardy's theorem will not hold in general for an arbitrary summability method unless the conditions for unimodality are satisfied. Fortunately, the distributions that are addressed by Hardy's theorem obey this property. If a distribution is unimodal then there are two possibilities.

1. There exists an integer m such that $p_j < p_{j+1}$ for $j < m$ and $p_j > p_{j+1}$ for $j \geq m$ or
2. There exist integers m and n, $m < n$, such that $p_j < p_{j+1}$ for $j < m-1$, $p_j = p_m$ for $j = m, m+1, \cdots, n$ and $p_j > p_{j+1}$ for $j > n$.

The following shows that not only Hardy's result holds but they possess the more restrictive property of strong unimodality.

1. For the Borel matrix method $B = \{b_{n,k}\}$, where $b_{n,k} = \frac{e^{-n}n^k}{n!}$; for $k = 0, 1, 2, \cdots$ and $n = 1, 2, 3, \cdots$. The condition for log concavity of the nth row requires that

$$b_{n,j}{}^2 \geq b_{n,j-1}b_{n,j+1}, \text{ for } j = 0, 1, 2, \cdots.$$

Since n is fixed for the time being, let $p_j = b_{n,j}$, with $p_{-1} = 0$. We have

$$\frac{e^{-2n}n^{2j}}{(j!)^2} \geq \frac{e^{-n}n^{j-1}}{(j-1)!}\frac{e^{-n}n^{j+1}}{(j+1)!}$$

$$1 \geq \frac{j}{j+1} \quad \Leftrightarrow \quad j+1 \geq j, \text{ for } j = 0, 1, 2, \cdots.$$

Hence each row of the Borel matrix method is log concave.

2. For the Euler method $E_p = \{e_{nk}\}$, where $e_{n,k} = \binom{n}{j} p^j q^{n-j}$. The condition for log concavity of the nth row requires that

$$\begin{aligned}
{e_{n,j}}^2 &\geq e_{n,j-1}e_{n,j+1} \text{ for } j = 0, 1, 2, \cdots \\
\binom{n}{j}^2 p^{2j} q^{2n-2j} &\geq \binom{n}{j-1} p^{j-1} q^{n-j+1} \binom{n}{j+1} p^{j+1} q^{n-j-1} \\
\binom{n}{j}^2 &\geq \binom{n}{j-1}\binom{n}{j+1} \\
1 &\geq \Big(\frac{j}{j+1}\Big)\Big(\frac{n-j}{n-j+1}\Big) \text{ for } j = 0, 1, 2, \cdots, n.
\end{aligned}$$

Since, this is always true, we have the log concavity.

3. For the Taylor method $T_k = \{t_{n,j}\}$, where $t_{n,k} = k^{n+1} \binom{j}{n} (1-k)^{j-n}$ for $j \geq n$. The condition for log concavity of the nth row requires that

$${t_{n,j}}^2 \geq t_{n,j-1}t_{n,j+1}$$

gives

$$\begin{aligned}
\binom{j}{j+1}^2 &\geq \binom{j-1}{n}\binom{j+1}{n} \text{ for } j \geq n \\
1 &\geq \Big(\frac{j+1}{j}\Big)\Big(\frac{j-n}{j-n+1}\Big) \text{ for } j \geq n.
\end{aligned}$$

Since this is a triviality, we have once again the log concavity.

The following is a direct consequence of a theorem of [10] and gives a partial (but essentially best possible) result for arbitrary regular random-walk methods. We omit its straightforward proof.

Theorem 6 *For a sequence of random variables $\{X_i\}_{i\geq 0}$ having a probability distribution $\{p_j\}$ on integers with $p_j = 0$ for $j < 0$ and $j > d$ while $p_0 > 0,\ p_1 > 0,\ p_{d-1} > 0,\ p_d > 0$, then there exists an integer n_0 such that for $n > n_0$, the entries of the nth rows of the random-walk method generated by the above sequence are strongly unimodal.*

VI. LARGE DEVIATIONS FOR RANDOM-WALK METHOD

We now turn our attention towards the large deviation aspects of Hardy's three theorems. As we will see later, these results are modified to both establish the probabilistic theory and to obtain the exact closed form of the constants as well. Since the Borel exponential method is not a matrix method, translating the same to a probability language will undergo a separate treatment in dealing with the related results. The result for the Euler method follows without much difficulty, where as that of the Taylor method will be carried out by transforming to a random-walk method.

Theorem 7 *Let $X_1, X_2, \cdots$ be independent and identically distributed random variables having a finite moment generating function and positive variance. Let $S_n = X_1 + X_2 + \cdots + X_n$ and let*

$$u_m = P(S_n = m); \ m = 0, 1, 2, \cdots.$$

If $m = M + h$, where $M = nE(X_1)$ and $\delta > 0$, then

$$\sum_{|h|>\delta n} u_m = O\Big(e^{-n\gamma}\Big), \ \textit{where } \gamma = \gamma(\delta) > 0.$$

Proof.

Let $\mu = E(X_1)$. Note that

$$\begin{aligned}\sum_{|h|>\delta n} u_m &= P(|S_n - n\mu| > \delta n) \\ &= P(S_n - n\mu > \delta n) + P(S_n - n\mu < -\delta n).\end{aligned}$$

Note also that we need only prove the result for $\delta > 0$ in a neighborhood of zero. We consider only the first term, since the second term can be analyzed similarly by considering $Y_i = -X_i$ for $i \geq 1$. Now

$$P(S_n - n\mu > \delta n) = P(S_n > n(\mu + \delta)),$$

and Theorem 5 will be applicable once we verify its conditions. Let $\phi(\theta) = E\Big(e^{\theta X_1}\Big)$ and $\theta_+ = \sup\{\theta : \phi(\theta) < \infty\}$. Our assumption concerning the existence of the moment generating function gives that $\theta_+ > 0$. And $\mathrm{Var}(X_1) > 0$ implies that the random variable X_1 is not degenerate at a point $a = \mu + \delta$. We need to show the existence of $\theta_a \in (0, \theta_+)$, so that

$$a = \frac{\phi'(\theta_a)}{\phi(\theta_a)}.$$

First note that $\frac{\phi'(0)}{\phi(0)} = \mu > 0$. Let $g(\theta) = \frac{\phi'(\theta)}{\phi(\theta)}$, and note that

$$\frac{d}{d\theta} g(\theta) = \frac{\phi(\theta)\phi''(\theta) - (\phi'(\theta))^2}{(\phi(\theta))^2}.$$

This gives that

$$g'(0) = \frac{d}{d\theta} g(\theta)\Big|_{\theta=0} = E(X_1^2) - (E(X_1))^2 = \mathrm{Var}(X_1) > 0.$$

Therefore, $g(\theta)$ is a strictly increasing function of θ in a neighborhood of 0. This gives that for any $\delta > 0$, in a neighborhood of 0, and $a = \mu + \delta$, there exists a θ_a such that $P(S_n - n\mu > \delta n) = O\Big(e^{-n\gamma}\Big)$, where $\gamma = \gamma(\delta) > 0$ is given by Theorem 5. The case $P(S_n - n\mu < -n\delta)$ is similarly taken care of by using $Y_i = -X_i$ for $i \geq 1$. This finishes the proof.

∎

The following corollary deals with what is known as the Poisson distribution. A random variable X, taking on values $0, 1, 2, \cdots$ is said to be a Poisson random variable with parameter λ, if for $\lambda > 0$,

$$P(X = m) = \frac{e^{-\lambda}\lambda^m}{m!}, \ m = 0, 1, 2, \cdots.$$

We will denote this by $X \sim \text{Pois}(\lambda)$. The following corollary is a Borel matrix (instead of Borel exponential method) version of Theorem 137(2), p. 200, [6].

Corollary 1 *Suppose that*

$$u_m = u_m(x) = \frac{e^{-x}x^m}{m!}; \ m = 0, 1, 2, \cdots,$$

where $x = 1, 2, \cdots,$ *so that* $\sum_m u_m = 1$. *If* $m = M + h,$ *where* $M = x$ *and* $\delta > 0$, *then*

$$\sum_{|h|>\delta x} u_m = O(e^{-\gamma x}), \ \textit{where } \gamma = \gamma(\delta) > 0.$$

Proof.

Let $S_n = X_1 + X_2 + \cdots + X_n \overset{\text{i.i.d.}}{\sim} \text{Pois}(1)$ in Theorem 7. We evaluate the function $\gamma(\delta)$ more explicitly as follows. The moment generating function $\varphi(t)$ is given by

$$\varphi(t) = \exp\{(e^t - 1)\} \ \Rightarrow \ \varphi'(t) = e^t \exp\{(e^t - 1)\}.$$

This gives that $a = \frac{\varphi'(t)}{\varphi(t)} = e^t > \mu$. Let us now solve for $a = \frac{\varphi'(t)}{\varphi(t)} \iff 1 + \delta = e^{\theta_a} \iff \theta_a = \log(1+\delta)$. Then by the large deviation result, Theorem 5,

$$n^{-1} \log P\{S_n > n(1+\delta)\} \sim -(1+\delta)\log(1+\delta) + (1+\delta) - 1 = -\gamma_1(\delta).$$

Note also that

$$\begin{aligned} \gamma_1(\delta) &= -\delta + (1+\delta)\log(1+\delta) \text{ and } \gamma(0) = 0 \\ \gamma_1'(\delta) &= -1 + \log(1+\delta) + \frac{1+\delta}{1+\delta} \\ &= \log(1+\delta) > 0 \text{ for all } \delta > 0. \end{aligned}$$

Hence, $\gamma_1'(\delta) > 0$ for any $\delta > 0$. We will identify the rate function $\gamma_1(\delta)$ in this case as

$$-\delta + (1+\delta)\log(1+\delta) \text{ for } \delta > 0 \text{ and for large values of } x.$$

And note that $\gamma_1(\delta)$ is a strictly increasing function of δ.

∎

The following corollary deals with what is known as the binomial distribution. Suppose that n independent trials are to be performed; each trial results in a "success" with probability $\frac{1}{1+q}$, for $q > 0$, and in a "failure" with probability $\frac{q}{1+q}$. If X represents the number of successes that occur in n trials, then X is said to be a binomial random variable with parameter $(n, \frac{1}{1+q})$. The probability mass function of a binomial random variable, having parameters $(n, \frac{1}{1+q})$ is given by

$$P(X = m) = (q+1)^{-n} \binom{n}{m} q^{n-m}, \ (0 \leq m \leq n).$$

Theorem 138(2), p. 201, [6] will be modified to accommodate the probabilistic nature of the problem using x instead of $[x]$, which will also make the proof simpler.

Corollary 2 *Suppose that $q > 0$ and*

$$u_m = u_m(n) = (q+1)^{-n}\binom{n}{m}q^{n-m},\ (0 \le m \le n),$$

so that $\sum_m u_m = 1$. If $m = M + h,$ where $M = x = \frac{n+1}{q+1}$ and $\delta > 0$, then

$$\sum_{|h|>\delta x} u_m = O(e^{-\gamma x}),\ \textit{where } \gamma = \gamma(\delta, q) > 0.$$

Proof.

Let $S_n = X_1 + X_2 + \cdots + X_n$. We now consider the random variables defined as

$$X_i = \begin{cases} 1 & \text{with probability} \quad 1/(q+1),\ q > 0 \\ 0 & \text{with probability} \qquad q/(q+1), \end{cases}$$

for $i = 1, 2, \cdots n$. Then $EX_1 = \frac{1}{q+1}$ and $E(X_1)^2 = \frac{1}{q+1} \Rightarrow \operatorname{Var}(X_1) = \frac{q}{(q+1)^2}$. The moment generating function $\varphi(t) = Ee^{tX_1} = \frac{e^t+q}{q+1}$ and $\varphi'(t) = \frac{e^t}{q+1} \Rightarrow \frac{\varphi'(t)}{\varphi(t)} = \frac{e^t}{e^t+q}$. Now we have

$$\begin{aligned}
\sum_{\left[\left|\frac{m-x}{x}\right|>\delta\right]} P\{S_n = m\} &= P\Big\{\Big|S_n - \frac{n+1}{q+1}\Big| > \delta\Big(\frac{n+1}{q+1}\Big)\Big\} \\
&= P\Big\{S_n - \frac{n+1}{q+1} > \Big(\frac{n+1}{q+1}\Big)\delta\Big\} + P\Big\{S_n - \frac{n+1}{q+1} < -\Big(\frac{n+1}{q+1}\Big)\delta\Big\}.
\end{aligned}$$

As before, we only proceed with the first term. The second term has a similar bound. Since $\frac{\varphi'(\theta_a)}{\varphi(\theta_a)} = \frac{e^{\theta_a}}{e^{\theta_a}+q} = \frac{(1+\delta)}{q+1}$, we obtain

$$\Rightarrow qe^{\theta_a} + e^{\theta_a} = e^{\theta_a} + q + \delta e^{\theta_a} + \delta q \Rightarrow e^{\theta_a}(q-\delta) = (\delta+1)q \Rightarrow e^{\theta_a} = \frac{(\delta+1)q}{q-\delta}.$$

This gives us $\theta_a = \log\Big(\frac{(\delta+1)q}{q-\delta}\Big) = \log\Big(1 + \frac{\delta(1+q)}{q-\delta}\Big)$, which is well defined for $q - \delta > 0$. Otherwise, the probability of the event is zero as $S_n \le n$ with probability 1. Then by the large deviation result, Theorem 5,

$$\begin{aligned}
\frac{1}{n}\log P\Big\{S_n > \frac{n+1}{q+1}(1+\delta)\Big\} &\le \frac{1}{n}\log P\Big\{S_n > \frac{n}{q+1}(1+\delta)\Big\} \\
&= -\frac{(1+\delta)}{1+q}\theta_a + \log\varphi(\theta_a) \\
&= -\frac{1+\delta}{1+q}\log\Big\{\frac{q(1+\delta)}{q-\delta}\Big\} + \log\Big\{\Big(\frac{q(1+\delta)}{q-\delta} + q\Big)/(q+1)\Big\} \\
&= -\frac{1+\delta}{1+q}\log\Big\{\frac{q(1+\delta)}{q-\delta}\Big\} + \log\Big\{\frac{q}{q-\delta}\Big\} \\
&= -\frac{1+\delta}{1+q}\Big\{\log\frac{q}{q-\delta} + \log(1+\delta)\Big\} + \log\Big\{\frac{q}{q-\delta}\Big\} \\
&= \Big(1 - \frac{1+\delta}{1+q}\Big)\log\frac{q}{q-\delta} - \frac{1+\delta}{1+q}\log(1+\delta) \\
&= \frac{q-\delta}{q+1}\log\frac{q}{q-\delta} - \frac{1+\delta}{1+q}\log(1+\delta)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{q}{1+q}\Big(\frac{q-\delta}{q}\Big)\log\frac{q-\delta}{q} - \frac{1+\delta}{1+q}\log(1+\delta) \\
&= -\frac{q}{1+q}\Big(1-\frac{\delta}{q}\Big)\log\Big(1-\frac{\delta}{q}\Big) - \frac{1}{q+1}(1+\delta)\log(1+\delta) \\
&= -\frac{q}{1+q}(1-s)\log(1-s) - \frac{1}{q+1}(1+\delta)\log(1+\delta),
\end{aligned}
$$

where $sq = \delta$. We will identify the rate function $\gamma_2(\delta, q)$ in this case as

$$\gamma_2(\delta, q) = \frac{q}{1+q}\Big(1-\frac{\delta}{q}\Big)\log\Big(1-\frac{\delta}{q}\Big) + \frac{1+\delta}{1+q}\log(1+\delta) \text{ for } 0 < \delta < q.$$

Lemma 9.4, p. 71 of [4] guarantees that $\gamma_2(\delta, q) > 0$ for $a > \mu$ and small θ. For the required assertion; the relation,

$$P\Big\{S_n > \frac{n+1}{q+1}(1+\delta)\Big\} = O(e^{-x\gamma}), \text{ where } \gamma = \gamma_2(\delta, q) > 0,$$

completes the proof.

∎

The next corollary is about the negative binomial distribution. A coin having a probability k of coming up heads is successively flipped until the nth head appears. If Y is the number of flips required to obtain n heads, then,

$$P(Y = m) = \binom{m-1}{n-1} k^n (1-k)^{m-n}, \; m \geq n.$$

We call Y a negative binomial random variable having parameter $k \in (0, 1)$. When $n = 1$, Y is called a geometric random variable.

Corollary 3 *(Theorem 139(2), p. 200, [6]) Suppose that* $0 < k < 1$ *is fixed and*

$$u_m = u_m(n) = k^{n+1}\binom{m}{n}(1-k)^{m-n}; \; (m \geq n),$$

so that $\sum_m u_m = 1$. *If* $m = M + h$, *where* $M = [x] = [\frac{n}{k}]$ *and* $\delta > 0$, *then*

$$\sum_{|h| > \delta x} u_m = O(e^{-\gamma x}), \text{ where } \gamma = \gamma(\delta, k) > 0.$$

Proof.

Consider a geometric random variable with parameter $k \in (0, 1)$,

$$P(X = x) = k(1-k)^{x-1}, \; x = 1, 2, 3, \cdots.$$

Then $EX = \frac{1}{k}$. The n^{th} row of the Taylor method is $P(Y + \sum_{i=1}^n X_i = m)$, where $X_0, X_1, X_2, \cdots \sim X$ and $Y = X_0 - 1$. To verify this, note that

$$
\begin{aligned}
P(Y + \sum_{i=1}^{n} X_i = m) &= P(X_0 - 1 + \sum_{i=1}^{n} X_i = m) \\
&= P(\sum_{i=1}^{n+1} X_i = m+1) \\
&= k^{n+1}\binom{m}{n}(1-k)^{m-n} \text{ for } m \geq n.
\end{aligned}
$$

We now apply the large deviation results. Let $S_{n+1} = \sum_{i=1}^{n+1} X_i$. We need to consider

$$\begin{aligned}
\sum_{|h|>\delta x} P\{S_{n+1} = m+1\} &= P\Big\{\Big|S_{n+1} - 1 - \Big[\frac{n}{k}\Big]\Big| > \frac{n}{k}\delta\Big\} \\
&= P\Big\{S_{n+1} > 1 + \Big[\frac{n}{k}\Big] + \frac{n}{k}\delta\Big\} + P\Big\{S_{n+1} < 1 + \Big[\frac{n}{k}\Big] - \frac{n}{k}\delta\Big\}.
\end{aligned}$$

As before, we only proceed with the first term since it can be shown that the second term has a similar bound.

$$\begin{aligned}
& P\Big\{S_{n+1} > 1 + \Big[\frac{n}{k}\Big] + \frac{n}{k}\delta\Big\} \le P\Big\{S_{n+1} > \frac{n}{k}(1+\delta)\Big\} \\
=\ & P\Big\{S_{n+1} > \Big(\frac{n}{n+1}\Big)\frac{n+1}{k}(1+\delta)\Big\} \le P\Big\{S_{n+1} > \frac{n+1}{k}(1+\delta/2)\Big\}, \text{ for all } n > \frac{2+\delta}{\delta}, \\
=\ & P\Big\{S_{n+1} > (n+1)a\Big\}, \text{ where } a = \frac{1+\delta/2}{k}.
\end{aligned}$$

The moment generating function $\varphi(t) = Ee^{tX} = \frac{ke^t}{1-(1-k)e^t}$,
and $\varphi'(t) = \frac{(1-(1-k)e^t)ke^t + ke^t(1-k)e^t}{(1-(1-k)e^t)^2} = \frac{ke^t}{(1-(1-k)e^t)^2} \Rightarrow \frac{\varphi'(t)}{\varphi(t)} = \frac{1}{(1-(1-k)e^t)}$.
Solution of $\frac{\varphi'(\theta_a)}{\varphi(\theta_a)} = \frac{1}{(1-(1-k)e^{\theta_a})} = \frac{1+\delta/2}{k}$ gives $e^{\theta_a} = \frac{1-k+\delta/2}{(1+\delta/2)(1-k)}$. This gives us that $\theta_a = \log\Big(1 + k\frac{\delta/2}{(1+\delta/2)(1-k)}\Big)$, which is well defined. Then by the large deviation result, Theorem 5,

$$\begin{aligned}
& \frac{1}{n+1}\log P\{S_{n+1} < (n+1)\frac{1+\delta/2}{k}\} \\
\sim\ & -\frac{(1+\delta/2)}{k}\log\frac{(1-k+\delta/2)}{(1+\delta/2)(1-k)} \\
& + \log\{k\frac{(1-k+\delta/2)}{(1+\delta/2)(1-k)}/(1-(1-k)\frac{(1-k+\delta/2)}{(1+\delta/2)(1-k)})\} \\
\sim\ & -\frac{(1+\delta/2)}{k}\log\frac{(1-k+\delta/2)}{(1+\delta/2)(1-k)} + \log\frac{(1-k+\delta/2)}{1-k} \\
\sim\ & -\frac{(1+\delta/2)}{k}[\log\frac{(1-k+\delta/2)}{1-k} + \log\frac{1}{1+\delta/2}] + \log\frac{(1-k+\delta/2)}{1-k} \\
\sim\ & \frac{(1+\delta/2)}{k}\log(1+\delta/2) + (1 - \frac{1+\delta/2}{k})\log\frac{(1-k+\delta/2)}{1-k} \\
\sim\ & \frac{(1+\delta/2)}{k}\log(1+\delta/2) - \frac{1-k}{k}(\frac{1-k+\delta/2}{1-k})\log\frac{(1-k+\delta/2)}{1-k} \\
\sim\ & \frac{(1+\delta/2)}{k}\log(1+\delta/2) - \frac{1-k}{k}(1+\frac{\delta/2}{1-k})\log\{1+\frac{\delta/2}{1-k}\}.
\end{aligned}$$

We will once again identify the rate function $\gamma_3(\delta, k)$ in this case as

$$\gamma_3(\delta,k) = -\frac{(1+\delta/2)}{k}\log(1+\delta/2) + \frac{1-k}{k}(1+\frac{\delta/2}{1-k})\log(1+\frac{\delta/2}{1-k}) \text{ for } \delta > 0.$$

Lemma 9.4, p. 71 of [4] guarantees that $\gamma_3(\delta,k) > 0$. For the required assertion; the relation,

$$P\{S_{n+1} > \frac{(1+\delta)n}{k}\} = O(e^{-n\gamma}), \text{ where } \gamma = \gamma_3(\delta,k) > 0,$$

completes the proof.

