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MULTI-VALUED FIXED POINT THEOREMS BY ALTERING DISTANCE BETWEEN THE POINTS

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Abstract. In this paper we establish some results concerning fixed point of multivalued maps defined in a complete metric space, we then recover the results in [2] and [4].

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1 Preliminaries

Let (X, d) be a metric space, we denote by 2^X the collection of subsets of X and by $CB(X)$ the set of nonempty closed bounded subsets of X provided with the Hausdorff metric d_H defined by

$$d_H(A, B) = \max\Big(\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\Big),$$

where $A, B \in CB(X)$ and $d(x, A) = \inf_{y\in A} d(x, y)$. We define also $\delta(A, B)$ by

$$\delta(A, B) = \sup\Big\{d(a, b), \quad a \in A, \quad b \in B\Big\},$$

it follows immediately from the definition of δ that

$$\delta(A, B) = 0 \quad \Longleftrightarrow \quad A = B = \{.\} \quad \text{and} \quad \delta(\{.\}, B) = d_H(\{.\}, B).$$

We shall use the following definitions

Definition 1.1. *A function $\psi : [0, +\infty[\longrightarrow [0, 1[$ will be said to have property (P) if for every $t > 0$, there exists $\eta(t) > 0$ such that*

$$0 \le s - t < \eta(t) \quad \Longrightarrow \quad \psi(s) \le \gamma(t) < 1.$$

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Definition 1.2. *Let $F : X \longrightarrow 2^X$ be a multi-valued map, then a point $x \in X$ is called a fixed point of F if $x \in Fx$, it is called absolutely fixed if $Fx = \{x\}$.*

Now let us recall the two results which we shall extend

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ satisfying the following condition*

$$(C) \quad \phi(d(Tx,Ty)) \leq a(d(x,y))\phi(d(x,y)) + b(d(x,y))\Big[\phi(d(x,Tx)) + \phi(d(y,Ty))\Big] + c(d(x,y))\min\Big\{\phi(d(x,Ty)), \phi(d(y,Tx))\Big\}$$

where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous and strictly increasing function such that $\phi(0) = 0$ and $a, b, c : \mathbb{R}^+ \longrightarrow [0, 1[$ are three functions such that $a + 2b$ and $a + c$ satisfies the property (P), then T has a unique fixed point.

Theorem 1.2. *([4] p 206 theorem 34.5) Let (X, d) be a complete metric space, $T : X \longrightarrow X$ a generalized contraction, that is :*

$$d(T(x), T(y)) \leq q(\alpha, \beta)d(x, y)$$

where $\alpha \leq d(x, y) \leq \beta$ with $0 < \alpha \leq \beta < +\infty$ and $q(\alpha, \beta) < 1$, then T has a unique fixed point.

In the extension of theorem 1.1, we study the case where a, b, c are constants and the space is complete and the case where a, b, c are functions and the space is compact.

2 Case where a, b, c are constants and the space is complete

We begin by proving the following lemmas

Lemma 2.1. *Let A, B are two compact subsets of a metric space, then for all $x \in A$, there exists $y \in B$ such that $d(x, y) \leq d_H(A, B)$.*

Proof. It is an immediate consequence of the definition of the Hausdorff distance d_H. □

Lemma 2.2. *Let X be a metric space, $F : X \longrightarrow 2^X$ a multi-valued map with nonempty bounded values, $p \in [0, 1[$ and $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ a continuous function, then*

$$\forall x \in X, \quad \exists y_x \in Fx \quad \text{such that} \quad p\phi(\delta(x, Fx)) \leq \phi(d(x, y_x)).$$

Proof. We can suppose that $\delta(x, Fx) > 0 \quad \forall x \in X$, because if there exists $x \in X$ such that $\delta(x, Fx) = 0$ then $Fx = \{x\}$, so we take then $y_x = x$. Using the continuity of ϕ and the definition of δ, we deduce that for

$$\varepsilon = \phi(\delta(x, Fx)) - p\phi(\delta(x, Fx)) > 0,$$

there exists $y_x \in Fx$ such that

$$\phi(\delta(x, Fx)) < \varepsilon + \phi(d(x, y_x)),$$

hence $p\phi(\delta(x, Fx)) \leq \phi(d(x, y_x))$. □

Theorem 2.1. *Let (X,d) be a complete metric space, $T : X \longrightarrow CB(X)$ a multi-valued map satisfying the following condition :*