∎

We now proceed to the remaining parts of Hardy's theorem. We quote the following theorem of Feller (p. 517, [5]) in order to generalize the next assertion.

Theorem 8 *(Feller [5]) Let $X, \{X_i\}_{i\geq 1}$ be a sequence of independent, identically distributed random variables with Ee^{tX} converges for $|t| \leq t_0$ and x varies with n in such a way that $x = o(n^{1/6})$. Let $S_n = X_1 + X_2 + \cdots + X_n$. Then*

$$\frac{1 - F_n(x)}{1 - \Phi(x)} \to 1, \text{ where } F_n(x) = P\Big(\frac{S_n - ES_n}{\sigma\sqrt{n}} \leq x\Big),$$

and $\Phi(x)$ is the standard normal distribution.

Remark 2 We now make some comments about the values of ξ and two related theorems.

1. The upper and the lower bounds of ξ, namely $\xi = \frac{1}{2}$ and $\xi = \frac{2}{3}$ in the next three corollaries are determined by theorems in probability theory.

2. The cases $\xi = \frac{1}{2}$ and $\xi = 1$ correspond to the central limit theorem (CLT) and the strong law of large numbers (SLLN) respectively, under some moment conditions on the sequence of random variable, (namely, finiteness of second moment for the CLT and the existence of mean for the SLLN). The probability distributions considered by these corollaries obviously have these conditions satisfied.

3. As stated in [5], the proof of Feller's Theorem 8 carries through without difficulty, if the condition $x = o(n^{1/6})$ is replaced by $x = o(n^{1/2})$. We require $\frac{1}{6} > \xi - \frac{1}{2} > 0$. This justifies the range of ξ, namely $\frac{2}{3} > \xi > \frac{1}{2}$.

4. According to the CLT, for a sequence of independent and identically distributed non-degenerate random variables $\{X_i\}_{i\geq 1}$ with $EX^2 < \infty$, we have

$$P\Big\{\frac{S_n - ES_n}{\sqrt{\mathrm{Var}(S_n)}} \leq x\Big\} \to \Phi(x),\ x \in \mathbb{R},$$

where $S_n = X_1 + X_2 + \cdots + X_n$, and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$. In other words,

$$P\Big\{\frac{S_n - n\mu}{\sqrt{n\sigma^2}} > x\Big\} \to 1 - \Phi(x).$$

5. SLLN gives

$$P\Big\{\frac{S_n}{n} \to^n \mu\Big\} = 1,$$

provided $\mu = EX_1$ exists.

Various other equivalent forms are available for $1 - \Phi(x)$ [5].

Lemma 1 *(Feller [5]) As $x \to \infty$,*

$$1 - \Phi(x) \sim \frac{1}{(2\pi)^{1/2} x} e^{\frac{-1}{2}x^2};$$

and more precisely, for every $x > 0$ the double inequality

$$\frac{1}{(2\pi)^{1/2}} e^{\frac{-1}{2}x^2} \{\frac{1}{x} - \frac{1}{x^3}\} < 1 - \Phi(x) < \frac{1}{(2\pi)^{1/2}} e^{\frac{-1}{2}x^2} \frac{1}{x} \text{ holds.}$$

We have three immediate corollaries accordingly. Note that Corollary 4 concerns the Borel matrix method instead of the Borel exponential method treated in Theorem 137(3), p. 200, [6].

Corollary 4 *Suppose that u_m are defined as in Corollary 1.*
If $m = M + h$, where $M = x$ and $\frac{1}{2} < \xi < \frac{2}{3}$, then

$$\sum_{|h| > x^{\xi}} u_m = O(e^{-x^{\eta}}), \text{ where } \eta \text{ is any number less than } 2\xi - 1.$$

Proof.

Let $S_n = X_1 + X_2 + \cdots + X_n$, where $X_i \sim^{\text{i. i. d.}}$ Pois(1), $x = n$ and $h = m - x$. Note that $\mu = \sigma^2 = 1$ so that $n\mu = x$. We now write

$$\sum_{|h| > x^{\xi}} u_m = P\{|S_n - x| > x^{\xi}\} = P\{S_n - x > x^{\xi}\} + P\{S_n - x < -x^{\xi}\}.$$

We will deal only with the first term as the second term has a similar bound. Hence

$$P\Big\{\frac{S_n - n\mu}{\sigma\sqrt{n}} > \frac{x - x + x^{\xi}}{\sqrt{x}}\Big\} \leq 1 - \Phi\Big(\frac{x^{\xi}}{\sqrt{x}}\Big)$$

by Theorem 8.

By Lemma 1,

$$\begin{aligned} P\Big\{\frac{S_n - n\mu}{\sigma\sqrt{n}} > x^{\xi - \frac{1}{2}}\Big\} &\sim \frac{1}{x'\sqrt{2\pi}} e^{-x'^2/2}, \text{ where } x' = x^{\xi - \frac{1}{2}} \\ &\sim \frac{1}{\sqrt{2\pi}}\Big(\frac{1}{x^{\xi - \frac{1}{2}}}\Big) e^{-\frac{1}{2}(x^{2\xi - 1})}. \end{aligned}$$

Since $\frac{1}{\sqrt{2\pi}}\Big(\frac{1}{x^{\xi - \frac{1}{2}}}\Big)$ is a decreasing function of x, it is bounded. Hence

$$\sum_{h > x^{\xi}} u_m = O(e^{-\frac{1}{2}x^{2\xi - 1}}).$$

A similar argument takes care of the other term.

∎

The following two corollaries follow similarly.

Corollary 5 *(Theorem 138(3), p. 201, [6]) Suppose that u_m are defined as in Corollary 2. If $m = M + h$, where $M = [x] = [\frac{n+1}{q+1}]$ and $\frac{1}{2} < \xi < \frac{2}{3}$, then*

$$\sum_{|h|>x^{\xi}} u_m = O(e^{-x^{\eta}}), \text{ where } \eta \text{ is any number less than } 2\xi - 1.$$

Corollary 6 *(Theorem 139(3), p. 201, [6]) Suppose that u_m are defined as in Corollary 3. If $m = M + h$, where $M = [x] = [\frac{n}{k}]$ and $\frac{1}{2} < \xi < \frac{2}{3}$, then*

$$\sum_{|h|>x^{\xi}} u_m = O(e^{-x^{\eta}}), \text{ where } \eta \text{ is any number less than } 2\xi - 1.$$

Tust as Theorem 8 carries through without difficulty, if the condition $x = o(n^{1/6})$ is replaced by $x = o(n^{1/2})$, we have:

Theorem 9 *(Feller [5]) Under the assumptions of Theorem 8, if the condition $x = o(n^{1/6})$ is replaced by $x = o(n^{1/2})$, then*

$$1 - F_n(x) \sim 1 - \Phi(x),$$

where $F_n(x) = P\Big(\frac{S_n - ES_n}{\sigma\sqrt{n}} \leq x\Big)$ and $\Phi(x)$ is the standard normal distribution.

The following three corollaries give Hardy's results. Once again, we give the result for the Borel matrix method instead of the Borel exponential method as given in Theorem 137(4), p. 200, [6].

Corollary 7 *Suppose that u_m are defined as in Corollary 1. If $m = M + h$, where $M = x$ and for every $\epsilon > 0$, there exist $x_0(\epsilon)$ and $\lambda_0(\epsilon) > 0$ such that*

$$\sum_{|h|>\lambda x^{1/2}} u_m < \epsilon, \text{ for all } x > x_0(\epsilon) \text{ and } \lambda > \lambda_0(\epsilon).$$

Proof.

Let $S_n = X_1 + X_2 + \cdots + X_n$ and let F be the distribution of $X_1 - 1$ with each $X_i \sim$ Pois(1). Let $x = n$ and $h = m - x$. We now write

$$\sum_{|h|>\lambda x^{1/2}} u_m = P\{|S_n - x| > \lambda x^{1/2}\} = P\{S_n - x > \lambda x^{1/2}\} + P\{S_n - x < -\lambda x^{1/2}\}.$$

As before, for the first term of the above

$$P\{S_n - x > \lambda(n)^{\frac{1}{2}}\} \leq P\{\frac{S_n - n}{\sqrt{n}} > \lambda\}.$$

Since $\lim_{\lambda\to\infty} 1 - \Phi(\lambda) = 0$, for every $\epsilon > 0$, there exists some $\lambda_0(\epsilon) > 0$ such that $1 - \Phi(\lambda) < \epsilon/2$ for all $\lambda \geq \lambda_0(\epsilon)$. We also have

$$\sum_{h>\lambda x^{1/2}} u_m \leq \sum_{h>\lambda_0(\epsilon)x^{1/2}} u_m \text{ for all } \lambda \geq \lambda_0(\epsilon).$$

Now from Theorem 9 for fixed $\lambda_0(\epsilon)$

$$\sum_{h>\lambda_0(\epsilon)x^{1/2}} u_m \sim 1-\Phi(\lambda_0(\epsilon)) \text{ for all large values of } n.$$

We now obtain, for any small $\epsilon > 0$ an $N(\epsilon)$ so that

$$\sum_{h>\lambda_0(\epsilon)x^{1/2}} u_m \leq (1+\epsilon/2)\Big\{1-\Phi(\lambda_0(\epsilon))\Big\} < \epsilon$$

for all $n \geq N(\epsilon)$ and $\lambda \geq \lambda_0(\epsilon)$. The above follows from

$$\Big|\frac{1-F_n(x)}{1-\Phi(x)} - 1\Big| < \epsilon/2 \ \Rightarrow\ 1-F_n(x) < (1-\Phi(x)) + \frac{(1-\Phi(x))}{2}\epsilon.$$

This yields that for every small $\epsilon > 0$, there exist some numbers $x_0(\epsilon)$ and $\lambda_0(\epsilon) > 0$ such that

$$\sum_{h>\lambda x^{1/2}} u_m < \epsilon$$

for all $x \geq x_0(\epsilon)$ and $\lambda \geq \lambda_0(\epsilon) > 0$, as required. A similar bound is obtained for the second term to complete the proof.

∎

The two corollaries given below have essentially the same proof.

Corollary 8 *(Theorem 138(4), p. 201, [6]) Suppose that u_m are defined as in Corollary 2. If $m = M+h$, where $M = [x] = [\frac{n+1}{q+1}]$ and for and every $\epsilon > 0$, there exist $x_0(\epsilon)$ and $\lambda_0(\epsilon) > 0$ such that*

$$\sum_{|h|>\lambda x^{1/2}} u_m < \epsilon, \textit{ for all } x > x_0(\epsilon) \textit{ and } \lambda > \lambda_0(\epsilon).$$

Corollary 9 *(Theorem 139(4), p. 201, [6]) Suppose that u_m are defined as in Corollary 3. If $m = M+h$, where $M = [x] = [\frac{n}{k}]$ and for every $\epsilon > 0$, there exist $x_0(\epsilon)$ and $\lambda_0(\epsilon) > 0$ such that*

$$\sum_{|h|>\lambda x^{1/2}} u_m < \epsilon, \textit{ for all } x > x_0(\epsilon) \textit{ and } \lambda > \lambda_0(\epsilon).$$

VII. LOCAL BEHAVIOR OF RANDOM-WALK METHODS

The succeeding theorems are some examples of local limit theorems, since they concern densities of the probability distributions, while the previous results provided limit theorems involving distribution functions.

The following approach makes use of two important local limit theorems. One is due to Petrov [11] and is called Petrov's approximation, and the other one is due to Richter [12].

Theorem 10 *(Richter [12]) Let $Y, \{Y_i\}_{i\geq 1}$ be a sequence of independent and identically distributed random variables with mean zero and positive second moment, which take* *values only in a certain arithmetic progression with probabilities* $p_k = P\{Y = a + kl\}$, l *the maximal span of the distribution,* k *an integer,* a *a fixed real number. Let us put*

$$P_n(k) = P\{\sum_{j=1}^{n} Y_j = an + kl\}.$$

Assume that there exists a positive number A *such that for every real number* s, $|s| < A$, Ee^{sY} *is convergent. If we put* $t = (an + kl)/(\sigma\sqrt{n})$, *then for* $t > 1$, $t = o(\sqrt{n})$ *as* $n \to \infty$,

$$\frac{\frac{\sigma\sqrt{n}}{l} P_n(k)}{\phi(t)} = e^{(t^3/\sqrt{n})\lambda((t/\sqrt{n}))}\Big[1 + O\Big(\frac{t}{\sqrt{n}}\Big)\Big],$$

where $\lambda(t)$ *is a power series converging for all sufficiently small values of* $|t|$ *and* $\phi(t)$ *is the standard normal density.*

We now prove Hardy's theorem for the random-walk method.

Theorem 11 *Let* $X, \{X_i\}_{i \geq 1}$ *be a sequence of independent and identically distributed non-negative random variables with positive variance and finite moment generating function. Then for* $|h| = |i - x| \leq x^{\xi}$ *for any fixed* ξ *with* $\frac{1}{2} < \xi < \frac{2}{3}$, *we have*

$$P\{S_n = i\} = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h| + 1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} \text{ as } n \to \infty,$$

where $t = \frac{\sqrt{\mu}}{\sigma}\frac{h}{\sqrt{x}}$, $x = n\mu$, $EX = \mu$ *and* $S_n = X_1 + X_2 + \cdots + X_n$.

Proof.

There are two cases. One is $|t| \leq 1$ and other is $|t| > 1$. We will apply Richter's theorem with $a = -EX$ and $l = 1$ for one of these two cases. First, we will consider the case $|t| \leq 1$. From Petrov's approximation [11] for any positive $k \geq 3$, (where k stands for assumed finiteness of the kth moments) we have

$$P\{S_n = i\} = \frac{\phi(t)}{\sigma\sqrt{n}} + \sum_{j=1}^{k-2} \frac{Q_j(t)\phi(t)}{n^{\frac{1}{2}+\frac{j}{2}}} + \frac{o(1)}{n^{\frac{1}{2}(k-1)}(1 + |t|^k)}$$

uniformly in i (or t) as $n \to \infty$, where $Q_j's$ are polynomials of degrees at most $3j$ depending only on the probability law P, (through its moments or cumulants) $t = \frac{i - n\mu}{\sigma\sqrt{n}}$, j being integer and ϕ is the standard normal density.

Since the distribution under consideration has at least third finite moment, substituting $k = 3$, we obtain

$$P\{S_n = i\} = \frac{\phi(t)}{\sigma\sqrt{n}} + \frac{Q_1(t)\phi(t)}{n} + \frac{o(1)}{n(1 + |t|^3)}.$$

For the appropriate constants A, B, C and D, this is precisely

$$P\{S_n = i\} = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + \frac{A + Bt + Ct^2 + Dt^3}{\sqrt{n}}\Big\} + \frac{o(1)}{n(1 + |t|^3)} \text{ as } n \to \infty.$$

The restriction $|h| = |i-x| \leq x^{\xi}$ for the fixed ξ with $\frac{1}{2} < \xi < \frac{2}{3}$ implies that $\frac{h}{\sqrt{x}} = o(x^{1/6})$, so that $h = o(x^{2/3})$. In view of these assumptions, we will show that the rate of $\frac{A}{\sqrt{n}} = o\Big(\frac{1}{x^{1/2}}\Big)$ is faster than that of $\frac{Bt}{\sqrt{n}}$, while the rate of $\frac{Ct^2}{\sqrt{n}}$ is faster than that of $\frac{Dt^3}{\sqrt{n}}$. This will be accomplished as follows;

$$\frac{Bt}{\sqrt{n}} = \frac{B'|h|}{x} = \frac{o(1)}{x^{1/3}} = o\Big(\frac{1}{x^{1/3}}\Big).$$

Hence the term that corresponds to the slower rate $\frac{Bt}{\sqrt{n}}$ is retained in the expression. Similarly

$$\frac{Ct^2}{\sqrt{n}} = \frac{C'|h^2|}{x^{3/2}} = \frac{o(1)}{x^{1/6}} = o\Big(\frac{1}{x^{1/6}}\Big),$$

where as

$$\frac{Dt^3}{\sqrt{n}} = \frac{D'|h|^3}{x^2} = o(1).$$

The contribution due to the term $\frac{Dt^3}{\sqrt{n}}$ will be retained as compared to $\frac{Ct^2}{\sqrt{n}}$. Taking the above into consideration, the expression now takes the form

$$P\{S_n = i\} = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} + \frac{o(1)}{n(1+|t|^3)} \text{ as } n \to \infty.$$

In order to prevent the term $O\Big(\frac{|h|}{x}\Big)$ from vanishing, it will be replaced by $O\Big(\frac{|h|+1}{x}\Big)$. This leads to the assertion

$$P\{S_n = i\} = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} + \frac{o(1)}{n(1+|t|^3)} \text{ as } n \to \infty,$$

where $t = \frac{\sqrt{\mu}}{\sigma}\frac{h}{\sqrt{x}}$, $x = n\mu$ and $EX = \mu$.

Now it remains to show that under condition $|t| \leq 1$, the last order condition can be absorbed by one of the other terms in the expansion. For this, we will consider

$$\frac{o(1)}{n(1+|t|^3)} \simeq \frac{o(1)}{x} = \frac{\phi(t)}{\sigma\sqrt{n}} \cdot \frac{\sigma\sqrt{n}}{\phi(t)}\frac{o(1)}{x} = \frac{\phi(t)}{\sigma\sqrt{n}}\frac{o(1)}{\sqrt{x}}$$

as $\phi(t)$ is bounded away from zero for $|t| \leq 1$. This order term, namely $\frac{o(1)}{\sqrt{x}}$ tends to zero faster than $\frac{|h|}{x} = \frac{o(1)}{x^{1/3}}$ and will be absorbed by $\frac{|h|+1}{x}$. Hence for $|t| \leq 1$, we obtain

$$P\{S_n = i\} = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} \text{ as } n \to \infty$$

as required.

To prove the assertion for the remaining case ($|t| > 1$), we apply Theorem 10 with $a = -EY$ and $l = 1$. The theorem is stated only for $t > 1$. The other case, namely $t < -1$ will be obtained by replacing $\{Y_i\}_{i\geq 1}$ by $\{-Y_i\}_{i\geq 1}$. Let us consider

$$P\{S_n = k\} = P\{\sum_{j=1}^{n} X_j - nEX = -nEX + kl\} = P\{\sum_{j=1}^{n}(X_j - EX) = na + kl\}.$$

The sequence of random variables $\{Y_j = X_j - EX\}_{j\geq 1}$ has mean zero, positive variance and finite moment generating function as required by Theorem 10. The order condition $\frac{h}{\sqrt{x}} = o(x^{1/6})$ will imply the condition $t = o(x^{1/2})$. Hence, by Theorem 10,

$$P\{S_n = k\} = \frac{\phi(t)}{\sigma\sqrt{n}} e^{(t^3/\sqrt{n})\lambda((t/\sqrt{n}))}\Big(1 + O\Big(\frac{t}{\sqrt{n}}\Big)\Big) \text{ as } n \to \infty.$$

Since $\frac{t^3}{\sqrt{n}} = O\Big(\frac{|h|^3}{x^2}\Big) = o(1)$, $\frac{t}{\sqrt{n}} = O\Big(\frac{1}{x^{1/3}}\Big)$ and $\lambda((t/\sqrt{n}))$ is convergent for all large values of n, we obtain

$$\begin{aligned} P\{S_n = k\} &= \frac{\phi(t)}{\sigma\sqrt{n}}\Big(1 + O\Big(\frac{|h|^3}{x^2}\Big)\Big)\Big(1 + O\Big(\frac{t}{\sqrt{n}}\Big)\Big) \\ &= \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} \text{ as } n \to \infty. \end{aligned}$$

In order to prevent the vanishing of the term $O\Big(\frac{|h|}{x}\Big)$, we replaced it by $O\Big(\frac{|h|+1}{x}\Big)$.

The assertion now is

$$P\{S_n = i\} = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} \text{ as } n \to \infty.$$

We complete the proof by combining the above two cases.

∎

Substituting for particular moments of the Borel matrix and Euler methods, two corresponding assertions now follow as stated in the corollaries. Since the Taylor method is not a member of the random-walk method, it requires additional work for proof of its corollary.

The Theorem 137(5), p. 200, [6] is in fact stated for the Borel exponential method. We will state it for the Borel Matrix method.

Corollary 10 *Suppose that u_m are defined as in Corollary 1. If $|h| \leq x^{\xi}$, then*

$$u_m = \frac{1}{\sqrt{2\pi x}} e^{-\frac{1}{2}(m-x)^2/x}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\}.$$

Proof.

Let $S_n = X_1 + X_2 + \cdots + X_n$. If each $X_i \sim$ Pois(1). Let $[x] = n$ and $h = m - n$. Note that $\mu = \sigma^2 = 1$ so that $n\mu = n$. We now write $t = \frac{m-n\mu}{\sigma\sqrt{n}}$. From Theorem 11, we have

$$P\{S_n = m\} = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} \text{ as } n \to \infty,$$

where $|h| = |m - n|$ and $[x] = n$.

This concludes the proof of the corollary.

∎

In the following two corollaries, we will use x instead of $[x]$ to address the corresponding probabilistic version of Theorem 138(5), p. 201 and Theorem 139(5), p. 201, [6].

Corollary 11 *Suppose that u_m are defined as in Corollary 2. If $|h| \le x^{\xi}$, then*

$$u_m = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\},$$

where $\sigma^2 = \frac{q}{(q+1)^2}$ and $|h| = |m - \frac{n+1}{q+1}|$.

Proof.

Let $S_n = X_1 + X_2 + \cdots + X_n$. We now consider the random variables defined as

$$X_i = \begin{cases} 1 & \text{with probability} \quad 1/(q+1),\ q > 0 \\ 0 & \text{with probability} \quad q/(q+1), \end{cases}$$

for $i = 1, 2, \cdots n$.
Now substitute $\mu = \frac{1}{(q+1)}$, $\sigma^2 = \frac{q}{(q+1)^2}$, and $t = \frac{m - \frac{n+1}{q+1}}{\sqrt{(n+1)\frac{q}{(q+1)^2}}}$ in Theorem 11 to get

$$u_m = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} \text{ as } n \to \infty,$$

where $|h| = |m - \frac{n+1}{q+1}|$ and $x = \frac{n+1}{q+1}$.
This concludes the proof of the corollary.
∎

Corollary 12 *Suppose that u_m are defined as in Corollary 3. If $|h| \le x^{\xi}$, then*

$$u_m = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\},$$

where $\sigma^2 = \frac{1}{k}$ and $|h| = |m - \frac{n}{k}|$.

Proof.

Let $X, X_0, X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables. Assume that X has a geometric distribution with parameter $0 < k < 1$. With $Y = X_0 - 1$, the nth row of the Taylor method is

$$P(Y + \sum_{i=1}^{n} X_i = m) = P(X_0 - 1 + \sum_{i=1}^{n} X_i = m) = P(\sum_{i=0}^{n} X_i = m + 1).$$

The t as defined in Theorem 11 now takes the form

$$\begin{aligned} t &= \frac{m + 1 - (n+1)\mu}{\sigma\sqrt{n+1}} \\ &= \sqrt{\frac{n}{n+1}}\frac{m - n\mu}{\sigma\sqrt{n}} - \sqrt{\frac{n}{n+1}}\frac{1-\mu}{\sigma\sqrt{n}} \\ &= o\Big(x^{1/6}\Big) - O\Big(\frac{1}{x^{1/2}}\Big) \\ &= o\Big(x^{1/6}\Big). \end{aligned}$$

This t has the right order condition as required by the theorem. Hence substituting $\mu = \frac{1}{k}$, $\sigma^2 = \frac{1-k}{k^2}$, and $t = \frac{m-\frac{n}{k}}{\sqrt{\frac{n(1-k)}{k^2}}}$, we get

$$u_m = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} \text{ as } n \to \infty,$$

where $|h| = |m - \frac{n}{k}|$ and $x = \frac{n}{k}$.

This concludes the proof of the corollary.

∎

VIII. GENERALIZATIONS FOR CONVOLUTION SUMMABILITY

The theory developed in the preceding sections can be generalized to independent, but not identically distributed of random variables. Indeed, every random-walk method can be obtained from a sequence $\{q\}$ and the convolution method. To start, we need a theorem from (p. 524, [5]).

Theorem 12 *(Feller [5]) Suppose there exists an interval $(-a, a)$ in which all the characteristic functions ω_k are analytic; that $E(|X_n|^3) \leq M\sigma_n^2$, where M is independent of n. If x and n vary so that $s_n \to \infty$ and $x = o(\sqrt{s_n})$, then*

$$\frac{1 - F_n(x)}{1 - \Phi(x)} \to 1 \text{ with an error } O(x^3/s_n) \text{ uniformly in } x > 1.$$

We now state and prove the theorem, which generalizes Hardy's theorem for convolution summability method.

Theorem 13 *Let $C = \{C_{n,k}\}_{n,k\geq 1}$ be a regular convolution summability matrix generated by an independent sequence Y, $\{X_i, i \geq 1\}$ with $E(e^{\theta Y}) < \infty$, and $\{X_i, i \geq 1\}$ of identically distributed integer-valued random variables with $E(e^{\theta X_i}) < \infty$ for all $i \geq 1$ and for some $\theta > 0$. The following assertions hold.*

1. *(a) If $\{q_j\}$ is unimodal and $\{p_j\}$ is strongly unimodal, then the entries of the nth row of the convolution method generated by the sequences $\{q_j\}$ and $\{p_j\}$ is unimodal for each n.*

   (b) If $\{q_j\}$ is unimodal and $p_j = 0$ for $j < 0$ and $j > d$ while $p_0 > 0$, $p_1 > 0$, $p_{d-1} > 0$, $p_d > 0$, then there exists an integer n_0 such that for $n > n_0$, the entries of the nth row of the convolution method generated by the sequences $\{q_j\}$ and $\{p_j\}$ is unimodal, where Y has the probability distribution $\{q_j\}_{i\geq 0}$ and $\{X_i\}_{i\geq 1}$ have probability distribution $\{p_j\}_{i\geq 0}$.

2. *If $k = [\mu_Y + n\mu_X] + h$ and $\delta > 0$, then*

$$\sum_{|h|>\delta x} C_{n,k} = O\Big(e^{-\gamma x}\Big), \text{ where } \gamma > 0.$$

3. *If $\frac{1}{2} < \xi < \frac{2}{3}$, then*

$$\sum_{|h|>x^{\xi}} C_{n,k} = O\Big(e^{-x^{\eta}}\Big), \text{ where } \eta \text{ is any number less than } 2\xi - 1.$$

4. *For every $\epsilon > 0$, there exist $x_0(\epsilon)$ and $\lambda_0(\epsilon) > 0$ such that*
$$\sum_{|h|>\lambda x^{\frac{1}{2}}} C_{n,k} < \epsilon, \text{ for all } x > x_0(\epsilon) \text{ and } \lambda > \lambda_0(\epsilon).$$

5. *If $|h| \leq x^{\xi}$ for $1/2 < \xi < 2/3$, then*
$$R_{n,k} = \sqrt{\left(\frac{C}{\pi x}\right)} e^{-Ch^2/x}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} \text{ as } t \to 0,$$
where C is a constant; $t = \frac{\sqrt{\mu}}{\sigma}\frac{h}{\sqrt{x}}$, $x = n\mu$ and $EX = \mu$, where R is the random-walk method generated by $\{X_i\}_{i\geq 1}$ and Y is as defined before. It is also assumed that $E(e^{Yt}) < \infty$ for $|t| \leq t_0$.

6. *The estimates (2) and (3) remain valid, if $C_{n,k}$ is multiplied by any fixed power of k.*

Proof.
(1). (a) A discrete distribution $\{p_j\}$ on integers is strongly unimodal if $\{p_j\} \star \{q_j\}$ is unimodal for any unimodal discrete distribution $\{q_j\}$. Hence $\{p_j\}^{\star n} \star \{q_j\}$ is unimodal for all values of n, provided $\{p_j\}$ is strongly unimodal while $\{q_j\}$ is unimodal.

(b) Theorem 1 of [10] guarantees that there exists an integer n_0 such that for all $n > n_0$, the $n-$fold convolution $\{p_j\}^{\star n}$ is strongly unimodal. If in addition $\{q_j\}$ is unimodal, then $\{p_j\}^{\star n} \star \{q_j\}$ is unimodal for all values of $n > n_0$. This completes the proof.
(2).

We begin by considering
$$\sum_{|h|>\delta x} C_{n,k} = P\Big\{\Big|\frac{Y + S_n - [\mu_Y + n\mu_X]}{\mu_Y + n\mu_X}\Big| > \delta\Big\}$$
$$= P\{Y + S_n - [\mu_Y + n\mu_X] \geq \delta(\mu_Y + n\mu_X)\} + P\{Y + S_n - [\mu_Y + n\mu_X] \leq -\delta(\mu_Y + n\mu_X)\}.$$
We will obtain the appropriate bound for the first term, and the same will be obtained for the second term using a similar technique. Consider
$$P\{Y + S_n - [\mu_Y + n\mu_X] \geq \delta(\mu_Y + n\mu_X)\} \leq P\{Y + S_n \geq \delta(\mu_Y + n\mu_X) + \mu_Y + n\mu_X - 1\}.$$
For some $\theta > 0$, we know that $Ee^{\theta X} < \infty$ and $Ee^{\theta Y} < \infty$. Apply Chebyshev's theorem to get
$$\begin{aligned} P\Big\{e^{\theta(Y+S_n)} \geq e^{\theta\{\delta(\mu_Y+n\mu_X)+\mu_Y+n\mu_X-1\}}\Big\} &\leq \frac{Ee^{\theta(Y+S_n)}}{e^{n\theta\mu_X(1+\delta)}e^{\theta(\delta\mu_Y+\mu_Y-1)}} \\ &= \frac{Ee^{\theta Y}\Big(Ee^{\theta X}\Big)^n}{e^{n\theta\mu_X(1+\delta)}e^{\theta(\delta\mu_Y+\mu_Y-1)}}. \end{aligned}$$
Letting $\phi(\theta) = Ee^{\theta X}$, we have
$$\begin{aligned} P\Big\{e^{\theta(Y+S_n)} \geq e^{\theta\{\delta(\mu_Y+n\mu_X)+\mu_Y+n\mu_X-1\}}\Big\} &\leq \frac{C(\theta)\Big(\phi(\theta)\Big)^n}{e^{n\theta\mu_X(1+\delta)}} \\ &= \frac{C(\theta)\Big(\phi(\theta)\Big)^n}{e^{na\theta}} \\ &= C(\theta)\exp\{-n(\theta a - \log\phi(\theta))\}, \end{aligned}$$

where $a = \mu_X(1+\delta) > \mu_X$ for $\delta > 0$ and $C(\theta)$ is a constant depending on $\theta > 0$. Lemma 9.4 of [4] proves that if $a > \mu$ and θ is small, then $a\theta - \log\phi(\theta) > 0$.

Since $\gamma(\theta) = \theta a - \log\phi(\theta) > 0$, we have

$$\sum_{h > \delta x} C_{n,k} = O\Big(e^{-n\gamma}\Big) \text{ for some } \gamma > 0.$$

To obtain a similar bound for the second term, we write

$$\begin{aligned} P\{Y + S_n - [\mu_Y + n\mu_X] &\leq -\delta(\mu_Y + n\mu_X)\} \\ &= P\{-Y - S_n \geq \delta(\mu_Y + n\mu_X) - [\mu_Y + n\mu_X]\} \\ &\leq P\{-Y - S_n \geq \delta(\mu_Y + n\mu_X) - (\mu_Y + n\mu_X)\}. \end{aligned}$$

Let $S_n^* = \sum_{i=1}^n (-X_i) = \sum_{i=1}^n X_i^*$ and $Y^* = -Y$, so that $EY^* = -\mu_Y$ and $EX^* = -\mu_X$. By writing in terms of Y^* and X^*, we obtain

$$P\{Y^* + S_n^* \geq \delta(\mu_Y + n\mu_X) - (\mu_Y + n\mu_X)\} = P\{Y^* + S_n^* \geq (\mu_{Y^*} + n\mu_{X^*})(1-\delta)\}.$$

As before, we now have by Chebyshev's theorem for some $\theta > 0$, So this last term is

$$\leq \frac{C'(\theta)\Big(\phi'(\theta)\Big)^n}{e^{\theta(1-\delta)(\mu_{Y^*} + n\mu_{X^*})}} = \frac{C'(\theta)\Big(\phi'(\theta)\Big)^n}{e^{na'\theta}} = C'(\theta)\exp\{-n(\theta a' - \log\phi'(\theta))\},$$

where $\phi'(\theta) = Ee^{\theta X^*}$, $a' = \mu_{X^*}(1-\delta) > \mu_{X^*}$ and $C'(\theta)$ is a constant depending on $\theta > 0$. As before, if we let $\gamma(\theta) = a'\theta - \log\phi'(\theta) > 0$ for $a' > \mu_{X^*}$ and θ is small.

Combining these two bounds, the assertion is

$$\sum_{|h| > \delta x} C_{n,k} = O\Big(e^{-n\gamma}\Big) \text{ where for some } \gamma > 0.$$

(3).

As computed before

$$\sum_{|h| > x^\xi} C_{n,k} = P\{|Y + S_n - \mu_Y - n\mu_X| > (\mu_Y + n\mu_X)^\xi\}$$

$$= P\{Y + S_n - \mu_Y - n\mu_X > (\mu_Y + n\mu_X)^\xi\} + P\{Y + S_n - \mu_Y - n\mu_X < -(\mu_Y + n\mu_X)^\xi\}.$$

Now consider the first term:

$$P\{Y + S_n - \mu_Y - n\mu_X > (\mu_Y + n\mu_X)^\xi\} = P\Big\{\frac{Y + S_n - \mu_Y - n\mu_X}{\sqrt{\sigma_Y^2 + n\sigma_X^2}} > \frac{(\mu_Y + n\mu_X)^\xi}{\sqrt{\sigma_Y^2 + n\sigma_X^2}}\Big\}$$

$$= P\Big\{\frac{Y + S_n - \mu_Y - n\mu_X}{\sqrt{\sigma_Y^2 + n\sigma_X^2}} > x'\Big\}, \text{ where } x' = \frac{(\mu_Y + n\mu_X)^\xi}{\sqrt{\sigma_Y^2 + n\sigma_X^2}}.$$

Let $s_n = \sqrt{\sigma_Y^2 + n\sigma_X^2}$ and $s_n \to^n \infty$. Then

$$x'/\sqrt{s_n} = \frac{(\mu_Y + n\mu_X)^\xi}{\sqrt{\sigma_Y^2 + n\sigma_X^2}} \cdot \frac{1}{(\sigma_Y^2 + n\sigma_X^2)^{\frac{1}{4}}} = \frac{(\mu_Y + n\mu_X)^\xi}{(\sigma_Y^2 + n\sigma_X^2)^{\frac{3}{4}}} \to 0 \text{ for } \xi < \frac{3}{4}.$$

We also require that $\xi > \frac{1}{2}$ in order to prevent x' itself tends to 0 or a constant. By Theorem 12, we obtain

$$P\{Y + S_n - \mu_Y - n\mu_X > (\mu_Y + n\mu_X)^\xi\} \sim 1 - \Phi(x')$$

with an error $O\left(x'^3/s_n\right)$ uniformly in $x' > 1$. That is,

$$\begin{aligned}
P\{Y + S_n - \mu_Y - n\mu_X > (\mu_Y + n\mu_X)^\xi\} &\sim \frac{1}{x'\sqrt{2\pi}} e^{-\frac{1}{2}x'^2} \\
&= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\sigma_Y^2 + n\sigma_X^2}}{(\mu_Y + n\mu_X)^\xi} e^{-\frac{1}{2}\left[\frac{(\mu_Y + n\mu_X)^\xi}{\sqrt{\sigma_Y^2 + n\sigma_X^2}}\right]^2} \\
&= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\sigma_Y^2 + n\sigma_X^2}}{(\mu_Y + n\mu_X)^\xi} e^{-\frac{1}{2}\frac{n^{2\xi}\left(\frac{\mu_Y}{n} + \mu_X\right)^{2\xi}}{n\left(\frac{\sigma_Y^2}{n} + \sigma_X^2\right)}} \\
&= O\left(e^{-n^\eta}\right),
\end{aligned}$$

where η is any number less than $2\xi - 1$. The second term has the same bound, thus proving (3).

(4).

We first consider, for $\lambda > 0$,

$$\begin{aligned}
\sum_{|h| > \lambda x^{\frac{1}{2}}} C_{n,k} &= P\{|Y + S_n - \mu_Y - n\mu_X| > \lambda(\mu_Y + n\mu_X)^{\frac{1}{2}}\} \\
&= P\{Y + S_n - \mu_Y - n\mu_X > \lambda(\mu_Y + n\mu_X)^{\frac{1}{2}}\} \\
&\quad + P\{Y + S_n - \mu_Y - n\mu_X < -\lambda(\mu_Y + n\mu_X)^{\frac{1}{2}}\}.
\end{aligned}$$

Begining with the first term, we see that

$$P\{Y + S_n - \mu_Y - n\mu_X > \lambda(\mu_Y + n\mu_X)^{\frac{1}{2}}\} = P\Big\{\frac{Y + S_n - \mu_Y - n\mu_X}{\sqrt{\sigma_Y^2 + n\sigma_X^2}} > \frac{\lambda(\mu_Y + n\mu_X)^{\frac{1}{2}}}{\sqrt{\sigma_Y^2 + n\sigma_X^2}}\Big\}.$$

Let $x' = \frac{(\mu_Y + n\mu_X)^{\frac{1}{2}}}{\sqrt{\sigma_Y^2 + n\sigma_X^2}}$. Now consider the following steps:

Step 1: Since

$$\lim_{n\to\infty} x' = \lim_{n\to\infty} \frac{(\mu_Y + n\mu_X)^{\frac{1}{2}}}{\sqrt{\sigma_Y^2 + n\sigma_X^2}} = \frac{\sqrt{\mu_X}}{\sigma_X} < \infty,$$

let $k = \min_n x'$.

Step 2: Since $1 - \Phi(k\lambda) \to 0$ as $\lambda \to \infty$; for every $\epsilon > 0$, there exists $\lambda_0(\epsilon)$ such that

$$1 - \Phi(k\lambda) < \frac{\epsilon}{2} \text{ for all } \lambda \geq \lambda_0(\epsilon) > 0.$$

In particular, we have $1 - \Phi(k\lambda_0(\epsilon)) < \frac{\epsilon}{2}$.

Step 3:

$$\sum_{h > \lambda x^{\frac{1}{2}}} C_{n,k} \leq \sum_{h > \lambda_0(\epsilon) x^{\frac{1}{2}}} C_{n,k} \text{ for all } \lambda \geq \lambda_0(\epsilon).$$

By the Central Limit Theorem, we now obtain

$$\begin{aligned}\sum_{h>\lambda_0(\epsilon)x^{\frac{1}{2}}} C_{n,k} &\leq 1-\Phi(\lambda_0(\epsilon)x')+\frac{\epsilon}{2} \text{ for all } n \geq N_0(\epsilon) \\ &\leq 1-\Phi(\lambda_0(\epsilon)k)+\frac{\epsilon}{2} < \epsilon \text{ for all } n \geq N_0(\epsilon) \text{ for all } \lambda \geq \lambda_0(\epsilon).\end{aligned}$$

Step 4: Let $x_0(\epsilon) = \mu_Y + N_0(\epsilon)\mu_X$. Now it follows that for every $\epsilon > 0$, there exist numbers $x_0(\epsilon)$ and $\lambda_0(\epsilon) > 0$ such that

$$\sum_{h>\lambda x^{\frac{1}{2}}} C_{n,k} \leq \sum_{h>\lambda_0(\epsilon)x^{\frac{1}{2}}} C_{n,k} < \epsilon$$

for all $x \geq x_0(\epsilon)$ and for all $\lambda \geq \lambda_0(\epsilon)$ as required. To finish, we carry out the calculations for the second term in a similar manner.
(5).