$$(H_1) \quad \phi(\delta(Tx,Ty)) \leq a\phi(d(x,y)) + b\Big[\phi(\delta(x,Tx)) + \phi(\delta(y,Ty))\Big] + \\ + c\min\Big\{\phi(d(x,Ty)), \phi(d(y,Tx))\Big\} \qquad \forall x,y \in X,$$

where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and strictly increasing such that $\phi(0) = 0$ and a, b, c are three positive constants such that $a + 2b < 1$ and $a + c < 1$, then T has a unique absolutely fixed point.

Proof. **Existence** Put $p = \sqrt{a+2b}$, then $p \in [0,1[$, so from lemma 2.2 we have

$$\forall x \in X \quad \exists y_x \in Tx, \quad \text{such that} \quad p\phi(\delta(x,Tx)) \leq \phi(d(x,y_x)).$$

Using the axiom of choice, we consider the map $f : X \longrightarrow X$ defined by $f(x) = y_x$, then $f(x) \in Tx \quad \forall x \in X$. Now $\forall x, y \in X$, we have

$$\begin{aligned}\phi(d(f(x),f(y))) &\leq \phi(\delta(Tx,Ty)) \\ &\leq a\phi(d(x,y)) + b\Big[\phi(\delta(x,Tx)) + \phi(\delta(y,Ty))\Big] + \\ &+ c\min\Big\{\phi(d(x,Ty)), \phi(d(y,Tx))\Big\} \\ &\leq a\phi(d(x,y)) + \frac{b}{p}\Big[\phi(d(x,f(x))) + \phi(d(y,f(y)))\Big] + \\ &+ c\min\Big\{\phi(d(x,f(y))), \phi(d(y,f(x)))\Big\},\end{aligned}$$

but

$$a + \frac{2b}{p} \leq \frac{a+2b}{p} = p < 1,$$

so from theorem 1.1 f has a fixed point, hence $\exists x \in X$ such that $f(x) = x \in Tx$, now using the fact that $p\phi(\delta(x,Tx)) \leq \phi(d(x,f(x))) = 0$, we deduce that $Tx = \{x\}$.

Unicity

Let x and y are two absolutely fixed point of T, then from the condition (H_1) we deduce that

$$\begin{aligned}\phi(d(x,y)) = \phi(\delta(Tx,Ty)) &\leq a\phi(d(x,y)) + b\Big[\phi(\delta(x,Tx)) + \phi(\delta(y,Ty))\Big] + \\ &+ c\min\Big\{\phi(d(x,Ty)), \phi(d(y,Tx))\Big\} \\ &\leq (a+c)\phi(d(x,y)) \\ &< \phi(d(x,y)),\end{aligned}$$

this is a contradiction. □

Remark 2.1. *In the proof of the last theorem, the use of the axiom of choice is indispensable to conclude that f is a map. If we don't use this axiom, we must suppose that the element y_x, given by lemma 2.2, is the unique element of the subset Fx satisfying some property, for example, if the multivalued map F is of closed and convex values in a strictly convex Banach space, then we can take for y_x the element of Fx with minimal norm.*

Theorem 2.2. *Let (X,d) be a complete metric space , $T : X \longrightarrow 2^X/\emptyset$ a multi-valued map with compact values satisfying the following condition*

$$(H_2) \quad \phi(d_H(Tx,Ty)) \leq a\phi(d(x,y)) + b\Big[\phi(d(x,Tx)) + \phi(d(y,Ty))\Big] + \\ + c\min\Big\{\phi(d(x,Ty)), \phi(d(y,Tx))\Big\} \qquad \forall x,y \in X,$$

where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and strictly increasing such that $\phi(0) = 0$ and a, b, c are three positive constants such that $a + 2b < 1$, we assume also that there exists $\alpha, \beta > 0$ such that $\phi(t) \geq \alpha t^\beta$ $\forall t > 0$, then T has a fixed point.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$, then $\exists x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq d_H(Tx_0, Tx_1)$$