Let us consider $C_{n,k} = P\{Y + S_n = k\}$. We write

$$P\{Y+S_n=k\} = \sum_{j=0}^{\infty} q_j P\{S_n = k-j\},$$

where $\{q_j\}_{j\geq 1}$ is the probability density of the random variable Y. For $|k-\mu_Y-n\mu_X| \leq (\mu_Y+n\mu_X)^{\xi}$ with $1/2 < \xi < 2/3$,

$$\begin{aligned}P\{Y+S_n=k\} = &\sum_{\{j:|k-j-n\mu_x|\leq(n\mu_X)^{\xi}\}} q_j P\{S_n=k-j\}+ \\ &\sum_{\{j:|k-j-n\mu_x|>(n\mu_X)^{\xi}\}} q_j P\{S_n=k-j\}.\end{aligned}$$

Using the results of the previous section the Random-walk method, the first sum above is equal to

$$\sum_{j=0}^{\infty} q_j \Big[\frac{\phi(t)}{\sigma_X\sqrt{n}}\Big\{1+O\Big(\frac{|h|+1}{x}\Big)+O\Big(\frac{|h|^3}{x^2}\Big)\Big\}\Big], \text{ where } t=\frac{k-j-n\mu_X}{\sigma_X\sqrt{n}} \text{ and } x=n\mu_X.$$

We have $|k-j-n\mu_X| \leq (n\mu_X)^{\xi}$ and $k-j \geq 0$. For $|k-\mu_Y-n\mu_X| \leq (\mu_Y+n\mu_X)^{\xi'}$ for every $k \geq 0$ and $\frac{1}{2} < \xi' < \frac{2}{3}$, we obtain $|j-\mu_Y| \leq |k-j-n\mu_X| + |k-\mu_Y-n\mu_X| \leq (n\mu_X)^{\xi} + (\mu_Y+n\mu_X)^{\xi'} < 2(\mu_Y+n\mu_X)^{2/3}$.

Let $h = |k-j-n\mu_X| \leq |k-\mu_Y-n\mu_X| + |j-\mu_Y|$ and $h' = |k-\mu_Y-n\mu_X|$. We then have $|h| < |h'| + |j-\mu_Y|$ for all $j \geq 0$.

Consider

$$\sum_{j=0}^{\infty} q_j \Big[\frac{\phi(t)}{\sigma_X\sqrt{n}}\Big\{1+O\Big(\frac{|h|+1}{x}\Big)+O\Big(\frac{|h|^3}{x^2}\Big)\Big\}\Big]$$

$$\begin{aligned}= \sum_{j=0}^{\infty} q_j \Big[&\frac{1}{\sqrt{2\pi(\sigma_Y^2+n\sigma_X^2)}}\Big(\frac{\sigma_Y^2+n\sigma_X^2}{n\sigma_X^2}\Big)^{\frac{1}{2}} e^{-\frac{1}{2}\left(\frac{k-j-\mu_Y-n\mu_X+\mu_Y}{\sqrt{\sigma_Y^2+n\mu_X^2}}\right)^2\left(\frac{\sigma_Y^2+n\sigma_X^2}{n\sigma_X^2}\right)} \\ &\Big\{1+O\Big(\frac{|h'|+|j-\mu_Y|+1}{\mu_Y+n\mu_X}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)\end{aligned}$$

$$+O\Big(\frac{|h'|^3+|j-\mu_Y|^3+3|h'|^2|j-\mu_Y|+3|h'||j-\mu_Y|^2}{(\mu_Y+n\mu_X)^2}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)^2\Big\}\Big]$$

$$=\sum_{j=0}^{\infty}q_j\Big[\frac{1}{\sqrt{2\pi(\sigma_Y^2+n\sigma_X^2)}}\Big(\frac{\sigma_Y^2+n\sigma_X^2}{n\sigma_X^2}\Big)^{\frac{1}{2}}$$

$$e^{-\frac{1}{2}\Big(\frac{|k-\mu_Y-n\mu_X|^2+|j-\mu_Y|^2-2|k-\mu_Y-n\mu_X||j-\mu_Y|}{\sigma_Y^2+n\mu_X^2}\Big)\Big(\frac{\sigma_Y^2+n\sigma_X^2}{n\sigma_X^2}\Big)}$$

$$\Big\{1+O\Big(\frac{|h'|+|j-\mu_Y|+1}{\mu_Y+n\mu_X}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)$$

$$+O\Big(\frac{|h'|^3+|j-\mu_Y|^3+3|h'|^2|j-\mu_Y|+3|h'||j-\mu_Y|^2}{(\mu_Y+n\mu_X)^2}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)^2\Big\}\Big].$$

Since $|k-\mu_Y-n\mu_X|<(\mu_Y+n\mu_X)^{\xi'}=(x')^{\xi'}$ and noting $0\le|j-\mu_Y|\le j+\mu_Y$, we rewrite

$$=\sum_{j=0}^{\infty}q_j\Big[\frac{1}{\sqrt{2\pi(\sigma_Y^2+n\sigma_X^2)}}\Big(\frac{\sigma_Y^2+n\sigma_X^2}{n\sigma_X^2}\Big)^{\frac{1}{2}}$$

$$e^{-\frac{1}{2}\Big(\frac{|h'|^2+|k-\mu_Y-n\mu_X||j-\mu_Y|}{\sigma_Y^2+n\mu_X^2}\Big)\Big(\frac{\sigma_Y^2+n\sigma_X^2}{n\sigma_X^2}\Big)}$$

$$\Big\{1+O\Big(\frac{|h'|+|j-\mu_Y|+1}{\mu_Y+n\mu_X}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)$$

$$+O\Big(\frac{|h'|^3+|j-\mu_Y|^3+3|h'|^2|j-\mu_Y|+3|h'||j-\mu_Y|^2}{(\mu_Y+n\mu_X)^2}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)^2\Big\}\Big].$$

Since $\frac{|k-\mu_Y-n\mu_X||j-\mu_Y|}{\sigma_Y^2+n\mu_X^2}\le\beta j+p$ for some β and $p>0$, we obtain, taking $\sum_{j=0}^{\infty}q_j j^k<\infty$ for all $k\ge 1$ and $\sum_{j=0}^{\infty}q_j e^{\beta j}<\infty$ into consideration that

$$C_{n,k}=\sqrt{\Big(\frac{C}{\pi M}\Big)}e^{-Ch'^2/M}\Big\{1+\Big(\frac{|h'|+1}{x'}\Big)+\Big(\frac{|h'|^3}{x'^2}\Big)\Big\},\ \text{where } C \text{ is a constant.}$$

Note that the omitted terms are of following order, namely

$$O\Big(\frac{|j-\mu_Y|}{\mu_Y+n\mu_X}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)=o\Big(\frac{1}{x'^{1/3}}\Big),$$

$$O\Big(\frac{|j-\mu_Y|^3}{(\mu_Y+n\mu_X)^2}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)^2=o(1),$$

$$O\Big(\frac{|h'|^2|j-\mu_Y|}{(\mu_Y+n\mu_X)^2}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)^2=o(1),$$

$$O\Big(\frac{|h'||j-\mu_Y|^2}{(\mu_Y+n\mu_X)^2}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)^2=o(1)$$

and are faster than other terms present in the expression.

We will find an appropriate bound for the second sum, namely

$$\sum_{\{j:|k-j-n\mu_x|>(n\mu_X)^{\xi}\ \&\ |k-\mu_Y-n\mu_X|\le(\mu_Y+n\mu_X)^{\xi}\}}q_jP\{S_n=k-j\}.$$

Subject to the finiteness of the second moments of $\{X_i\}_{i\geq 1}$, this is equal to

$$\sum_{\{j:|k-j-n\mu_x|>(n\mu_X)^\xi \ \& \ |k-\mu_Y-n\mu_X|\leq(\mu_Y+n\mu_X)^\xi\}} q_j \frac{1}{\sqrt{2\pi n\sigma_X}} e^{\frac{-1}{2}(k-j-n\mu_X)^2/n\sigma_X^2} + o(1\sqrt{n}).$$

The last term of this is

$$o(1\sqrt{n}) = \frac{\phi(t')}{\sqrt{n\sigma_X^2+\sigma_Y^2}} \frac{\sqrt{n\sigma_X^2+\sigma_Y^2}}{\phi(t')} o(1\sqrt{n}) = \frac{\phi(t')}{\sqrt{n\sigma_X^2+\sigma_Y^2}} \frac{o(1)}{\phi(t')} = \frac{\phi(t')}{\sqrt{n\sigma_X^2+\sigma_Y^2}} o(1),$$

where $t' = \frac{k-n\mu_x\mu_Y}{\mu_Y+n\mu_X}$. We next show that $\phi(t') > 0$. But,

$$\phi(t') = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}(k-n\mu_X\mu_Y)^2/(n\sigma_X^2+\sigma_Y^2)} > \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}n^{2\xi-1}} > \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}n^\epsilon} > 0,$$

where $2\xi - 1 < \epsilon$ and $1/2 < \xi < 2/3$. The first term

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi(n\sigma_X^2+\mu_Y^2)}} \sqrt{\frac{n\sigma_X^2+\mu_Y^2}{n\sigma_X^2}} \sum_{\{j:|k-j-n\mu_x|>(n\mu_X)^\xi \ \& \ |k-\mu_Y-n\mu_X|\leq(\mu_Y+n\mu_X)^\xi\}} q_j \\
\times \ & e^{\frac{-1}{2}\frac{(k-n\mu_X-\mu_Y)^2}{(n\sigma_X^2+\sigma_Y^2)}\left(\frac{n\sigma_X^2+\sigma_Y^2}{n\sigma_X^2}\right)} e^{\frac{-1}{2}(j-\mu_Y)^2/n\sigma_X^2} e^{|k-n\mu_X-\mu_Y||j-\mu_Y|/n\sigma_X^2} \\
\leq \ & \frac{1}{\sqrt{2\pi(n\sigma_X^2+\mu_Y^2)}} e^{\frac{-1}{2}\frac{(k-n\mu_X-\mu_Y)^2}{(n\sigma_X^2+\sigma_Y^2)}} \sum_j q_j e^{(n\mu_X+\mu_Y)^{\xi-1}(j+\mu_Y)} \\
= \ & \frac{1}{\sqrt{2\pi(n\sigma_X^2+\mu_Y^2)}} e^{\frac{-1}{2}\frac{(k-n\mu_X-\mu_Y)^2}{(n\sigma_X^2+\sigma_Y^2)}} \sum_j q_j e^{\theta j},
\end{aligned}$$

where θ is small and $\theta \to 0$ as $n \to \infty$.

$$= \frac{1}{\sqrt{2\pi(n\sigma_X^2+\mu_Y^2)}} e^{\frac{-1}{2}\frac{(k-n\mu_X-\mu_Y)^2}{(n\sigma_X^2+\sigma_Y^2)}} O(1)$$

and this can be absorbed by the first expansion.

This completes the proof of (5).

(6).

For the first part of this assertion, we assume

$$\sum_{|h|>\delta x} C_{n,k} = P\Big\{\Big|\frac{Y+S_n-[\mu_Y+n\mu_X]}{\mu_Y+n\mu_X}\Big| > \delta\Big\} = O(e^{-n\gamma}) \text{ for some } \gamma > 0.$$

We want to show that

$$\sum_{|h|>\delta x} k^l C_{n,k} = O(e^{-n\alpha}) \text{ for fixed number } l \text{ and some } \alpha > 0.$$

Consider

$$E\{Z_n^l \chi_{A_n}\}; \text{ where } A_n = \{|Z_n - [\mu_Y + n\mu_X]| > \delta(\mu_Y + n\mu_X)\} \text{ and } Z_n = Y + S_n.$$

Now by the Cauchy-Schwarz inequality

$$E\{Z_n^l \chi_{A_n}\} \le \{E(Z_n^{2l})\}^{1/2}\{E(\chi_{A_n}^2)\}^{1/2}.$$

The necessary bounds for $\{E(Z_n^{2l})\}^{1/2}$ and $\{E(\chi_{A_n}^2)\}^{1/2}$ are now computed. As assumed, $\{E(\chi_{A_n}^2)\}^{1/2} = \{P(A_n)\}^{1/2} = O(e^{-n\gamma/2})$. By the strong law of large numbers,

$$\{E(Z_n^{2l})\}^{1/2} = O(n^{l/2}).$$

Therefore

$$E\{Z_n^l \chi_{A_n}\} \le O(n^l e^{-n\gamma/2}) = O(e^{-n\alpha}) \text{ for some } \alpha > 0.$$

This proves the first part of (6).

For the second part, we assume that

$$\sum_{|h|>x^\xi} C_{n,k} = P\Big\{\Big|Y + S_n - [\mu_Y + n\mu_X]\Big| > (\mu_Y + n\mu_X)^\xi\Big\} = O\Big(e^{-x^\eta}\Big),$$

where η is any number less than $2\xi - 1$. We want to show that

$$\sum_{|h|>x^\xi} k^l C_{n,k} = O(e^{-x^\alpha}) \text{ for fixed number } l,$$

where α is any number less than $2\xi - 1$. Consider

$$E\{Z_n^l \chi_{B_n}\}; \text{ where } B_n = \{|Y + S_n - [\mu_Y + n\mu_X]| > (\mu_Y + n\mu_X)^\xi\}$$

and $Z_n = Y + S_n$. Once again, by the Cauchy-Schwarz inequality

$$E\{Z_n^l \chi_{B_n}\} \le \{E(Z_n^{2l})\}^{1/2}\{E(\chi_{B_n}^2)\}^{1/2}.$$

The necessary bounds are $E(Z_n^{2l}) = O(n^{2l})$ and $E(\chi_{B_n}^2) = O\Big(e^{-n^\eta}\Big)$ for any $\eta < 2\xi - 1$ by assumption. Therefore, if $\frac{1}{2} < \xi < \frac{2}{3}$, then

$$E\{Z_n^l \chi_{B_n}\} = O\Big(e^{-n^\eta}\Big),$$

for any $\eta < 2\xi - 1$. This completes the proof of second part of (6).
∎

IX. ACKNOWLEDGEMENTS

The author wishes to thank the faculty and department of mathematics and computer sciences of the Kent State University, Kent, Ohio for all supports rendered to complete the relevent degree program. Professor M. Kazim Khan, under his direction the dissertation was undetaken, has helped in numerous ways. I am deeply indebted to him. The author also wishes to thank Professor Terrance J. Quinn of Texas A&M Int'l University for his comments and spending Sunday afternoons with us (in his own words). They immensely helped to improve this paper.

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September 10, 2000.

Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 1, July 2002, pp. 111–119.
Submitted: January 23, 2002. Published: July 1 2002.

ON ANALYTIC CONTINUATION OF SUMMABILITY TRANSFORMS

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Abstract. The question of possibility of extending the definition of an analytic function to a larger domain is the concept of analytic continuation. This will be the subject of this paper for various summability transforms of random variables.

1991 A.M.S. (MOS) Subject Classification Codes. 30, 40, 46, 60

Key Words and Phrases. Analytic Continuation, The Okada Theorem, Summability, Uniform Convergence, Random-walk Methods, Convolution Summability Methods

I. INTRODUCTION

The question of the possibility of extending the definition of an analytic function to a larger domain is the concept of analytic continuation. This will be the subject of the present paper.

At the outset, we recall that the series $\sum_{k=0}^{\infty} a_k z^k$ with $\{a_k\}_{k\geq 0}$ being constants and the positive radius of convergence, $R > 0$, converges for $|z| < R$, where it represents an analytic function. The series diverges for $|z| \geq R$ and so that the domain $|z| < R$ is the largest possible domain of convergence of the series. Over the domain $|z| < R$, the power series converges in general and hence represents an analytic function. In this connection, we now appeal to:

Theorem 1 (Markushevich, Theorem 6.1 [1]). *If $\{g_n(z)\}_{n=0}^{\infty}$ is a sequence of functions, each analytic on a domain D, and this sequence converges uniformly on compact subsets of D, then the limiting function $g(z)$ is analytic on D.*

In the event, the transformation to be chosen is reasonably well-behaved then the limiting function $g(z)$ agrees with $f(z)$ on $\{z : |z - z_0| < R\}$, and hence, $g(z)$ will yields the analytic continuation of $f(z)$ to the domain D. As an application of summability to analytic continuation, we consider the following theorem [1]. Before begin this task, we recall the definition of the Borel polygon, which plays a significant importance in the discussion to follow.

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Definition 1. Let f be analytic at the origin. If s is a singular point of f, then let $l(s)$ be the line perpendicular to the line segment $[0,s]$ passing through the point s, and let $H_{l(s)}$ be the open half-plane containing the origin with boundary $l(s)$. The Borel polygon with respect to f is the set

$$B(f) = \cap H_{l(s)},$$

where the intersection is taken over all singular points of f.

$B(f)$ is a domain, since it is the intersection of open, arc-wise connected set.

Example 1. The Borel polygon of $f(z) = 1/(1-z)$ is the half-plane $\{z : \mathrm{Re}(z) < 1\}$, since $z = 1$ is the only point of singularity of f.

II. SUMMABILITY ANALOGUES

The summability analogues of Theorem 1 for Borel exponential method is described in the following theorem.

Theorem 2 (Powel & Shah [1]). *Let $s_n(z) = \sum_{k=0}^{n} a_k z^k, z \neq 1$, where $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence $R > 0$. If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ on $S_R(0)$ and $z \in B(f)$, then*

$$\lim_{x\to\infty} e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} s_k(z) = f(z),$$

where $S_R(0)$ denotes a disc of radius R centered at 0 and $B(f)$ is the Borel polygon with respect to f.

The proof of this theorem is given for Borel exponential method. The same proof works for the Borel matrix method. The proofs for the Euler and the Taylor transformations can be obtained with a minor adjustment to the proof as follows;

$$\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (\frac{z}{t})^k = (\frac{pz}{t} + q)^n = (1 + p[\frac{z}{t} - 1])^n \simeq \exp\{np[\frac{z}{t} - 1]\}.$$

Much of the discussion will now revolve around the following types of summability methods. Let $Y, X_1, X_2, \cdots$ be a sequence of independent non-negative integer valued random variables such that Y has probability function q and the $X_i's$ are identically distributed with probability function p. Let $S_0 = Y$ and $S_n = Y + X_1 + \cdots + X_n$ for $n \geq 1$. Let $\{p_j\}_{j\geq 0}$ and $\{q_j\}_{j\geq 0}$ be the probability distributions of $X_1, X_2, \cdots$ and Y respectively. The n^{th} row k^{th} column entry of the convolution summability matrix C is the probability $C_{n,k} = P(S_n = k)$. The method C is regular if and only if $P(X_1 = 0) < 1$. Some classical summability methods are examples of the method C. For instance, when $Y = 0$ and $X_1 \sim \text{Binomial}(1,1)$, then C becomes the Euler method denoted by E_r. When $Y \sim X_1 \sim \text{Poisson}(1)$ we get the Borel matrix method. When $Y \sim \text{Geometric}(1-r)$ and $X_1 \sim Y + 1$, then we get the Taylor method. And when $Y \sim X_1 \sim \text{Geometric}(1-r)$ we get the Meyer-Kónig method. We shall call C a convolution method and when $Y = 0$ with probability 1, it is called the random-walk method. The method C can be extended easily to non-identically distributed random variables; however, it will serve our purpose adequately, as it is. Hence, it is suspected that the regular convolution summability matrix transformation $\{C_{n,k}\}_{n,k\geq 0}$ obeys this theorem. The regular convolution summability matrix referred to everywhere in this paper has the above construction with appropriate moment conditions.

Theorem 3. *Let $s_n(z) = \sum_{k=0}^{n} a_k z^k$, where $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence $R > 0$. If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ on $S_R(0)$, $z \in B(f)$, and $\{p_j\}_{j\geq 0}$ has the property that $\sum_{j=0}^{\infty} p_j \theta^j < \infty$ for $0 \leq \theta < R < \infty$, then*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} C_{n,k} s_k(z) = f(z).$$

Proof.