(the last inequality follows from lemma 2.1), hence

$$\begin{aligned} \phi(d(x_1,x_2)) &\leq \phi(d_H(Tx_0,Tx_1)) \\ &\leq a\phi(d(x_0,x_1)) + b\Big[\phi(d(x_0,Tx_0)) + \phi(d(x_1,Tx_1))\Big] + \\ &+ c\min\Big\{\phi(d(x_0,Tx_1)), \phi(d(x_1,Tx_0))\Big\} \\ &\leq a\phi(d(x_0,x_1)) + b\Big[\phi(d(x_0,x_1)) + \phi(d(x_1,x_2))\Big] + \\ &+ c\min\Big\{\phi(d(x_0,Tx_1)), \phi(d(x_1,x_1))\Big\}, \end{aligned}$$

so

$$\phi(d(x_1,x_2)) \leq k\phi(d(x_0,x_1)) \qquad \text{where} \qquad k = \frac{a+b}{1-b} < 1.$$

Repeating this argument, we have for $x_2 \in Tx_1$, $\exists x_3 \in Tx_2$ such that

$$\phi(d(x_2,x_3)) \leq k\phi(d(x_1,x_2)) \leq k^2\phi(d(x_0,x_1)).$$

In general, for each integer $n \geq 0$ there exists x_n such that

$$x_{n+1} \in Tx_n \quad \text{and} \quad \phi(d(x_n,x_{n+1})) \leq k^n\phi(d(x_0,x_1)),$$

but from hypothesis there exists $\alpha, \beta > 0$ such that

$$\begin{aligned} \alpha[d(x_n,x_{n+1})]^\beta &\leq \phi(d(x_n,x_{n+1})) \\ &\leq k^n\phi(d(x_0,x_1)), \end{aligned}$$

therefore

$$d(x_n,x_{n+1}) \leq p^n\gamma \quad \text{where} \quad p = k^{1/\beta} \quad \text{and} \quad \gamma = \Big(\frac{\phi(d(x_0,x_1))}{\alpha}\Big)^{1/\beta}.$$

Since $p < 1$, we deduce that (x_n) is a Cauchy sequence, hence $\exists x \in X$ such that $\lim\limits_{n\to+\infty} x_n = x$. On the other hand,

$$|d(x_n,Tx) - d(x,Tx)| \leq d(x_n,x), \qquad \text{so} \quad \lim_{n\to+\infty} d(x_n,Tx) = d(x,Tx).$$

Now we have to show that x is a fixed point of T, that is $d(x, Tx) = 0$. Using the condition (H_2) we have

$$\begin{aligned}
\phi(d(x_{n+1}, Tx)) &\leq \phi(d_H(Tx_n, Tx)) \\
&\leq a\phi(d(x_n, x)) + b\Big[\phi(d(x_n, Tx_n)) + \phi(d(x, Tx))\Big] + \\
&+ c\min\Big\{\phi(d(x_n, Tx)), \phi(d(x, Tx_n))\Big\} \\
&\leq a\phi(d(x_n, x)) + b\Big[\phi(d(x_n, x_{n+1})) + \phi(d(x, Tx))\Big] + \\
&+ c\phi(d(x, x_{n+1})),
\end{aligned}$$

let $n \to +\infty$, we obtain

$$\phi(d(x, Tx)) \leq b\phi(d(x, Tx)) \leq \frac{1}{2}\phi(d(x, Tx)),$$

hence $\phi(d(x, Tx)) = 0$ and consequently $d(x, Tx) = 0$. $\square$

Remark 2.2. *If ϕ satisfies $\phi(t + s) \leq \phi(t) + \phi(s)$ $\forall t, s \in \mathbb{R}^+$, then we can eliminate the hypothesis $\phi(t) \geq \alpha t^\beta$, indeed, from the inequality*

$$\phi(d(x_n, x_{n+1})) \leq k^n\phi(d(x_0, x_1)),$$

we deduce that $\lim\limits_{n,p\to+\infty} \phi(d(x_n, x_p)) = 0$, *hence* $\lim\limits_{n,p\to+\infty} d(x_n, x_p) = 0.$

3 Case where a, b, c are functions and the space is compact

Theorem 3.1. *Let (X, d) be a compact metric space, $T : X \longrightarrow 2^X/\emptyset$ a multi-valued map with closed values, $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ continuous and strictly increasing such that $\phi(0) = 0$ and let a, b, c are three functions defined on $\mathbb{R}^+$ with values in $[0, 1[$ such that $a + 2b$ satisfies the property (P), suppose also that T satisfies the following condition*

$$(H_3) \quad \phi(d_H(Tx, Ty)) \leq a(d(x, y))\phi(d(x, y)) + b(d(x, y))\Big[\phi(d(x, Tx)) + \phi(d(y, Ty))\Big] +$$
$$+ c(d(x, y))\min\Big\{\phi(d(x, Ty)), \phi(d(y, Tx))\Big\} \qquad \forall x, y \in X,$$

then T has a fixed point.

Proof. Select $x_0 \in X$ and $x_1 \in Tx_0$, then $\exists x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq d_H(Tx_0, Tx_1),$$

so

$$\begin{aligned}
\phi(d(x_1, x_2)) &\leq \phi(d_H(Tx_0, Tx_1)) \\
&\leq a(d(x_0, x_1))\phi(d(x_0, x_1)) + b(d(x_0, x_1))\Big[\phi(d(x_0, Tx_0)) + \phi(d(x_1, Tx_1))\Big] + \\
&+ c(d(x_0, x_1))\min\Big\{\phi(d(x_0, Tx_1)), \phi(d(x_1, Tx_0))\Big\} \\
&\leq a(d(x_0, x_1))\phi(d(x_0, x_1)) + b(d(x_0, x_1))\Big[\phi(d(x_0, x_1)) + \phi(d(x_1, x_2))\Big] + \\
&+ c(d(x_0, x_1))\min\Big\{\phi(d(x_0, Tx_1)), \phi(d(x_1, x_1))\Big\},
\end{aligned}$$

hence $\phi(d(x_1,x_2)) \le q(d(x_0,x_1))\phi(d(x_0,x_1))$ where

$$q(d(x_0,x_1)) = \frac{a(d(x_0,x_1)) + b(d(x_0,x_1))}{1 - b(d(x_0,x_1))}.$$

By induction there exists x_n such that $x_{n+1} \in Tx_n$ and

$$\phi(d(x_n,x_{n+1})) \le q(d(x_{n-1},x_n))\phi(d(x_{n-1},x_n)) < \phi(d(x_{n-1},x_n)). \quad (*)$$

Put $l_n = d(x_n,x_{n+1})$, then we have $\phi(l_n) < \phi(l_{n-1})$, but ϕ is increasing, so l_n is decreasing, hence $\lim_{n\to+\infty} l_n = l$ exists . Now we show that $l = 0$, suppose that $l > 0$, then since q satisfies the property (P)

$$\exists\eta(l) > 0 \quad \text{such that} \quad 0 \le s - l < \eta(l) \quad \Longrightarrow \quad q(s) \le \gamma(l) < 1,$$

but $l_n \ge l$ and $\lim_{n\to+\infty} l_n = l$, so there exists n_0 such that

$$n \ge n_0 \quad \Longrightarrow \quad 0 \le l_n - l < \eta(l),$$

consequently $q(l_n) \le \gamma(l) < 1$. From the inequality (*) we deduce that

$$\phi(l_n) \le \gamma(l)\phi(l_{n-1}) \quad \forall n \ge n_0,$$

hence when $n \to +\infty$ we obtain

$$\phi(l) \le \gamma(l)\phi(l) < \phi(l),$$

this is a contradiction, so $\lim_{n\to+\infty} d(x_n,x_{n+1}) = 0$. X is compact, so x_n has a convergent subsequence which we denote by y_n, let $y = \lim_{n\to+\infty} y_n$, we have to show that y is a fixed point of T. Since $\lim_{n\to+\infty} d(y_n,Ty) = d(y,Ty)$, we deduce by the condition (H_3) that

$$\begin{aligned}
\phi(d(y_{n+1},Ty)) &\le \phi(d_H(Ty_n,Ty)) \\
&\le a(d(y_n,y))\phi(d(y_n,y)) + b(d(y_n,y))\Big[\phi(d(y_n,Ty_n)) + \phi(d(y,Ty))\Big] \\
&+ c(d(y_n,y))\min\Big\{\phi(d(y_n,Ty)), \phi(d(y,Ty_n))\Big\} \\
&\le a(d(y_n,y))\phi(d(y_n,y)) + b(d(y_n,y))\Big[\phi(d(y_n,y_{n+1})) + \phi(d(y,Ty))\Big] \\
&+ c(d(y_n,y))\phi(d(y,y_{n+1})) \\
&< \phi(d(y_n,y)) + \frac{1}{2}\Big[\phi(d(y_n,y_{n+1})) + \phi(d(y,Ty))\Big] + \phi(d(y,y_{n+1})),
\end{aligned}$$