We proceed as in [1]. Let $z \in B(f)$. If $z = 0$, the result follows immediately. It is therefore assumed that $z \neq 0$. By lemma 6.4 of [1], we have that f is analytic inside and on the circle C having diameter $[0, z]$. Let Γ_1 and Γ be two circles, each concentric to C, where f is analytic on $\mathrm{int}(\Gamma_1)$, $C \subseteq \mathrm{int}(\Gamma)$, and $\Gamma \subseteq \mathrm{int}(\Gamma_1)$. Now

$$\begin{aligned}
s_n(z) &= \sum_{k=0}^{n} a_k z^k \\
&= \sum_{k=0}^{n} \Big(\frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t^{k+1}}dt\Big) z^k \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t}\Big(\sum_{k=0}^{n}\Big(\frac{z}{t}\Big)^k\Big)dt \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t}\Big[\frac{1-(z/t)^{n+1}}{1-(z/t)}\Big]dt \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\Big(1-\Big(\frac{z}{t}\Big)^{n+1}\Big)dt \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}dt - \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\Big(\frac{z}{t}\Big)^{n+1}dt \\
&= f(z) - \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\Big(\frac{z}{t}\Big)^{n+1}dt.
\end{aligned}$$

Taking the above into consideration, we consider

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=0}^{\infty} C_{n,k}s_k(z) &= \lim_{n\to\infty}\Big\{\sum_{k=0}^{\infty} C_{n,k}f(z) - \sum_{k=0}^{\infty} C_{n,k}\Big(\frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\Big(\frac{z}{t}\Big)^{k+1}dt\Big)\Big\} \\
&= f(z) - \lim_{n\to\infty}\frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\sum_{k=0}^{\infty} C_{n,k}\Big(\frac{z}{t}\Big)^{k+1}dt.
\end{aligned}$$

Thus in order to obtain the required assertion, we will show that

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} P\{S_n = k\}(\frac{z}{t})^{k+1} = 0.$$

This will be done in the following manner.

$$\sum_{k=0}^{\infty} P\{S_n = k\}(\frac{z}{t})^{k+1} = (\frac{z}{t})\sum_{k=0}^{\infty} P\{S_n = k\}(\frac{z}{t})^{k}.$$

The assumption $\sum_{j=0}^{\infty} p_j\theta^j < \infty$ for $0 \le \theta < R < \infty$, guarantees the convergence of the series.

$$\sum_{k=0}^{\infty} P\{S_n = k\}(\frac{z}{t})^{k+1} = (\frac{z}{t})E(\frac{z}{t})^{S_n}; \text{ by the definition of the expectation.}$$

$$= (\frac{z}{t})E\{(\frac{z}{t})^{Y}(\frac{z}{t})^{\sum_{i=1}^{n} X_i}\}$$

$$= (\frac{z}{t})\{E(\frac{z}{t})^{Y}\Pi_{i=1}^{n}E(\frac{z}{t})^{X_1}\}$$

as r. v.'s are independent and $X_1's$ are i. i. d..

$$= (\frac{z}{t})E\{(\frac{z}{t})^{Y}(E(\frac{z}{t})^{X_1})^n \to 0,$$

in view of the inclusion;

$$\{z : |z| < t\} \supseteq \{z : |z|^{X_i} < t^{X_1} \text{ a. s.}\} \supseteq \{z : E|\frac{z}{t}|^{X_1} < 1\};$$

which implies that $E(|\frac{z}{t}|^{X_1})^n \to 0$. This completes the proof of the theorem. □

We see that the convolution summability matrix transformation of the sequence of partial sums of $\sum_{k=0}^{\infty} a_k z^k$ gives us the analytical continuation of the power series $\sum_{k=0}^{\infty} a_k z^k$ to the Borel polygon $B(f)$ (since, $B(f)$ is a domain and $S_R(0) \subset B(f)$.) However, we are very much interested in obtaining a theorem in the following setting.

III. THE OKADA THEOREM

We now examine the above theorem in a more general setting, in particular; we will use a general matrix transformation of the sequence of partial sum of $\sum_{k=1}^{\infty} a_k z^k$, which has a positive radius of convergence.

As shown in [1], the Okada theorem holds in general for a regular infinite matrix provided that the three assumptions therein are satisfied. Condition (iii) of the Silverman-Toeplitz theorem need not hold. We will show that the convolution summability method satisfies the rigorous conditions stipulated in the Okada theorem. This is worth considering, as the convolution summability method includes Borel, Euler, Meyer-König and Taylor transformation methods and much more.

We will denote the complement of a set A of complex numbers by $C(A)$ and that of cA by $cA = \{cz : z \in A\}$.

Theorem 4 (Okada Theorem [1]). *Let $A = \{a_{n,k}\}$ be an infinite matrix of complex entries and denote the k^{th} partial sum of the geometric series, $\sum_{j=0}^{k} z^j$, by $s_k(z)$.*

Let $R' = \{z : \lim_{n\to\infty} \tau_n(z) := \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}s_k(z) = \frac{1}{(1-z)}\}$ and assume that $B' = \{z : |z| < 1\} \subseteq R', R'$ is open and $\lim_{n\to\infty} \tau_n(z) = \frac{1}{(1-z)}$ uniformly on compact subsets of R'. Let $R = \{\omega = 1/z : z \in C(R')\} \cup \{0\}$. Let D be a fixed domain, where $0 \in D$ and denote by $\mathcal{F}$, the collection of all functions analytic on D. For $f \in \mathcal{F}$ denote the power series representation of f about the origin by $\sum_{k=0}^{\infty} b_k(f)z^k$.

If $U(D) = \{z : \lim_{n\to\infty} \sigma_n^f(z) = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}(\sum_{j=0}^{k} b_j(f)z^j) = f(z)$ for all $f \in \mathcal{F}\}$, then

(1) **(i)** *$U(D) = \cap_{c\notin D} cR' = \{c : cR \subseteq D\}$ and*

(2) **(ii)** *$\lim_{n\to\infty} \sigma_n^f(z)$ converges uniformly on compact subsets of $U(D)$.*

Note 1. *The set $U(D)$ is some sense a set of $z's$, where the transformation reproduces the function f under consideration. Thus, the transform matrix behaves as an identity matrix.*

Theorem 5. *For the regular convolution summability transformation matrix given by $C = \{C_{n,k}\}_{n,k\geq 0}$ with $\sum_{j=0}^{\infty} p_j j < \infty$, we have*

(1) $B' = \{z : |z| < 1\} \subseteq R'$,
(2) R' *is open, and*
(3) $\lim_{n\to\infty} \tau_n(z) = \sum_{k=0}^{\infty} C_{n,k} s_k(z) = \frac{1}{(1-z)}$ *uniformly on compact subsets of* R', *where* $s_k(z) = \sum_{j=0}^{k} z^j$ *and* $R' = \{z : \lim_{n\to\infty} \tau_n(z) := \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} s_k(z) = \frac{1}{(1-z)}\}$.

Proof.
(1). For $B' \subseteq R'$, we ask, if $|z| < 1$, then $z \in R'$? We have two cases.
(a). The series

$$\tau_n(z) = \sum_{k=0}^{\infty} C_{n,k} s_k(z) = \sum_{k=0}^{\infty} P\{S_n = k\} \sum_{j=0}^{k} z^j, \text{ where } S_n = Y + \sum_{i=1}^{n} X_i,$$

converges. This will be shown as follows: Since $\sum_{k=0}^{\infty} P\{S_n = k\} = 1$, and for fixed $z \in B', z^{k+1} \to^k 0$, there exists $M > 0$ such that $\left|\frac{1-z^{k+1}}{1-z}\right| \leq M$. Hence,

$$\sum_{k=0}^{\infty} \left| P\{S_n = k\} \frac{1 - z^{k+1}}{1 - z} \right| \leq M \sum_{k=0}^{\infty} P\{S_n = k\} \leq M < \infty.$$

(b). Let $z \in R'$, and consider

$$\tau_n(z) = \sum_{k=0}^{\infty} C_{n,k} s_k(z) = \sum_{k=0}^{\infty} P\{S_n = k\} \sum_{j=0}^{k} z^j, \text{ where } S_n = Y + \sum_{i=1}^{n} X_i.$$

By the formula for the sum of the geometric series

$$\begin{aligned}
\tau_n(z) &= \sum_{k=0}^{\infty} P\{S_n = k\} \frac{1 - z^{k+1}}{1 - z} \\
\lim_{n\to\infty} \tau_n(z) &= \frac{1}{1 - z} \{\lim_{n\to\infty} [\sum_{k=0}^{\infty} P\{S_n = k\} - \sum_{k=0}^{\infty} P\{S_n = k\} z^{k+1}]\} \\
&= \frac{1}{1 - z} \{1 - z \lim_{n\to\infty} \sum_{k=0}^{\infty} P\{S_n = k\} z^k\} \\
&= \frac{1}{1 - z} \{1 - z \lim_{n\to\infty} E z^{S_n}\} \\
&= \frac{1}{1 - z} \{1 - \lim_{n\to\infty} z E z^Y E z^{\sum_{i=1}^{n} X_1}\}, \text{ since } Y, X_i \text{ are independent.} \\
&= \frac{1}{1 - z} \{1 - z E z^Y \lim_{n\to\infty} \Pi_{i=1}^{n} E z^{X_1}\} \text{ since } X_i \text{ are i.i.d..} \\
&= \frac{1}{1 - z} \{1 - z E z^Y \lim_{n\to\infty} (E z^{X_1})^n\} \\
&= \frac{1}{1 - z} - \frac{z}{1 - z} E z^Y \lim_{n\to\infty} (E z^{X_1})^n.
\end{aligned}$$

Now $\lim_{n\to\infty}(Ez^{X_1})^n \to 0$ if and only if $|Ez^{X_1}| < 1$. Thus, the right hand side tends to $\frac{1}{1-z}$ uniformly on $\{z : |z| < 1\}$, as the following inclusion holds. Since C is regular, $p_0 < 1$ and therefore $E(|z|^{X_1}) = |z|^0 p_0 + |z|p_1 + |z|^2 p_2 + \cdots < p_0 + p_1 + p_2 + \cdots = 1$. If C is not regular, then $E(|z|^{X_1}) = 1$ is possible whenever $|z| < 1$. That is,

$$\{z : |Ez^{X_1}| < 1\} \supseteq \{z : E|z|^{X_1} < 1\} \supseteq \{z : |z| < 1\}.$$

So, $B' \subseteq R'$.
(2). To show that R' is open, we consider $R' = \{z : |Ez^{X_1}| < 1\}$. In other words, we will show that

$$R' = \Big\{z : \Big|\sum_{k=0}^{\infty} p_k z^k\Big| < 1, \sum_{k=0}^{\infty} p_k = 1, p_k \geq 0 \text{ with } p_0 < 1\Big\} \text{ is open.}$$

$$\Longleftrightarrow$$

The function $g : z \to \sum_{k=0}^{\infty} p_k z^k$ is continuous at all $z \in R'$.
For this, let $g(z) = \sum_{k=0}^{\infty} p_k z^k$. The series converges at $z = 1$. For any other $z, z_0 \in B'$ and $|z - z_0| < \delta$, consider,

$$\begin{aligned}
|g(z) - g(z_0)| &= \Big|\sum_{j=0}^{\infty} p_j z^j - p_j z_0^j\Big| \\
&\leq \sum_{j=0}^{\infty} p_j |z^j - z_0^j| \\
&= \sum_{j=0}^{\infty} p_j |z - z_0||z^{j-1} + z^{j-2} z_0 + \cdots + z_0^{j-1}| \\
&< |z - z_0| \sum_{j=0}^{\infty} p_j j \text{ as } |z|, \text{ and } |z_0| < 1 \\
&= |z - z_0| EX.
\end{aligned}$$

Since $E|X| = \sum_{j=0}^{\infty} p_j j < \infty$, we have $|g(z) - g(z_)| < \epsilon$.
(3). Now, we need to show that $\lim_{n\to\infty} \tau_n(z) = \frac{1}{(1-z)}$ uniformly on a compact subset κ of R'. Is this convergence at a point $z \in R'$? For this, consider

$$\Big|\tau_n(z) - \frac{1}{(1-z)}\Big| = \frac{|z|}{|1-z|}\Big|\sum_{k=0}^{\infty} P\{S_n = k\} z^k\Big|.$$

Suppose $g : R' \to^{\text{into}} B'$ is continuous at every point. Let κ be a compact subset of R'. Let $z \in \kappa$ and $\epsilon > 0$. Now,

$$\Big|\tau_n(z) - \frac{1}{(1-z)}\Big| = q(z)|Ez^Y|\Big|\sum_{k=0}^{\infty} p_k z^k\Big|^n,$$

where $q(z) = \frac{|z|}{|1-z|}$. We need to show that for $z \in \kappa \subseteq R'$, $g(z) \in B'$. Then, there exists $\delta(z, \epsilon)$ such that $|z_0 - z| < \delta(z, \epsilon)$, $\Rightarrow |g(z) - g(z_0)| < \epsilon$ and $|g(z_0)| < 1$. Let

$\{B(z, \delta(z, \epsilon_z)\}$ is an open covering for κ. Then there exist $z_1, z_2, \cdots, z_l$ for some $l \in \aleph$ such that

$$\cup_{j=l}^{l} B(z_j, \delta(z_j, \epsilon_{z_j})) \supseteq \kappa.$$

Therefore, there exist $\eta_\kappa < 1$ such that

$$|g(z)| \leq \eta_\kappa < 1 \text{ for all } z \in \kappa.$$

So, η_κ is fixed number and $0 \leq \eta_\kappa < 1$. Let us assume that there exits N such that $n \geq N \Rightarrow \eta_\kappa^n < \epsilon/g(z)|Ez^Y|$. Therefore, $\forall z \in \kappa$ and $n \geq N$,

$$\begin{aligned}\left|\tau_n(z) - \frac{1}{(1-z)}\right| &= q(z)|Ez^Y||g(z)|^n \\ &\leq q(z)|Ez^Y|\eta_\kappa^n \\ &< q(z)|Ez^Y|\frac{\epsilon}{g(z)|Ez^Y|} \\ &= \epsilon.\end{aligned}$$

Which asserts that the function $g : R' \to B'$ converges uniformly on κ. This together with the above gives that $\lim_{n\to\infty} \tau_n(z) = \frac{1}{1-z}$ uniformly on compact subsets of R', $\{z : |z| < 1\} \subseteq R'$ and R' is open. $\square$

Remark 1. As the main conditions of the theorem hold for convolution summability method, the conclusions of the Okada theorem follows immediately.

We now restate the Okada theorem for the convolution summability transformation matrix.

Theorem 6. *Let $C = \{C_{n,k}\}_{n,k\geq 0}$ be a regular convolution summability transformation matrix with $\sum_{j=o}^{\infty} p_j\theta^j < \infty$ for $0 \leq \theta < \infty$, and denote the k^{th} partial sum of the geometric series, $\sum_{j=0}^{k} z^j$, by $s_k(z)$.*

Let $R' = \{z : \lim_{n\to\infty} \tau_n(z) := \lim_{n\to\infty} \sum_{k=0}^{\infty} Ca_{n,k}s_k(z) = \frac{1}{1-z}\}$, and $R = \{\omega = 1/z : z \in C(R')\} \cup \{0\}$. Let D be a fixed domain, where $0 \in D$ and denote by $\mathcal{F}$, the collection of all functions analytic on D. For $f \in \mathcal{F}$ denote the power series representation of f about the origin by $\sum_{k=0}^{\infty} b_k(f)z^k$. If

$$U(D) = \{z : \lim_{n\to\infty} \sigma_n^f(z) = \lim_{n\to\infty} \sum_{k=0}^{\infty} C_{n,k}(\sum_{j=0}^{k} b_j(f)z^j) = f(z) \text{ for all } f \in \mathcal{F}\},$$

then

(1) **(i)** *$U(D) = \cap_{c\notin D} cR' = \{c : cR \subseteq D\}$ and*
(2) **(ii)** *$\lim_{n\to\infty} \sigma_n^f(z)$ converges uniformly on compact subsets of $U(D)$.*

Proof.
The proof follows from the Okada theorem as the three assumptions now hold true for a regular Convolution summability transformation matrix. $\square$

In Theorem 5, we have shown that the set R' is open. We will now investigate how does this set R' look like for a few known summability methods. It is also noted that the openedness of R' depends only on the random sequence $\{X_i\}_{i\geq 0}$.

Are there interesting cases ? For this, we consider some known examples of this method.

(1) Borel exponential method with $\lambda > 0$; Consider $Ez^X = \sum_{m=0}^{\infty} \frac{e^{-\lambda}\lambda^m}{m!} z^m = e^{-\lambda}\sum_{m=0}^{\infty} \frac{(z\lambda)^m}{m!} = e^{-\lambda}e^{z\lambda} = e^{-\lambda(1-z)} = e^{-\lambda(1-x-iy)}$. Thus, $|Ez^X| < 1 \Rightarrow x < 1$ as $\lambda > 0$. Hence, $R' = \{z : \mathrm{Re}(z) < 1\}$.

(2) Euler method with $0 < p < 1$; Consider $Ez^X = z^0 p + z^1(1-p) = p + (x+iy)(1-p) = p + x(1-p) + iy(1-p)$.
Since $|Ez^X| = \sqrt{(p + x(1-p))^2 + y^2(1-p)^2}$, we obtain $\left(x + \frac{p}{1-p}\right)^2 + y^2 < \frac{1}{(1-p)^2}$. The set R' is interior of a circle of radius $\frac{1}{1-p}$ and center at $\left(-\frac{p}{1-p}, 0\right)$.

(3) Meyer-König method with $0 < k < 1$; Since $Ez^X = \sum_{m=0}^{\infty} z^m p(1-p)^{m-1} = \left(\frac{p}{1-p}\right)\frac{1}{1-z(1-p)}$ for $|z| < 1$,

$$Ez^X = \left(\frac{p}{1-p}\right)\frac{1}{1-(x+iy)(1-p)} = \left(\frac{p}{1-p}\right)\frac{1 - x(1-p) + iy(1-p)}{[1-x(1-p)]^2 + y^2(1-p)^2}.$$

The condition $|Ez^X| < 1$ gives

$$\left(x - \frac{1}{1-p}\right)^2 + y^2 > \frac{p^2}{(1-p)^4}.$$

Hence R' is the exterior of a circle of radius $\frac{p}{(1-p)^2}$ and center at $\left(\frac{1}{1-p}, 0\right)$.

(4) Taylor method with $0 < r < 1$; Since $Ez^X = \sum_{m=1}^{\infty} z^m p(1-p)^{m-1} = \frac{p}{(1-p)}\frac{z(1-p)}{1-z(1-p)}$, the condition $|Ez^X| < 1$ gives

$$\left(x - \frac{1-p}{1-2p}\right)^2 + y^2 < \frac{p^2}{(1-2p)^2}.$$

Hence R' is the interior of a circle of radius $\frac{p}{(1-2p)}$ and center at $\left(\frac{1-p}{1-2p}, 0\right)$.

This shows that R' is not necessarily the same set for every member of the convolution method, but the interior of a circle of a positive radius that may be infinite and the center is on the real axis. Based on these observations, the following proposition is appropriate.

Proposition 1. *The condition* $\left|Ez^X\right| < 1$ *in the proof of Theorem 5, corresponds to an interior or exterior of a circle of a positive radius* r > 0, *possibly infinite and the center is on the real axis. If* $\beta > 1$, *then* $\left|Ez^X\right| < 1$ *is an interior of a circle, and if* $\beta < 1$, *then* $\left|Ez^X\right| < 1$ *is an exterior of a circle, where* $\sum_{k=0}^{\infty} z^k \sum_{j=k}^{\infty} p_j = \beta < \infty$.

Proof.

To prove the assertion, we will explore the condition $\Big|Ez^X\Big| < 1$.

$$\begin{aligned}
\Big|Ez^X\Big| &= \Big|\sum_{j=0}^{\infty} p_j z^j\Big| \\
&= \Big|\sum_{j=0}^{\infty} p_j \Big\{\sum_{k=0}^{j}(z^k - z^{k-1}) + \frac{1}{z}\Big\}\Big| \\
&= \Big|\sum_{k=0}^{\infty}(z^k - z^{k-1}) \sum_{j=k}^{\infty} p_j + \frac{1}{z}\Big| \\
&= \Big|\Big(1 - \frac{1}{z}\Big) \sum_{k=0}^{\infty} z^k \sum_{j=k}^{\infty} p_j + \frac{1}{z}\Big|.
\end{aligned}$$

We show that the sum $\sum_{k=0}^{\infty} z^k \sum_{j=k}^{\infty} p_j$, by the ratio test is absolutely convergent. If the sum is $\sum_{k=0}^{\infty} a_k z^k$, then $\Big|\frac{a_{k+1}}{a_k}\Big| = \Big(\sum_{j=k+1}^{\infty} p_j / \sum_{j=k}^{\infty} p_j\Big)|z| < 1$ as z is chosen to be $|z| < 1$. This means that the series converges for all values of z such that $|z| < 1$. Thus,

$$\begin{aligned}
\Big|Ez^X\Big| &= \Big|\Big(1 - \frac{1}{z}\Big)\beta + \frac{1}{z}\Big| \\
&= \Big|\beta\Big\{1 + \frac{\gamma}{z}\Big\}\Big| \text{ where, } \gamma = \Big(\frac{1-\beta}{\beta}\Big) \\
&= \Big|\beta\Big\{\frac{[(x+\gamma)+iy][x-iy]}{x^2+y^2}\Big\}\Big|.
\end{aligned}$$

Now, $\Big|Ez^X\Big| < 1$, implies

$$\begin{aligned}
\beta^2\Big\{[x(x+\gamma)+y^2]^2 + [xy - y(x+\gamma)]^2\Big\} &< (x^2+y^2)^2 \\
\beta^2\Big\{x^2 + y^2 + 2x\gamma + \gamma^2\Big\} &< (x^2+y^2) \\
\Big(x - \frac{\beta}{1+\beta}\Big)^2 + y^2 &< \frac{1}{(1+\beta)^2}.
\end{aligned}$$

Thus, the latter inequality concludes the assertion as it corresponds to a circle of a radius $\frac{1}{1+\beta}$, centered at $\Big(\frac{\beta}{1+\beta}, 0\Big)$. $\square$

IV. ACKNOWLEDGEMENTS

The author wishes to thank Professor M. Kazim Khan of the Kent State University for proposing this topic for further exploration with some pertinent insights to the problem in hand, and Professor Terrance J. Quinn, formally of Texas A&M Int'l University for many discussions over this topic, which eventually motivated to realize that there are more to this interesting topic. His patience for some of the questions is very appreciated. His comments and spending Sunday afternoons with us (in his own words) immensely helped to improve this paper.