therefore taking $n \to +\infty$, we obtain

$$\phi(d(y,Ty)) \le \frac{1}{2}\phi(d(y,Ty)), \quad \text{hence} \quad \phi(d(y,Ty)) = 0,$$

consequently $d(y,Ty) = 0$, and $y \in Ty$. $\square$

4 Generalization of Krasnosel'skii theorem

The generalized contraction mapping theorem which is due to M.A Krasnosels'kii is an extension of the well-known Banach's contraction principle. In the following theorem, we consider a multi-valued and generalized ϕ-contraction.

Theorem 4.1. *Let (X,d) be a complete metric space, $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ a continuous and strictly increasing map such that $\phi(0) = 0$, we suppose also that $\phi(t+s) \leq \phi(t) + \phi(s) \quad \forall s,t \in \mathbb{R}^+$. Let $T : X \longrightarrow 2^X/\emptyset$ a multi-valued map with compact values satisfying the following condition*

$$(H_4) \qquad \phi(d_H(Tx,Ty)) \leq q(\alpha,\beta)\phi(d(x,y))$$

where $0 < \alpha \leq \beta < +\infty$, $\alpha \leq d(x,y) \leq \beta$ and $q(\alpha,\beta) < 1$, then T has a fixed point.

Remarks 4.1. 1) *From the condition (H_4), we deduce that T is continuous.*
2) *The condition (H_4) is in particular satisfied if*

$$\phi(d_H(Tx,Ty)) \leq \phi(d(x,y)) - r(\phi(d(x,y))) \quad \text{where} \quad r : \mathbb{R}^+_* \longrightarrow \mathbb{R}^+_* \text{ is continuous,}$$

indeed $\phi(d(x,y)) - r(\phi(d(x,y))) = \phi(d(x,y))\Big[1 - \frac{r(\phi(d(x,y)))}{\phi(d(x,y))}\Big]$, *so we can take* $q(\alpha,\beta) = \sup\limits_{t\in[\alpha,\beta]} (1 - \frac{r(\phi(t))}{\phi(t)})$.

Proof of theorem 4.1. Select $x_0 \in X$ and $x_1 \in Tx_0$, then $\exists x_2 \in Tx_1$ such that $d(x_1,x_2) \leq d_H(Tx_0,Tx_1)$, so

$$\phi(d(x_2,x_1)) \leq \phi(d_H(Tx_0,Tx_1)) \leq q(\alpha,\beta)\phi(d(x_1,x_0)),$$

at the same $\exists x_3 \in Tx_2$ such that $\phi(d(x_3,x_2)) \leq q(\alpha,\beta)\phi(d(x_2,x_1))$ and by induction there exists a sequence x_n such that $x_{n+1} \in Tx_n$ and

$$\begin{aligned}\phi(d(x_{n+1},x_n)) &\leq \phi(d_H(Tx_n,Tx_{n-1})) \\ &\leq q(\alpha,\beta)\phi(d(x_n,x_{n-1})) \\ &< \phi(d(x_n,x_{n-1})),\end{aligned}$$

hence the sequence $\phi(d(x_{n+1},x_n))$ is decreasing, but ϕ is increasing, so we deduce that the sequence $\tau_n = d(x_{n+1},x_n)$ is decreasing. Let $\tau = \lim\limits_{n\to+\infty} \tau_n$, we have to show that $\tau = 0$, suppose that $\tau > 0$, then for $\varepsilon = 1$ there exists n_0 such that $n \geq n_0 \implies \tau_n \leq \tau + 1$, on the other hand $\tau \leq \tau_n$, so from some rank, the consecutive terms of the sequence x_n are in the crow with radius τ and $\tau + 1$, so from the condition (H_4) we have

$$\phi(\tau_{n+m}) \leq \Big[q(\tau,\tau+1)\Big]^m \phi(\tau+1) \qquad \forall m = 1,2,...$$

letting $m \to +\infty$ we obtain $\phi(\tau) \leq 0$, this is a contradiction. Hence

$$\lim_{n\to+\infty} d(x_{n+1},x_n) = 0 \quad \text{and consequently} \quad \lim_{n\to+\infty} \phi(d(x_{n+1},x_n)) = 0.$$