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Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 1, July 2002, pp. 120–125.
Submitted: November 22, 2001 . Published: July 1 2002.

A COMMON FIXED POINT THEOREM FOR NEAR-CONTRACTIVE MAPPINGS

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Abstract. In this work we establish a common fixed point theorem in complete metric spaces which generalizes some results in [1] and [2].

1991 A.M.S. (MOS) Subject Classification Codes. 47H10.

Key Words and Phrases. Common fixed points, complete metric speces, generalized contractive conditions.

1 Preliminaries

In 1984, M. S. khan [1] has established fixed point theorem for self-maps of complete metric spaces by altering the distances between the points with the use of a function $\phi : [0,\infty) \to [0,\infty)$ satisfying the following properties:

$$(i) \quad \phi : [0,\infty) \to [0,\infty) \quad \text{is continuous and stricly increasing in} \quad \Re^+,$$
$$(ii) \quad \phi(t) = 0 \quad \text{if} \quad t = 0.$$

They proved the following Theorem:

Theorem 1.1. *Let T be a self-map of a complete metric space (X,d) and ϕ satisfying (i) and (ii). Furthermore, let a,b,c be three decreasing functions from $(0,\infty)$ into $[0,1)$ such that*

$$a(t) + 2b(t) + c(t) < 1$$

for every $t > 0$.
Suppose that T satisfies the following condition:

$$(A) \quad \phi(d(Tx,Ty)) \le a(d(x,y))\phi(d(x,y)) + b(d(x,y))\Big[\phi(d(x,Tx)) + \phi(d(y,Ty))\Big] + c(d(x,y)) \min\Big\{\phi(d(x,Ty)), \phi(d(y,Tx))\Big\}$$

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Typeset by $\mathcal{AMS}$-TEX

where $x, y \in X$ and $x \neq y$. Then T has a unique fixed point.

In 1996 R.A.RASHWAN AND A.M.SADEEK [2] have established a common fixed point theorem for a apair of mappings satisfying the contractive conditions corresponding to (A). They proved the following Theorem 1.2 which generalizes Theorem 1.1 :

Theorem 1.2. *Let S and T be self-mappings on a complete metric space (X, d) and ϕ stisfying the conditions (i) and (ii). Furthermore, let a, b, c be three decreasing functoins from $[0, \infty)$ into $[0, 1)$ such that*

$$(B.1) \quad a(t) + 2b(t) + c(t) < 1$$

for every $t > 0$. Suppose that S and T satisfy the following condition:

$$(B.2) \quad \phi(d(Sx, Ty)) \leq a(d(x, y))\phi(d(x, y)) + b(d(x, y))\Big[\phi(d(x, Sx)) + \phi(d(y, Ty))\Big] \\ + c(d(x, y)) \min\Big\{\phi(d(x, Ty)), \phi(d(y, Sx))\Big\}$$

where $x, y \in X$ and $x \neq y$. Then T and S have a unique common fixed point in X

In this works we shall extend this last theorem by using a much extensive hypothesis than (B. 1) and a, b and c are three decreasing functions.
We shall use the following definition

Definition 1.1. *A function $\psi : [0, +\infty[\longrightarrow [0, 1[$ will be said to have property (P) if for every $t > 0$, there exists $\eta(t) > 0$ such that*

$$0 \leq s - t < \eta(t) \quad \Longrightarrow \quad \psi(s) \leq \gamma(t) < 1.$$

2 Main Results

Theorem 2.1. *Let T and S be a self-mappings on a complete metric space (X, d) and ϕ satisfying (i) and (ii). Furthermore, let a, b and c be three functions from $[0, \infty)$ into $[0, 1)$ satisfying at least one of the following conditions:*

(D_1) $\quad a + 2b$ *has property (P) and* $\quad a + c : (0, \infty) \to [0, 1)$ *is decreasing function.*

(D_2) $\quad a + 2b + c : (0, \infty) \to [0, 1)$ *is decreasing function.*

Suppose that T and S satisfy the condition $(B.2)$. Then T and S have a unique common fixed point in X.

Proof. Suppose first that a, b and c satisfy the condition (D_2) then a + 2b + c has property (P) which implies $a + 2b$ has property (P).
Let x_0 be some point in X, define:

$$x_{2n+2} = Sx_{2n+1}, \qquad n = 0, 1, 2, ...$$
$$x_{2n+1} = Tx_{2n}, \qquad n = 0, 1, 2, ...$$

Let $\tau_n = d(x_n, x_{n+1})$. We may assume that $\tau_n > 0$ for every n. Now for an even integer $n > 0$, We have

$$\phi(\tau_n) = \phi(d(x_n, x_{n+1})) \\ = \phi(d(S^{n-1}x, T^n x)) \leq A_1 + A_2 + A_3$$

Where

$$A_1 = a(d(x_{n-1}, x_n))\phi(d(x_{n-1}, x_n)),$$
$$A_2 = b(d(x_{n-1}, x_n))[\phi(d(x_{n-1}, Sx_{n-1})) + \phi(d(x_n, Tx_n))],$$
$$A_3 = c(d(x_{n-1}, x_n))min\{\phi(d(x_{n-1}, Tx_n)), \phi(d(x_n, Sx_{n-1}))\}.$$

So

$$\phi(\tau_n) \leq a(\tau_{n-1})\phi(\tau_{n-1}) + b(\tau_{n-1})[\phi(\tau_{n-1}) + \phi(\tau_n)],$$

Hence we have

$$\phi(\tau_n) \leq \frac{a(\tau_{n-1}) + b(\tau_{n-1})}{1 - b(\tau_{n-1})}\phi(\tau_{n-1}) < \phi(\tau_{n-1}) \quad (1)$$

Similary for an odd integer n one can show that n satisfies to (1). Then for all integer $n \geq 1$ we have:

$$\phi(\tau_n) \leq \frac{a(\tau_{n-1}) + b(\tau_{n-1})}{1 - b(\tau_{n-1})}\phi(\tau_{n-1}) < \phi(\tau_{n-1}) \quad (2)$$

Since ϕ is incressing, $\{\tau_n\}$ is a decreasing sequence. Let $lim_n \tau_n = \tau$, assume that $\tau > 0$. Since $a + 2b$ has property (P), then for $0 \leq$ s -$\tau < \eta(\tau)$,

$$a(s) + 2b(s) \leq \gamma(\tau) < 1,$$

it follows that for these s

$$\frac{a(s) + b(s)}{1 - b(s)} \leq \frac{\gamma(\tau) - b(s)}{1 - b(s)} \leq \gamma(\tau) \quad (3) \text{ too}$$

Since

$$\tau_n \geq \tau, \quad \lim_n \tau_n = \tau \qquad \text{and} \quad \lim_n \frac{\phi(\tau_n)}{\phi(\tau_{n-1})} = 1,$$

there exists an integer $p \geq 1$ such that

$$0 \leq \tau_{p-1} - \tau < \eta(\tau) \qquad (4) \text{ while} \quad \gamma(\tau) < \frac{\phi(\tau_p)}{\phi(\tau_{p-1})},$$

the last inequality with (2) imply that

$$\gamma(\tau) < \frac{a(\tau_{p-1}) + b(\tau_{p-1})}{1 - b(\tau_{p-1})}$$

and (3) with (4) imply that

$$\frac{a(\tau_{p-1}) + b(\tau_{p-1})}{1 - b(\tau_{p-1})} \leq \gamma(\tau),$$

which is a contradiction. Consequently $\tau = 0$. Now, we wish to show that $\{x_n\}$ is a cauchy sequence since $\tau = 0$ it suffices to prove that $\{x_{2n}\}$ is a cauchy sequence we proceed by contraction. Then there exists $\varepsilon > 0$, and two sequences of natural numbers $\{2m(i)\}, \{2n(i)\}$ such that

$$2n(i) > 2m(i) \geq 2i \qquad \text{and} \quad d(x_{2n(i)}, x_{2m(i)}) > \varepsilon \quad (5)$$

For each even integer $2i$, let $2m(i)$ be the least even a integer exceeding $2n(i)$ and satisfy (5) Then.

$$d(x_{2n(i)}, x_{2m(i)}) > \varepsilon \qquad \text{and} \quad d(x_{2n(i)-2}, x_{2m(i)}) \leq \varepsilon$$

Then we have

$$\begin{aligned} \varepsilon < d_{2i} &= d(x_{2n(i)}, x_{2m(i)}) \\ &\leq d(x_{2n(i)}, x_{2n(i)-1}) + d(x_{2n(i)-1}, x_{2n(i)-2}) + d(x_{2n(i)-2}, x_{m(i)}) \\ &\leq \tau_{2n(i)-1} + \tau_{2n(i)-2} + \varepsilon, \end{aligned}$$

since $\{\tau_n\}$ converges to 0, $d_{2i} \to \varepsilon$. Futhermore, by triangler inequality, it follows that

$$d_{2i} - \tau_{2m(i)} \leq d(x_{2n(i)}, x_{2m(i)+1}) \leq d_{2i} + \tau_{2m(i)}$$

and therefor the sequence $\{d(x_{2n(i)}, x_{2m(i)+1})\}$ converges to ε. Moreover, by triangle inquality we have:

$$d_{2i} - \tau_{2n(i)-1} \leq d(x_{2n(i)-1}, x_{2m(i)}) \leq d_{2i} + \tau_{2n(i)-1}$$

Thus

$$d(x_{2n(i)-1}, x_{2m(i)}) \to \varepsilon \qquad \text{as} \quad i \to \infty.$$

From (B. 2), we also deduce :

$$\begin{aligned} \phi(d(x_{2n(i)}, x_{2m(i)+1})) &= \phi(d(Sx_{2n(i)-1}, Tx_{2m(i)})) \\ &\leq C_1 + C_2 + C_3 \\ &\leq C_4(D_j) + C_5(D_j) + C_6(D_j) \quad (6.j) \quad j = 1, 2 \end{aligned}$$

Where

$$\begin{aligned} C_1 &= a(d(x_{2n(i)-1}, x_{2m(i)})\phi(d(x_{2n(i)-1}, x_{2m(i)}), \\ C_2 &= b(d(x_{2n(i)-1}, x_{2m(i)}))[\phi(\tau_{2n(i)-1}) + \phi(\tau_{2m(i)})], \\ C_3 &= c(d(x_{2n(i)-1}, x_{2m(i)}))min\{\phi(d(x_{2n(i)}, x_{2m(i)}), \phi(d(x_{2n(i)-1}, x_{2m(i)+1}))\}, \\ C_4(D_1) &= a(d(x_{2n(i)-1}, x_{2m(i)})\phi(d_{2i} + \tau_{2n(i)-1}), \\ C_5(D_1) &= [\phi(\tau_{2n(i)-1}) + \phi(\tau_{2m(i)})], \\ C_6(D_1) &= c(d(x_{2n(i)-1}, x_{2m(i)})\phi(d_{2i} + \tau_{2n(i)-1}), \end{aligned}$$

and

$$\begin{aligned} C_4(D_2) &= a(d(x_{2n(i)-1}, x_{2m(i)})\phi(d_{2i} + \tau_{2n(i)-1} + \tau_{2m(i)}), \\ C_5(D_2) &= 2b(d(x_{2n(i)-1}, x_{2m(i)})\phi(d_{2i} + \tau_{2n(i)-1} + \tau_{2m(i)}), \\ C_6(D_2) &= c(d(x_{2n(i)-1}, x_{2m(i)})(\phi(d_{2i} + \tau_{2n(i)-1}) + \tau_{2m(i)}). \end{aligned}$$

Since $d(x_{2n(i)-1}, x_{2m(i)}) \to \varepsilon$ as $i \to \infty$, there exists an integer $q \geq 1$ such that

$$d(x_{2n(i)-1}, x_{2m(i)}) > \frac{\varepsilon}{2} \quad (7).$$

for all $i \geq q$.
From (D_1), (6. 1) and (7) we have:

$$\phi(d(x_{2n(i)}, x_{2m(i)+1})) \leq [a(\frac{\varepsilon}{2}) + c(\frac{\varepsilon}{2})]\phi(d_{2i} + \tau_{2n(i)-1}) + [\phi(\tau_{2n(i)-1}) + \phi(\tau_{2m(i)})]$$

for all $i \geq q$. As $i \to \infty$, we have:

$$\phi(\varepsilon) \leq [a(\frac{\varepsilon}{2}) + c(\frac{\varepsilon}{2})]\phi(\varepsilon) < \phi(\varepsilon)$$

This is a contradiction.
From (D_2), (6. 2) and (7) we have:

$$\phi(d(x_{2n(i)}, x_{2m(i)+1})) \leq [a(\frac{\varepsilon}{2}) + c(\frac{\varepsilon}{2}) + 2b(\frac{\varepsilon}{2})]\phi(d_{2i} + \tau_{2n(i)-1} + \tau_{2m(i)})$$

for all $i \geq q$. As $i \to \infty$, we have:

$$\phi(\varepsilon) \leq [a(\frac{\varepsilon}{2}) + c(\frac{\varepsilon}{2}) + 2b(\frac{\varepsilon}{2})]\phi(\varepsilon) < \phi(\varepsilon)$$

This is a contradiction.
Hence $\{x_n\}$ is a cauchy sequence in a complete metric space X, then there exists a point $x \in X$ such that

$$x_n \to x \qquad \text{as} \quad n \to \infty.$$

Now we shall show that $Sx = Tx = x$. From (B. 2), we have:

$$\begin{aligned}
\phi(d(x_{2n+2}, Tx)) &= \phi(d(Sx_{2n+1}, Tx)) \\
&\leq a(d(x_{2n+1}, x))\phi((d(x_{2n+1}, x)) \\
&+ b((d(x_{2n+1}, x)))[\phi(d(x_{2n+1}, x_{2n+2}) + \phi(d(Tx, x))] \\
&+ c((d(x_{2n+1}, x)))min\{\phi((d(x_{2n+1}, Tx)), \phi(d(x_{2n+2}, x))\} \quad (8)
\end{aligned}$$

Using $b < \frac{1}{2}$ and letting $n \to \infty$ in (8), we have

$$\phi(d(x, Tx)) \leq \frac{1}{2}\phi(d(x, Tx)).$$

That is $\phi(d(x, Tx)) = 0$ and therefore $d(x, Tx) = 0$, i.e. $x = Tx$. Similarly, it can be shown that $Sx = x$. Hence S and T have a common fixed point $x \in X$. We claim that x is the unique common fixed point of S and T. For this suppose that y $(x \neq y)$ is another fixed point of T and S. Then

$$\begin{aligned}
\phi(d(x, y)) = \phi(\delta(Sx, Ty)) &\leq a\phi(d(x, y)) + b\Big[\phi(\delta(x, Sx)) + \phi(\delta(y, Ty))\Big] + \\
&+ c\min\Big\{\phi(d(x, Ty)), \phi(d(y, Sx))\Big\} \\
&\leq (a + c)\phi(d(x, y)) \\
&< \phi(d(x, y)),
\end{aligned}$$

a contradiction. This completes the proof of the theorem.

Remark 2.1. *We first note that Theorem 1.2 is a special case of each results of Theorem 2.1. Also, it is obvious that in case (D_1) of Theorem 2.1 the function $a + 2b + c$ can take values larger than 1.*

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Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 1, July 2002, pp. 126–134.
Submitted: January 14, 2002. Published: July 1 2002.

MULTI-VALUED FIXED POINT THEOREMS BY ALTERING DISTANCE BETWEEN THE POINTS

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Abstract. In this paper we establish some results concerning fixed point of multivalued maps defined in a complete metric space, we then recover the results in [2] and [4].

1991 A.M.S. (MOS) Subject Classification Codes. 47H10.

Key Words and Phrases. Multi-valued map , fixed point, complete metric space, generalized contractive condition.

1 Preliminaries

Let (X, d) be a metric space, we denote by 2^X the collection of subsets of X and by $CB(X)$ the set of nonempty closed bounded subsets of X provided with the Hausdorff metric d_H defined by

$$d_H(A, B) = \max\Big(\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\Big),$$

where $A, B \in CB(X)$ and $d(x, A) = \inf\limits_{y\in A} d(x, y)$. We define also $\delta(A, B)$ by

$$\delta(A, B) = \sup\Big\{d(a, b), \quad a \in A, \quad b \in B\Big\},$$

it follows immediately from the definition of δ that

$$\delta(A, B) = 0 \quad \Longleftrightarrow \quad A = B = \{.\} \quad \text{and} \quad \delta(\{.\}, B) = d_H(\{.\}, B).$$

We shall use the following definitions

Definition 1.1. *A function $\psi : [0, +\infty[\longrightarrow [0, 1[$ will be said to have property (P) if for every $t > 0$, there exists $\eta(t) > 0$ such that*

$$0 \le s - t < \eta(t) \quad \Longrightarrow \quad \psi(s) \le \gamma(t) < 1.$$

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Definition 1.2. *Let $F : X \longrightarrow 2^X$ be a multi-valued map, then a point $x \in X$ is called a fixed point of F if $x \in Fx$, it is called absolutely fixed if $Fx = \{x\}$.*

Now let us recall the two results which we shall extend

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ satisfying the following condition*

$$(C) \quad \phi(d(Tx,Ty)) \le a(d(x,y))\phi(d(x,y)) + b(d(x,y))\Big[\phi(d(x,Tx)) + \phi(d(y,Ty))\Big] + c(d(x,y)) \min\Big\{\phi(d(x,Ty)), \phi(d(y,Tx))\Big\}$$

where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous and strictly increasing function such that $\phi(0) = 0$ and $a, b, c : \mathbb{R}^+ \longrightarrow [0, 1[$ are three functions such that $a + 2b$ and $a + c$ satisfies the property (P), then T has a unique fixed point.

Theorem 1.2. *([4] p 206 theorem 34.5) Let (X, d) be a complete metric space, $T : X \longrightarrow X$ a generalized contraction, that is :*

$$d(T(x), T(y)) \le q(\alpha, \beta)d(x, y)$$

where $\alpha \le d(x, y) \le \beta$ with $0 < \alpha \le \beta < +\infty$ and $q(\alpha, \beta) < 1$, then T has a unique fixed point.

In the extension of theorem 1.1, we study the case where a, b, c are constants and the space is complete and the case where a, b, c are functions and the space is compact.

2 Case where a, b, c are constants and the space is complete

We begin by proving the following lemmas

Lemma 2.1. *Let A, B are two compact subsets of a metric space, then for all $x \in A$, there exists $y \in B$ such that $d(x, y) \le d_H(A, B)$.*

Proof. It is an immediate consequence of the definition of the Hausdorff distance d_H. □

Lemma 2.2. *Let X be a metric space, $F : X \longrightarrow 2^X$ a multi-valued map with nonempty bounded values, $p \in [0, 1[$ and $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ a continuous function, then*

$$\forall x \in X, \quad \exists y_x \in Fx \quad \text{such that} \quad p\phi(\delta(x, Fx)) \le \phi(d(x, y_x)).$$

Proof. We can suppose that $\delta(x, Fx) > 0 \quad \forall x \in X$, because if there exists $x \in X$ such that $\delta(x, Fx) = 0$ then $Fx = \{x\}$, so we take then $y_x = x$. Using the continuity of ϕ and the definition of δ, we deduce that for

$$\varepsilon = \phi(\delta(x, Fx)) - p\phi(\delta(x, Fx)) > 0,$$

there exists $y_x \in Fx$ such that

$$\phi(\delta(x, Fx)) < \varepsilon + \phi(d(x, y_x)),$$

hence $p\phi(\delta(x, Fx)) \le \phi(d(x, y_x))$. □

Theorem 2.1. *Let (X,d) be a complete metric space, $T : X \longrightarrow CB(X)$ a multi-valued map satisfying the following condition :*

$$(H_1) \quad \phi(\delta(Tx,Ty)) \leq a\phi(d(x,y)) + b\Big[\phi(\delta(x,Tx)) + \phi(\delta(y,Ty))\Big] + \\ + c\min\Big\{\phi(d(x,Ty)), \phi(d(y,Tx))\Big\} \qquad \forall x,y \in X,$$

where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and strictly increasing such that $\phi(0) = 0$ and a,b,c are three positive constants such that $a + 2b < 1$ and $a + c < 1$, then T has a unique absolutely fixed point.

Proof. **Existence** Put $p = \sqrt{a+2b}$, then $p \in [0,1[$, so from lemma 2.2 we have

$$\forall x \in X \quad \exists y_x \in Tx, \quad \text{such that} \quad p\phi(\delta(x,Tx)) \leq \phi(d(x,y_x)).$$

Using the axiom of choice, we consider the map $f : X \longrightarrow X$ defined by $f(x) = y_x$, then $f(x) \in Tx \quad \forall x \in X$. Now $\forall x,y \in X$, we have

$$\begin{aligned}\phi(d(f(x),f(y))) &\leq \phi(\delta(Tx,Ty)) \\ &\leq a\phi(d(x,y)) + b\Big[\phi(\delta(x,Tx)) + \phi(\delta(y,Ty))\Big] + \\ &+ c\min\Big\{\phi(d(x,Ty)), \phi(d(y,Tx))\Big\} \\ &\leq a\phi(d(x,y)) + \frac{b}{p}\Big[\phi(d(x,f(x))) + \phi(d(y,f(y)))\Big] + \\ &+ c\min\Big\{\phi(d(x,f(y))), \phi(d(y,f(x)))\Big\},\end{aligned}$$

but

$$a + \frac{2b}{p} \leq \frac{a+2b}{p} = p < 1,$$

so from theorem 1.1 f has a fixed point, hence $\exists x \in X$ such that $f(x) = x \in Tx$, now using the fact that $p\phi(\delta(x,Tx)) \leq \phi(d(x,f(x))) = 0$, we deduce that $Tx = \{x\}$.

Unicity

Let x and y are two absolutely fixed point of T, then from the condition (H_1) we deduce that

$$\begin{aligned}\phi(d(x,y)) = \phi(\delta(Tx,Ty)) &\leq a\phi(d(x,y)) + b\Big[\phi(\delta(x,Tx)) + \phi(\delta(y,Ty))\Big] + \\ &+ c\min\Big\{\phi(d(x,Ty)), \phi(d(y,Tx))\Big\} \\ &\leq (a+c)\phi(d(x,y)) \\ &< \phi(d(x,y)),\end{aligned}$$

this is a contradiction. □

Remark 2.1. *In the proof of the last theorem, the use of the axiom of choice is indispensable to conclude that f is a map. If we don't use this axiom, we must suppose that the element y_x, given by lemma 2.2, is the unique element of the subset Fx satisfying some property, for example, if the multivalued map F is of closed and convex values in a strictly convex Banach space, then we can take for y_x the element of Fx with minimal norm.*

Theorem 2.2. *Let (X,d) be a complete metric space , $T : X \longrightarrow 2^X/\emptyset$ a multi-valued map with compact values satisfying the following condition*

$$(H_2) \quad \phi(d_H(Tx,Ty)) \leq a\phi(d(x,y)) + b\Big[\phi(d(x,Tx)) + \phi(d(y,Ty))\Big]+ \\ + c\min\Big\{\phi(d(x,Ty)), \phi(d(y,Tx))\Big\} \qquad \forall x,y \in X,$$

where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and strictly increasing such that $\phi(0) = 0$ and a, b, c are three positive constants such that $a + 2b < 1$, we assume also that there exists $\alpha, \beta > 0$ such that $\phi(t) \geq \alpha t^\beta \quad \forall t > 0$, then T has a fixed point.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$, then $\exists x_2 \in Tx_1$ such that

$$d(x_1,x_2) \leq d_H(Tx_0,Tx_1)$$

(the last inequality follows from lemma 2.1), hence

$$\begin{aligned}\phi(d(x_1,x_2)) &\leq \phi(d_H(Tx_0,Tx_1))\\ &\leq a\phi(d(x_0,x_1)) + b\Big[\phi(d(x_0,Tx_0)) + \phi(d(x_1,Tx_1))\Big]+\\ &+ c\min\Big\{\phi(d(x_0,Tx_1)), \phi(d(x_1,Tx_0))\Big\}\\ &\leq a\phi(d(x_0,x_1)) + b\Big[\phi(d(x_0,x_1)) + \phi(d(x_1,x_2))\Big]+\\ &+ c\min\Big\{\phi(d(x_0,Tx_1)), \phi(d(x_1,x_1))\Big\},\end{aligned}$$

so

$$\phi(d(x_1,x_2)) \leq k\phi(d(x_0,x_1)) \qquad \text{where} \qquad k = \frac{a+b}{1-b} < 1.$$

Repeating this argument, we have for $x_2 \in Tx_1$, $\exists x_3 \in Tx_2$ such that

$$\phi(d(x_2,x_3)) \leq k\phi(d(x_1,x_2)) \leq k^2\phi(d(x_0,x_1)).$$

In general, for each integer $n \geq 0$ there exists x_n such that

$$x_{n+1} \in Tx_n \quad \text{and} \quad \phi(d(x_n,x_{n+1})) \leq k^n\phi(d(x_0,x_1)),$$

but from hypothesis there exists $\alpha, \beta > 0$ such that

$$\begin{aligned}\alpha[d(x_n,x_{n+1})]^\beta &\leq \phi(d(x_n,x_{n+1}))\\ &\leq k^n\phi(d(x_0,x_1)),\end{aligned}$$

therefore

$$d(x_n,x_{n+1}) \leq p^n\gamma \quad \text{where} \quad p = k^{1/\beta} \quad \text{and} \quad \gamma = \Big(\frac{\phi(d(x_0,x_1))}{\alpha}\Big)^{1/\beta}.$$

Since $p < 1$, we deduce that (x_n) is a Cauchy sequence, hence $\exists x \in X$ such that $\lim_{n\to+\infty} x_n = x$. On the other hand,

$$|d(x_n,Tx) - d(x,Tx)| \leq d(x_n,x), \qquad \text{so} \quad \lim_{n\to+\infty} d(x_n,Tx) = d(x,Tx).$$

Now we have to show that x is a fixed point of T, that is $d(x,Tx)=0$. Using the condition (H_2) we have

$$\begin{aligned}\phi(d(x_{n+1},Tx)) &\leq \phi(d_H(Tx_n,Tx))\\ &\leq a\phi(d(x_n,x))+b\Big[\phi(d(x_n,Tx_n))+\phi(d(x,Tx))\Big]+\\ &+c\min\Big\{\phi(d(x_n,Tx)),\phi(d(x,Tx_n))\Big\}\\ &\leq a\phi(d(x_n,x))+b\Big[\phi(d(x_n,x_{n+1}))+\phi(d(x,Tx))\Big]+\\ &+c\phi(d(x,x_{n+1})),\end{aligned}$$

let $n\to+\infty$, we obtain

$$\phi(d(x,Tx))\leq b\phi(d(x,Tx))\leq\frac{1}{2}\phi(d(x,Tx)),$$

hence $\phi(d(x,Tx))=0$ and consequently $d(x,Tx)=0$. $\square$

Remark 2.2. *If ϕ satisfies $\phi(t+s)\leq\phi(t)+\phi(s)$ $\forall t,s\in\mathbb{R}^+$, then we can eliminate the hypothesis $\phi(t)\geq\alpha t^\beta$, indeed, from the inequality*

$$\phi(d(x_n,x_{n+1}))\leq k^n\phi(d(x_0,x_1)),$$

we deduce that $\lim_{n,p\to+\infty}\phi(d(x_n,x_p))=0,$ *hence* $\lim_{n,p\to+\infty}d(x_n,x_p)=0.$

3 Case where a,b,c are functions and the space is compact

Theorem 3.1. *Let (X,d) be a compact metric space, $T:X\longrightarrow 2^X/\emptyset$ a multi-valued map with closed values, $\phi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ continuous and strictly increasing such that $\phi(0)=0$ and let a,b,c are three functions defined on $\mathbb{R}^+$ with values in $[0,1[$ such that $a+2b$ satisfies the property (P), suppose also that T satisfies the following condition*

$$(H_3)\quad \begin{aligned}\phi(d_H(Tx,Ty))&\leq a(d(x,y))\phi(d(x,y))+b(d(x,y))\Big[\phi(d(x,Tx))+\phi(d(y,Ty))\Big]+\\ &+c(d(x,y))\min\Big\{\phi(d(x,Ty)),\phi(d(y,Tx))\Big\}\qquad\forall x,y\in X,\end{aligned}$$

then T has a fixed point.

Proof. Select $x_0\in X$ and $x_1\in Tx_0$, then $\exists x_2\in Tx_1$ such that

$$d(x_1,x_2)\leq d_H(Tx_0,Tx_1),$$

so

$$\begin{aligned}\phi(d(x_1,x_2)) &\leq \phi(d_H(Tx_0,Tx_1))\\ &\leq a(d(x_0,x_1))\phi(d(x_0,x_1))+b(d(x_0,x_1))\Big[\phi(d(x_0,Tx_0))+\phi(d(x_1,Tx_1))\Big]+\\ &+c(d(x_0,x_1))\min\Big\{\phi(d(x_0,Tx_1)),\phi(d(x_1,Tx_0))\Big\}\\ &\leq a(d(x_0,x_1))\phi(d(x_0,x_1))+b(d(x_0,x_1))\Big[\phi(d(x_0,x_1))+\phi(d(x_1,x_2))\Big]+\\ &+c(d(x_0,x_1))\min\Big\{\phi(d(x_0,Tx_1)),\phi(d(x_1,x_1))\Big\},\end{aligned}$$

hence $\quad \phi(d(x_1,x_2)) \leq q(d(x_0,x_1))\phi(d(x_0,x_1))$ $\quad$ where

$$q(d(x_0,x_1)) = \frac{a(d(x_0,x_1)) + b(d(x_0,x_1))}{1 - b(d(x_0,x_1))}.$$

By induction there exists x_n such that $\quad x_{n+1} \in Tx_n \quad$ and

$$\phi(d(x_n,x_{n+1})) \leq q(d(x_{n-1},x_n))\phi(d(x_{n-1},x_n)) < \phi(d(x_{n-1},x_n)). \quad (*)$$

Put $\quad l_n = d(x_n,x_{n+1}),\quad$ then we have $\quad \phi(l_n) < \phi(l_{n-1}),\quad$ but ϕ is increasing, so l_n is decreasing, hence $\quad \lim_{n\to+\infty} l_n = l \quad$ exists . Now we show that $l = 0$, suppose that $l > 0$, then since q satisfies the property (P)

$$\exists\eta(l) > 0 \quad \text{such that} \quad 0 \leq s - l < \eta(l) \quad \Longrightarrow \quad q(s) \leq \gamma(l) < 1,$$

but $\quad l_n \geq l \quad$ and $\lim_{n\to+\infty} l_n = l, \quad$ so there exists n_0 such that

$$n \geq n_0 \quad \Longrightarrow \quad 0 \leq l_n - l < \eta(l),$$

consequently $\quad q(l_n) \leq \gamma(l) < 1$. From the inequality (*) we deduce that

$$\phi(l_n) \leq \gamma(l)\phi(l_{n-1}) \quad \forall n \geq n_0,$$

hence when $n \to +\infty$ we obtain

$$\phi(l) \leq \gamma(l)\phi(l) < \phi(l),$$

this is a contradiction, so $\quad \lim_{n\to+\infty} d(x_n,x_{n+1}) = 0$. X is compact, so x_n has a convergent subsequence which we denote by y_n, let $\quad y = \lim_{n\to+\infty} y_n, \quad$ we have to show that y is a fixed point of T. Since $\quad \lim_{n\to+\infty} d(y_n,Ty) = d(y,Ty), \quad$ we deduce by the condition (H_3) that

$$\begin{aligned}
\phi(d(y_{n+1},Ty)) &\leq \phi(d_H(Ty_n,Ty)) \\
&\leq a(d(y_n,y))\phi(d(y_n,y)) + b(d(y_n,y))\Big[\phi(d(y_n,Ty_n)) + \phi(d(y,Ty))\Big] \\
&+ c(d(y_n,y))\min\Big\{\phi(d(y_n,Ty)),\phi(d(y,Ty_n))\Big\} \\
&\leq a(d(y_n,y))\phi(d(y_n,y)) + b(d(y_n,y))\Big[\phi(d(y_n,y_{n+1})) + \phi(d(y,Ty))\Big] \\
&+ c(d(y_n,y))\phi(d(y,y_{n+1})) \\
&< \phi(d(y_n,y)) + \frac{1}{2}\Big[\phi(d(y_n,y_{n+1})) + \phi(d(y,Ty))\Big] + \phi(d(y,y_{n+1})),
\end{aligned}$$

therefore taking $n \to +\infty$, we obtain

$$\phi(d(y,Ty)) \leq \frac{1}{2}\phi(d(y,Ty)), \quad \text{hence} \quad \phi(d(y,Ty)) = 0,$$

consequently $d(y,Ty) = 0$, and $y \in Ty$. $\square$

4 Generalization of Krasnosel'skii theorem

The generalized contraction mapping theorem which is due to M.A Krasnosels'kii is an extension of the well-known Banach's contraction principle. In the following theorem, we consider a multi-valued and generalized ϕ-contraction.

Theorem 4.1. *Let (X,d) be a complete metric space, $\phi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ a continuous and strictly increasing map such that $\phi(0)=0$, we suppose also that $\phi(t+s)\leq\phi(t)+\phi(s)\quad\forall s,t\in\mathbb{R}^+$. Let $T:X\longrightarrow 2^X/\emptyset$ a multi-valued map with compact values satisfying the following condition*

$$(H_4)\qquad \phi(d_H(Tx,Ty))\leq q(\alpha,\beta)\phi(d(x,y))$$

where $0<\alpha\leq\beta<+\infty$, $\alpha\leq d(x,y)\leq\beta$ and $q(\alpha,\beta)<1$, then T has a fixed point.

Remarks 4.1. **1)** *From the condition (H_4), we deduce that T is continuous.*
2) *The condition (H_4) is in particular satisfied if*

$$\phi(d_H(Tx,Ty))\leq\phi(d(x,y))-r(\phi(d(x,y)))\quad \textit{where}\quad r:\mathbb{R}^+_*\longrightarrow\mathbb{R}^+_* \textit{ is continuous,}$$

indeed $\phi(d(x,y))-r(\phi(d(x,y)))=\phi(d(x,y))\Big[1-\frac{r(\phi(d(x,y)))}{\phi(d(x,y))}\Big]$, *so we can take* $q(\alpha,\beta)=\sup\limits_{t\in[\alpha,\beta]}(1-\frac{r(\phi(t))}{\phi(t)})$.

Proof of theorem 4.1. Select $x_0\in X$ and $x_1\in Tx_0$, then $\exists x_2\in Tx_1$ such that $d(x_1,x_2)\leq d_H(Tx_0,Tx_1)$, so

$$\phi(d(x_2,x_1))\leq\phi(d_H(Tx_0,Tx_1))\leq q(\alpha,\beta)\phi(d(x_1,x_0)),$$

at the same $\exists x_3\in Tx_2$ such that $\phi(d(x_3,x_2))\leq q(\alpha,\beta)\phi(d(x_2,x_1))$ and by induction there exists a sequence x_n such that $x_{n+1}\in Tx_n$ and

$$\begin{aligned}\phi(d(x_{n+1},x_n))&\leq\phi(d_H(Tx_n,Tx_{n-1}))\\&\leq q(\alpha,\beta)\phi(d(x_n,x_{n-1}))\\&<\phi(d(x_n,x_{n-1})),\end{aligned}$$

hence the sequence $\phi(d(x_{n+1},x_n))$ is decreasing, but ϕ is increasing, so we deduce that the sequence $\tau_n=d(x_{n+1},x_n)$ is decreasing. Let $\tau=\lim\limits_{n\to+\infty}\tau_n$, we have to show that $\tau=0$, suppose that $\tau>0$, then for $\varepsilon=1$ there exists n_0 such that $n\geq n_0\implies\tau_n\leq\tau+1$, on the other hand $\tau\leq\tau_n$, so from some rank, the consecutive terms of the sequence x_n are in the crow with radius τ and $\tau+1$, so from the condition (H_4) we have

$$\phi(\tau_{n+m})\leq\Big[q(\tau,\tau+1)\Big]^m\phi(\tau+1)\qquad\forall m=1,2,...$$

letting $m\to+\infty$ we obtain $\phi(\tau)\leq 0$, this is a contradiction. Hence

$$\lim_{n\to+\infty}d(x_{n+1},x_n)=0\quad\text{and consequently}\quad\lim_{n\to+\infty}\phi(d(x_{n+1},x_n))=0.$$

Now we have to show that x_n is a Cauchy sequence, to do this we show that $\lim\limits_{n,m\to+\infty} \phi(d(x_{n+m}, x_n)) = 0$. Let $\varepsilon > 0$, then for large n we have

$$\phi(d(x_{n+1}, x_n)) \leq \frac{\varepsilon}{2}\Big[1 - q(\phi^{-1}(\frac{\varepsilon}{2}), \phi^{-1}(\varepsilon))\Big]$$

where $q(\phi^{-1}(\frac{\varepsilon}{2}), \phi^{-1}(\varepsilon))$ will be specified later. We show that

$$\phi(d(x_{n+1}, x_n)) \leq \varepsilon \quad \Longrightarrow \quad \phi(d(x_{n+2}, x_n)) \leq \varepsilon,$$

and by the same way we show also that $\phi(d(x_{n+3}, x_n)) \leq \varepsilon$ and so on, we show that $\phi(d(x_{n+m}, x_n)) \leq \varepsilon \quad \forall m = 1, 2, ..$
Assume that $\phi(d(x_{n+1}, x_n)) \leq \varepsilon$, then
i) If $\phi(d(x_{n+1}, x_n)) \leq \frac{\varepsilon}{2}$, then

$$\begin{aligned}\phi(d(x_{n+2}, x_n)) &\leq \phi\Big(d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)\Big)\\ &\leq \phi(d(x_{n+2}, x_{n+1}) + \phi(d(x_{n+1}, x_n))\\ &\leq \phi(d(x_{n+1}, x_n)) + \phi(d(x_{n+1}, x_n))\\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

ii) If $\frac{\varepsilon}{2} \leq \phi(d(x_{n+1}, x_n)) \leq \varepsilon$, then $\phi^{-1}(\frac{\varepsilon}{2}) \leq d(x_{n+1}, x_n) \leq \phi^{-1}(\varepsilon)$, so from the condition (H_4) we deduce that

$$\begin{aligned}\phi(d(x_{n+2}, x_n)) &\leq \phi(d(x_{n+2}, x_{n+1})) + \phi(d(x_{n+1}, x_n))\\ &\leq \phi(d_H(Tx_{n+1}, Tx_n)) + \phi(d(x_{n+1}, x_n))\\ &\leq q(\phi^{-1}(\frac{\varepsilon}{2}), \phi^{-1}(\varepsilon))\phi(d(x_{n+1}, x_n)) + \frac{\varepsilon}{2}\Big(1 - q(\phi^{-1}(\frac{\varepsilon}{2}), \phi^{-1}(\varepsilon))\Big)\\ &\leq q(\phi^{-1}(\frac{\varepsilon}{2}), \phi^{-1}(\varepsilon))\varepsilon + \frac{\varepsilon}{2} - q(\phi^{-1}(\frac{\varepsilon}{2}), \phi^{-1}(\varepsilon))\frac{\varepsilon}{2}\\ &\leq \varepsilon.\end{aligned}$$

Hence $\exists x \in X$ such that $\lim\limits_{n\to+\infty} x_n = x$, now since T is continuous and

$$d(x_{n+1}, Tx) \leq d_H(Tx_n, Tx),$$

we deduce by letting $n \to +\infty$ that $d(x, Tx) = 0$ and consequently $x \in Tx$. $\square$

5 Consequences of theorem 4.1

1) If T is univalued and $\phi(t) = t$, then we recover Krasnosel'skii's theorem (see theorem 1.2 in section 1).
2) If T is univalued, $\phi(t) = t$ and $q(\alpha, \beta) = k < 1$, then we recover Banach' theorem (see e.g [3] p 7).
3) If T is univalued, $\phi(t) = t$ and $q(\alpha, \beta) = \sup\limits_{t\in[\alpha,\beta]} \frac{\varphi(t)}{t}$, then we recover Boyd-Wong' theorem (see e.g [5] p 117).
4) If T is univalued, $\phi(t) = t$ and $q(\alpha, \beta) = \sup\limits_{D\times D} \frac{d(f(x),f(y))}{d(x,y)}$, then we recover Edelstein' theorem (see e.g [3] p 12).

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Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 1, July 2002, pp. 135–142.
Submitted: December 7, 2001. Published: July 1 2002.

FIXED POINT THEOREM AND APPLICATION IN MODULAR SPACE

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Abstract. In this paper, by analogy with Banach contraction principle, we establish a result concerning fixed point in modular space, an application of the result is given in Orlicz space for the integral equation of Voltera type.

1991 A.M.S. (MOS) Subject Classification Codes. 47H10

Key Words and Phrases. Fixed point, modular space, integral equation.

1 Introduction and Preliminaries

In [1], the author has proved the following result using the fact that the modular ρ satisfies the Δ_2 condition and the Fatou property, the proof was based on the theorem of Cantor which he showed in a modular space. Also, let us note that the author used implicitly (without indicating it) the fact that B is ρ-bounded (for the notations and definitions, see the preliminaries below).
In this paper and by a simplified method, we show that the result remains true without supposing that the modular ρ verifies neither the Δ_2 condition nor the Fatou property.

Theorem 1.1. *Let B be a ρ-closed subset of a ρ-complete modular space X_ρ and $T : B \longrightarrow B$ satisfying*

$$\rho(2(Tx - Ty)) \leq k\rho(x - y) \quad \forall x, y \in B, \quad \textit{where} \quad 0 < k < 2,$$

we suppose also that ρ is convex and satisfies the Δ_2 condition and the Fatou property, then T has a unique fixed point.

Now let us recall some basic definitions in the theory of modular spaces. For more details we refer the reader to the book of Musielak [6].