Now we have to show that x_n is a Cauchy sequence, to do this we show that $\lim_{n,m\to+\infty} \phi(d(x_{n+m}, x_n)) = 0$. Let $\varepsilon > 0$, then for large n we have

$$\phi(d(x_{n+1}, x_n)) \leq \frac{\varepsilon}{2}\Big[1 - q(\phi^{-1}(\frac{\varepsilon}{2}), \phi^{-1}(\varepsilon))\Big]$$

where $q(\phi^{-1}(\frac{\varepsilon}{2}), \phi^{-1}(\varepsilon))$ will be specified later. We show that

$$\phi(d(x_{n+1}, x_n)) \leq \varepsilon \quad \Longrightarrow \quad \phi(d(x_{n+2}, x_n)) \leq \varepsilon,$$

and by the same way we show also that $\phi(d(x_{n+3}, x_n)) \leq \varepsilon$ and so on, we show that $\phi(d(x_{n+m}, x_n)) \leq \varepsilon \quad \forall m = 1, 2, ..$

Assume that $\phi(d(x_{n+1}, x_n)) \leq \varepsilon$, then

i) If $\phi(d(x_{n+1}, x_n)) \leq \frac{\varepsilon}{2}$, then

$$\begin{aligned}\phi(d(x_{n+2}, x_n)) &\leq \phi\Big(d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)\Big)\\ &\leq \phi(d(x_{n+2}, x_{n+1}) + \phi(d(x_{n+1}, x_n))\\ &\leq \phi(d(x_{n+1}, x_n)) + \phi(d(x_{n+1}, x_n))\\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

ii) If $\frac{\varepsilon}{2} \leq \phi(d(x_{n+1}, x_n)) \leq \varepsilon$, then $\phi^{-1}(\frac{\varepsilon}{2}) \leq d(x_{n+1}, x_n) \leq \phi^{-1}(\varepsilon)$, so from the condition (H_4) we deduce that

$$\begin{aligned}\phi(d(x_{n+2}, x_n)) &\leq \phi(d(x_{n+2}, x_{n+1})) + \phi(d(x_{n+1}, x_n))\\ &\leq \phi(d_H(Tx_{n+1}, Tx_n)) + \phi(d(x_{n+1}, x_n))\\ &\leq q(\phi^{-1}(\frac{\varepsilon}{2}), \phi^{-1}(\varepsilon))\phi(d(x_{n+1}, x_n)) + \frac{\varepsilon}{2}\Big(1 - q(\phi^{-1}(\frac{\varepsilon}{2}), \phi^{-1}(\varepsilon))\Big)\\ &\leq q(\phi^{-1}(\frac{\varepsilon}{2}), \phi^{-1}(\varepsilon))\varepsilon + \frac{\varepsilon}{2} - q(\phi^{-1}(\frac{\varepsilon}{2}), \phi^{-1}(\varepsilon))\frac{\varepsilon}{2}\\ &\leq \varepsilon.\end{aligned}$$

Hence $\exists x \in X$ such that $\lim_{n\to+\infty} x_n = x$, now since T is continuous and

$$d(x_{n+1}, Tx) \leq d_H(Tx_n, Tx),$$

we deduce by letting $n \to +\infty$ that $d(x, Tx) = 0$ and consequently $x \in Tx$. $\square$

5 Consequences of theorem 4.1

1) If T is univalued and $\phi(t) = t$, then we recover Krasnosel'skii's theorem (see theorem 1.2 in section 1).

2) If T is univalued, $\phi(t) = t$ and $q(\alpha, \beta) = k < 1$, then we recover Banach' theorem (see e.g [3] p 7).

3) If T is univalued, $\phi(t) = t$ and $q(\alpha, \beta) = \sup_{t\in[\alpha,\beta]} \frac{\varphi(t)}{t}$, then we recover Boyd-Wong' theorem (see e.g [5] p 117).

4) If T is univalued, $\phi(t) = t$ and $q(\alpha, \beta) = \sup_{D\times D} \frac{d(f(x), f(y))}{d(x,y)}$, then we recover Edelstein' theorem (see e.g [3] p 12).

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