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Definition 1.1. *Let X be a vector space over K ($K = \mathbb{R}$ or C), a functional $\rho : X \longrightarrow [0, +\infty]$ is called a modular if for all x and y in X, there holds*

a) $\rho(x) = 0$ *if and only if* $x = 0$,

b) $\rho(\alpha x) = \rho(x)$ *for every* $\alpha \in K$ *with* $|\alpha| = 1$,

c) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ *for every* $\alpha, \beta \geq 0$, *and* $\alpha + \beta = 1$.

If we replace c) by $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$, *then the modular ρ is called convex.*

A modular ρ defines a corresponding modular space X_ρ given by

$$X_\rho = \Big\{ x \in X \quad / \quad \lim_{\lambda \to 0} \rho(\lambda x) = 0 \Big\}.$$

Definition 1.2. *Let X_ρ be a modular space.*

a) *A sequence x_n is called ρ-convergent to x and write $x_n \to_\rho x$, if and only if $\rho(x_n - x) \to 0$ as $n \to +\infty$.*

b) *A sequence x_n is called ρ-Cauchy whenever $\rho(x_n - x_m) \to 0$ as $n, m \to +\infty$.*

c) *The modular space X_ρ is called ρ-complete if any ρ-Cauchy sequence is ρ-convergent.*

d) *The modular ρ is said to satisfy the Δ_2 condition if $\rho(2x_n) \to 0$ whenever $\rho(x_n) \to 0$ as $n \to +\infty$.*

e) *ρ is said to have the Fatou property if $\rho(x) \leq \liminf\limits_{n \to +\infty} \rho(x_n)$ whenever $x_n \to_\rho x$.*

f) *A subset $B \subset X_\rho$ is called ρ-closed if for any sequence $x_n \subset B$ ρ-convergent to $x \in X_\rho$, we have $x \in B$.*

g) *A subset $B \subset X_\rho$ is said to be ρ-bounded if* $\sup\{\rho(x - y) : x, y \in B\} < +\infty$.

2 Main result

Theorem 2.1. *Let B be a ρ-closed and ρ-bounded subset of a ρ-complete modular space X_ρ and $T : B \longrightarrow B$ satisfying*

$$\rho(2(Tx - Ty)) \leq k\rho(x - y) \quad \forall x, y \in B, \quad \textit{where} \quad 0 < k < 2,$$

we suppose also that ρ is convex, then T has a unique fixed point.

Proof. Select $x_0 \in B$ and put $x_{n+1} = Tx_n, (n \geq 0)$ and $\alpha_{n+1} = \rho(x_{n+1} - x_n)$, then

$$\begin{aligned} \alpha_{n+1} &= \rho(x_{n+1} - x_n) \\ &= \rho(Tx_n - Tx_{n-1}) \\ &\leq \frac{1}{2}\rho(2(Tx_n - Tx_{n-1})) \\ &\leq \frac{k}{2}\rho(x_n - x_{n-1}) \\ &= \frac{k}{2}\alpha_n, \end{aligned}$$

so $\alpha_{n+1} \leq (\frac{k}{2})^{n+1}\alpha_0$, hence $\lim\limits_{n \to +\infty} \rho(x_{n+1} - x_n) = 0$.

Now we have to show that x_n is ρ-Cauchy.

Let $\varepsilon > 0$, if we show that $\rho(x_{n+1} - x_n) \leq \varepsilon \implies \rho(x_{n+2} - x_n) \leq \varepsilon$, then by the same way we show that $\rho(x_{n+3} - x_n) \leq \varepsilon$ and so on we show that $\rho(x_{n+m} - x_n) \leq \varepsilon \quad \forall m = 1, 2, ...$

We have $\lim\limits_{n\to+\infty} \rho(x_{n+1}-x_n)=0$, so for n large we have $\rho(x_{n+1}-x_n)\leq\varepsilon$ and $\rho(x_n-x_{n-1})\leq\varepsilon(1-\frac{k}{2})$, thus

$$\begin{aligned}
\rho(x_{n+2}-x_n) &= \rho(Tx_{n+1}-Tx_{n-1})\\
&= \rho(Tx_{n+1}-Tx_n+Tx_n-Tx_{n-1})\\
&\leq \frac{1}{2}\rho(2(Tx_{n+1}-Tx_n))+\frac{1}{2}\rho(2(Tx_n-Tx_{n-1}))\\
&\leq \frac{k}{2}\rho(x_{n+1}-x_n)+\frac{k}{2}\rho(x_n-x_{n-1})\\
&\leq \frac{k}{2}\rho(x_{n+1}-x_n)+\rho(x_n-x_{n-1})\\
&\leq \frac{k}{2}\varepsilon+\varepsilon(1-\frac{k}{2})\\
&= \varepsilon,
\end{aligned}$$

repeating the same argument, we have

$$\begin{aligned}
\rho(x_{n+3}-x_n) &= \rho(Tx_{n+2}-Tx_n+Tx_n-Tx_{n-1})\\
&\leq \frac{1}{2}\rho(2(Tx_{n+2}-Tx_n))+\frac{1}{2}\rho(2(Tx_n-Tx_{n-1}))\\
&\leq \frac{k}{2}\rho(x_{n+2}-x_n)+\frac{k}{2}\rho(x_n-x_{n-1})\\
&\leq \frac{k}{2}\rho(x_{n+2}-x_n)+\rho(x_n-x_{n-1})\\
&\leq \frac{k}{2}\varepsilon+\varepsilon(1-\frac{k}{2})\\
&= \varepsilon.
\end{aligned}$$

So x_n is ρ-Cauchy and consequently there exists $x\in B$ such that $x_n\to_\rho x$, that is $\lim\limits_{n\to+\infty}\rho(x_n-x)=0$.

Let us show that x is a fixed point of T, we have

$$\begin{aligned}
\rho(\frac{x-Tx}{2}) &= \rho(\frac{x-Tx_n}{2}+\frac{Tx_n-Tx}{2}))\\
&\leq \rho(x-Tx_n)+\rho(Tx_n-Tx)\\
&\leq \rho(x-x_{n+1})+\rho(2(Tx_n-Tx))\\
&\leq \rho(x-x_{n+1})+k\rho(x_n-x),
\end{aligned}$$

thus, letting $n\to+\infty$, we obtain

$$\rho(\frac{x-Tx}{2})=0, \quad \text{hence} \quad x=Tx.$$

Now assume that there is $y\in B$ such that $y=Ty$, then

$$\begin{aligned}
\rho(x-y) &= \rho(Tx-Ty)\\
&\leq \frac{1}{2}\rho(2(Tx-Ty))\\
&\leq \frac{k}{2}\rho(x-y),
\end{aligned}$$

since $k<2$ and B is ρ-bounded, we obtain a contradiction $\square$

Remarks 2.1.

1) *If we suppose in the previous theorem that $0 < k < 1$, then we can suppress the hypothesis : ρ is convex.*

2) *If we suppose that there is $x_0 \in B$ such that $\rho(x_0 - Tx_0) < +\infty$, then we can eliminate the hypothesis : B is ρ-bounded, but in this case, the fixed point of T is not necessarily unique. If in particular the modular ρ is with values in $[0, +\infty[$, then the fixed point is unique without assuming that B is ρ-bounded.*

From theorem 2.1 we deduce the following corollary,

Corollary 2.1. *Under the same hypothesis of theorem 2.1, suppose that T satisfies*

$$\rho(\theta(Tx - Ty)) \leq k\rho(x - y) \quad \textit{with} \quad \theta \geq 2 \quad \textit{and} \quad k < \theta,$$

then T has a unique fixed point

Proof. We have

$$\begin{aligned} \rho(2(Tx - Ty)) &= \rho(\frac{2\theta}{\theta}(Tx - Ty)) \\ &\leq \frac{2}{\theta}\rho(\theta(Tx - Ty)) \\ &\leq \frac{2k}{\theta}\rho(x - y) \\ &= k'\rho(x - y) \end{aligned}$$

where $\quad k' = \dfrac{2k}{\theta} < 2,\quad$ so from theorem 2.1, T has a unique fixed point. $\square$

3 Application to the integral equation of Voltera type

We look for a solution of the integral equation

$$x(t) = x_0(t) + \mu \int_0^t K(t, s)Rx(s)ds \qquad (*)$$

in Orlicz space L^φ, where $x_0(.)$ and $K(.,.)$ and R are given. We consider the vector space X defined by

$$X = \Big\{f : [0, 1] \longrightarrow \mathbb{R} \quad \text{measurable}\Big\},$$

we define in X the following modular : for $f \in X$, we put

$$\rho(f) = \int_0^1 \varphi(t, |f(t)|)dt,$$

where $\varphi : [0, 1] \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ satisfies :

i) φ is continuous and increasing and convex with respect to the second variable.

ii) $\lim\limits_{x \to +\infty} \varphi(t, x) = +\infty, \quad \varphi(t, 0) = 0, \quad \varphi(t, x) > 0 \quad \forall x > 0.$

iii) φ is measurable with respect to the first variable.

Then it is known (see for example [6]) that ρ is a convex modular and :

1) $L^\varphi = \Big\{f \in X \quad / \quad \lim\limits_{\lambda \to 0} \rho(\lambda f) = 0\Big\}$ is a modular space.

2) L^φ is ρ-complete.
3) L^φ is a Banach space equipped with the Luxembourg norm defined by

$$\|x\|_\rho = \inf\Big\{\alpha > 0 \quad / \quad \rho(\frac{x}{\alpha}) \leq 1\Big\}.$$

4) ρ satisfies the Δ_2 condition and the Fatou property.
5) $\lim\limits_{n\to+\infty} \|x_n - x\|_\rho = 0 \iff \lim\limits_{n\to+\infty} \rho(x_n - x) = 0,$ note that for an arbitrary modular ρ which does not satisfy the Δ_2 condition, we have

$$\lim_{n\to+\infty} \|x_n - x\|_\rho = 0 \iff \lim_{n\to+\infty} \rho(\alpha(x_n - x)) = 0 \quad \forall\alpha > 0.$$

To resolve the integral equation (*), we construct another complete modular space on which the operator T defined by

$$(Tx)(t) = x_0(t) + \mu \int_0^t K(t,s)Rx(s)ds$$

admits a fixed point which corresponds to a solution of (*).
We consider the vector space Y defined by

$$Y = \Big\{x : [0,1] \longrightarrow L^\varphi \quad \text{continuous with respect the norm} \quad \|.\|_\rho\Big\},$$

we define on Y the functional ρ_∞ by $\rho_\infty(x) = \sup\limits_{t\in[0,1]} \rho(x(t)),$ where ρ is the modular of L^φ, we can verify easily that $\rho_\infty : Y \longrightarrow [0,+\infty]$ satisfies all the properties of a convex modular, we obtain then the modular space

$$Y_{\rho_\infty} = \Big\{x \in Y \quad / \quad \lim_{\lambda\to 0} \rho_\infty(\lambda x) = 0\Big\}.$$

lemma 3.1. *Y_{ρ_∞} is ρ_∞-complete.*

Proof. Let x_n be a ρ_∞-Cauchy sequence in Y_{ρ_∞}, then

$$\forall\varepsilon > 0 \quad \exists n_0, \quad \forall n, p \geq n_0, \quad \forall t \in [0,1] \quad \rho(x_n(t) - x_p(t)) \leq \varepsilon, \qquad (**)$$

so $\forall t \in [0,1]$ $x_n(t)$ is ρ-Cauchy in L^φ, hence $x_n(t) \to_\rho x(t)$ and $x(t) \in L^\varphi$.
In the relation (**), we fix $n \geq n_0$, we obtain $\liminf\limits_{p\to+\infty} \rho(x_n(t) - x_p(t)) \leq \varepsilon$, but ρ satisfies the Fatou property, thus

$$\forall t \in [0,1] \quad \rho(x_n(t) - x(t)) \leq \liminf_{p\to+\infty} \rho(x_n(t) - x_p(t)) \leq \varepsilon,$$

so $\rho_\infty(x_n - x) \leq \varepsilon,$ hence $x_n \to_{\rho_\infty} x$.
Now we have to show that $x \in Y_{\rho_\infty}$, that is, x is continuous and $\lim\limits_{\lambda\to 0} \rho_\infty(\lambda x) = 0$.
i) $x : [0,1] \longrightarrow L^\varphi$ is continuous, indeed

$$\lim_{n\to+\infty} \rho_\infty(x_n - x) = 0 \implies \forall t \in [0,1] \quad \lim_{n\to+\infty} \rho(x_n(t) - x(t)) = 0,$$

since ρ satisfies the Δ_2 condition, we deduce that $\lim\limits_{n\to+\infty} \|x_n(t) - x(t)\|_\rho = 0$, hence

$$\forall \varepsilon > 0 \quad \exists n_0, \quad \forall n \geq n_0 \quad \forall t \in [0,1], \quad \|x_n(t) - x(t)\|_\rho \leq \varepsilon.$$

Let $t_0 \in [0,1]$, then $\exists \eta > 0$ such that $|t - t_0| \leq \eta \implies \|x_{n_0}((t) - x_{n_0}(t_0)\|_\rho \leq \varepsilon$, so

$$\begin{aligned}\|x(t) - x(t_0)\|_\rho &\leq \|x(t) - x_{n_0}(t)\|_\rho + \|x_{n_0}(t) - x_{n_0}(t_0)\|_\rho + \|x_{n_0}(t_0) - x(t_0)\|_\rho \\ &\leq 3\varepsilon.\end{aligned}$$

ii) $\lim\limits_{\lambda\to 0} \rho_\infty(\lambda x) = 0$, indeed

$$\begin{aligned}\rho_\infty(\lambda x) &= \rho_\infty(\lambda(x - x_n) + \lambda x_n) \\ &\leq \rho_\infty(2\lambda(x - x_n)) + \rho_\infty(2\lambda x_n),\end{aligned}$$

we can suppose that $2\lambda < 1$, so $\rho_\infty(\lambda x) \leq \rho_\infty(x - x_n) + \rho_\infty(2\lambda x_n)$, therefore $\limsup\limits_{\lambda\to 0} \rho_\infty(\lambda x) \leq \rho_\infty(x - x_n)$, since this inequality is satisfied for all n, we deduce that $\lim\limits_{\lambda\to 0} \rho_\infty(\lambda x) = 0$. $\square$

Now we return to the integral equation (*) :

$$x(t) = x_0(t) + \mu \int_0^t K(t,s)Rx(s)ds$$

where $\mu > 0$, $x_0 \in Y_{\rho_\infty}$, $K : [0,1] \times [0,1] \longrightarrow \mathbb{R}^+$ is continuous and $R : L^\varphi \longrightarrow L^\varphi$ satisfies $\rho(Rx - Ry) \leq \theta\rho(x - y)$ $\forall x, y \in L^\varphi$ with $\theta \geq 2$ and $\mu\theta < 1$, then we have the following result

Theorem 3.1. *We suppose also that :*

a) $\sup\limits_{t\in[0,1]} \int_0^t K(t,s)ds \leq 1$.

b) *There exists* $f_0 \in Y_{\rho_\infty}$ *such that* $\rho_\infty(f_0 - Tf_0) < +\infty$.

c) $\rho(R(0)) < +\infty$ *and* $\rho(\varepsilon Rx) \leq \rho(R(\varepsilon x))$ $\forall x \in L^\varphi, \forall \varepsilon$ *with* ε *sufficiently small. Then we have :*

1) $\forall x \in Y_{\rho_\infty}$, $Tx \in Y_{\rho_\infty}$.

2) $\exists k < \theta$ *such that* $\rho_\infty(\theta(Tx - Ty)) \leq k\rho_\infty(x - y)$.

We deduce that the operator $T : Y_{\rho_\infty} \longrightarrow Y_{\rho_\infty}$ *has a fixed point, and consequently the equation (*) has a solution.*

Proof. Without loss of generality, taking $\tilde{R}(x) = R(x) - R(0)$, we can suppose that $R(0) = 0$.

1) **i)** Let $x \in Y_{\rho_\infty}$, then the map $(Tx)(.)$ is continuous on $[0,1]$, indeed, it suffices to show that the map $t \longmapsto Rx(t)$ is continuous.

Let $t_0 \in [0,1]$, then since $x(.)$ is continuous, we have $\lim\limits_{t\to t_0} \|x(t) - x(t_0)\|_\rho = 0$, but since ρ satisfies the Δ_2 condition, we deduce that $\lim\limits_{t\to t_0} \rho(x(t) - x(t_0)) = 0$, hence $\lim\limits_{t\to t_0} \rho(Rx(t) - Rx(t_0)) = 0$, and so $\lim\limits_{t\to t_0} \|Rx(t) - Rx(t_0)\|_\rho = 0$.

ii) $\lim\limits_{\lambda\to 0} \rho_\infty(\lambda Tx) = 0$, indeed $\forall t \in [0,1]$, we have

$$\begin{aligned}\rho(\lambda(Tx)(t)) &= \rho\Big(\lambda x_0(t) + \lambda\mu \int_0^t K(t,s)Rx(s)ds\Big)\\ &\le \rho(2\lambda x_0(t)) + \rho\Big(2\lambda\mu \int_0^t K(t,s)Rx(s)ds\Big)\\ &\le \rho_\infty(2\lambda x_0) + \rho\Big(\lambda \int_0^t K(t,s)Rx(s)ds\Big).\end{aligned}$$

To conclude, it is enough to show that $\rho\Big(\lambda \int_0^t K(t,s)Rx(s)ds\Big) \le \theta\rho_\infty(\lambda x)$.
We have $\rho\Big(\lambda \int_0^t K(t,s)Rx(s)ds\Big) = \rho\Big(\int_0^t K(t,s)(\lambda Rx(s))ds\Big)$,
On the other hand, from the mean value theorem,(see for example [5]), we have

$$\begin{aligned}\int_0^t K(t,s)(\lambda Rx(s))ds &\in \Big(\int_0^t K(t,s)ds\Big)\overline{\text{Co}}\Big\{\lambda Rx(\tau), \quad \tau\in[0,t]\Big\}\\ &\subset \Big(\int_0^t K(t,s)ds\Big)\overline{\text{Co}}\Big\{\lambda Rx(\tau), \quad \tau\in[0,1]\Big\},\end{aligned}$$

so $\int_0^t K(t,s)(\lambda Rx(s))ds = \Big(\int_0^t K(t,s)ds\Big)y(\tau)$ with
$y(\tau) \in \overline{\text{Co}}\Big\{\lambda Rx(\tau), \quad \tau\in[0,1]\Big\}$, hence

$$y(\tau) = \lim_{p\to+\infty}\Big(\sum_{i=0}^{n_p}\lambda_i^p\lambda Rx(\tau_i^p)\Big) \quad \text{with} \quad \sum_{i=1}^{n_p}\lambda_i^p = 1,$$

now since ρ satisfies the Δ_2 condition and the Fatou property, we deduce that

$$\lim_{p\to+\infty}\rho\Big(y(\tau) - \sum_{i=0}^{n_p}\lambda_i^p\lambda Rx(\tau_i^p)\Big) = 0 \quad \text{and} \quad \rho(y(\tau)) \le \liminf_{p\to+\infty}\rho\Big(\sum_{i=0}^{n_p}\lambda_i^p\lambda Rx(\tau_i^p)\Big),$$

but we have

$$\begin{aligned}\rho\Big(\sum_{i=0}^{n_p}\lambda_i^p\lambda Rx(\tau_i^p)\Big) &\le \sum_{i=0}^{n_p}\lambda_i^p\rho(\lambda Rx(\tau_i^p))\\ &\le \sum_{i=0}^{n_p}\lambda_i^p\rho(R(\lambda x(\tau_i^p)))\\ &\le \sum_{i=0}^{n_p}\lambda_i^p\theta\rho(\lambda x(\tau_i^p))\\ &\le \sum_{i=0}^{n_p}\lambda_i^p\theta\sup_{t\in[0,1]}\rho(\lambda x(t))\\ &= \theta\rho_\infty(\lambda x).\end{aligned}$$

2) We have $Tx(t) - Ty(t) = \mu\int_0^t K(t,s)Rx(s)ds$, so

$$\rho\Big(\theta(Tx(t) - Ty(t))\Big) = \rho\Big(\theta\mu\int_0^t K(t,s)(Rx(s) - Ry(s))ds\Big).$$

Put $L(t_0) = \sup_{t\in[0,1]} \int_0^t K(t,s)ds$, then using the mean value theorem and the same technique as the previous one, we obtain

$$\rho\Big(\theta(Tx(t) - Ty(t))\Big) \leq \theta^2 \mu L(t_0) \rho_\infty(x-y) \quad \forall t \in [0,1],$$

so

$$\rho_\infty\Big(\theta(Tx - Ty)\Big) \leq k\rho_\infty(x-y) \quad \text{with} \quad k = \theta^2\mu L(t_0) < \theta.$$

Hence we conclude from corollary 2.1 and remark 2.1 that T has a fixed point. □

Remark 3.1. *If R is reduced to the identity, then the solution of the integral equation (*) can be obtained directly using the Banach fixed point theorem.*

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