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A COMMON FIXED POINT THEOREM FOR NEAR-CONTRACTIVE MAPPINGS

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Abstract. In this work we establish a common fixed point theorem in complete metric spaces which generalizes some results in [1] and [2].

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1 Preliminaries

In 1984, M. S. khan [1] has established fixed point theorem for self-maps of complete metric spaces by altering the distances between the points with the use of a function $\phi : [0, \infty) \to [0, \infty)$ satisfying the following properties:

$$(i) \quad \phi : [0, \infty) \to [0, \infty) \quad \text{is continuous and stricly increasing in} \quad \Re^+,$$
$$(ii) \quad \phi(t) = 0 \quad \text{if} \quad t = 0.$$

They proved the following Theorem:

Theorem 1.1. *Let T be a self-map of a complete metric space (X, d) and ϕ satisfying (i) and (ii). Furthermore, let a, b, c be three decreasing functions from $(0, \infty)$ into $[0, 1)$ such that*

$$a(t) + 2b(t) + c(t) < 1$$

for every $t > 0$.
Suppose that T satisfies the following condition:

$$(A) \quad \phi(d(Tx, Ty)) \le a(d(x, y))\phi(d(x, y)) + b(d(x, y))\Big[\phi(d(x, Tx)) + \phi(d(y, Ty))\Big] + c(d(x, y)) \min\Big\{\phi(d(x, Ty)), \phi(d(y, Tx))\Big\}$$

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where $x, y \in X$ and $x \neq y$. Then T has a unique fixed point.

In 1996 R.A.RASHWAN AND A.M.SADEEK [2] have established a common fixed point theorem for a apair of mappings satisfying the contractive conditions corresponding to (A). They proved the following Theorem 1.2 which generalizes Theorem 1.1 :

Theorem 1.2. *Let S and T be self-mappings on a complete metric space (X, d) and ϕ stisfying the conditions (i) and (ii). Furthermore, let a, b, c be three decreasing functoins from $[0, \infty)$ into $[0, 1)$ such that*

$$(B.1) \quad a(t) + 2b(t) + c(t) < 1$$

for every $t > 0$. Suppose that S and T satisfy the following condition:

$$(B.2) \quad \phi(d(Sx, Ty)) \leq a(d(x,y))\phi(d(x,y)) + b(d(x,y))\Big[\phi(d(x,Sx)) + \phi(d(y,Ty))\Big] + c(d(x,y)) \min\Big\{\phi(d(x,Ty)), \phi(d(y,Sx))\Big\}$$

where $x, y \in X$ and $x \neq y$. Then T and S have a unique common fixed point in X

In this works we shall extend this last theorem by using a much extensive hypothesis than (B. 1) and a, b and c are three decreasing functions.
We shall use the following definition

Definition 1.1. *A function $\psi : [0, +\infty[\longrightarrow [0, 1[$ will be said to have property (P) if for every $t > 0$, there exists $\eta(t) > 0$ such that*

$$0 \leq s - t < \eta(t) \quad \Longrightarrow \quad \psi(s) \leq \gamma(t) < 1.$$

2 Main Results

Theorem 2.1. *Let T and S be a self-mappings on a complete metric space (X, d) and ϕ satisfying (i) and (ii). Furthermore, let a, b and c be three functions from $[0, \infty)$ into $[0, 1)$ satisfying at least one of the following conditions:*

(D_1) $a + 2b$ *has property (P) and* $a + c : (0, \infty) \to [0, 1)$ *is decreasing function.*

(D_2) $a + 2b + c : (0, \infty) \to [0, 1)$ *is decreasing function.*

Suppose that T and S satisfy the condition $(B.2)$. Then T and S have a unique common fixed point in X.

Proof. Suppose first that a, b and c satisfy the condition (D_2) then a + 2b + c has property (P) which implies $a + 2b$ has property (P).
Let x_0 be some point in X, define:

$$x_{2n+2} = Sx_{2n+1}, \qquad n = 0, 1, 2, ...$$
$$x_{2n+1} = Tx_{2n}, \qquad n = 0, 1, 2, ...$$

Let $\tau_n = d(x_n, x_{n+1})$. We may assume that $\tau_n > 0$ for every n. Now for an even integer $n > 0$, We have

$$\phi(\tau_n) = \phi(d(x_n, x_{n+1})) = \phi(d(S^{n-1}x, T^n x)) \leq A_1 + A_2 + A_3$$

Where

$$A_1 = a(d(x_{n-1}, x_n))\phi(d(x_{n-1}, x_n)),$$
$$A_2 = b(d(x_{n-1}, x_n))[\phi(d(x_{n-1}, Sx_{n-1})) + \phi(d(x_n, Tx_n))],$$
$$A_3 = c(d(x_{n-1}, x_n))min\{\phi(d(x_{n-1}, Tx_n)), \phi(d(x_n, Sx_{n-1}))\}.$$

So

$$\phi(\tau_n) \leq a(\tau_{n-1})\phi(\tau_{n-1}) + b(\tau_{n-1})[\phi(\tau_{n-1}) + \phi(\tau_n)],$$

Hence we have

$$\phi(\tau_n) \leq \frac{a(\tau_{n-1}) + b(\tau_{n-1})}{1 - b(\tau_{n-1})}\phi(\tau_{n-1}) < \phi(\tau_{n-1}) \quad (1)$$

Similary for an odd integer n one can show that n satisfies to (1). Then for all integer $n \geq 1$ we have:

$$\phi(\tau_n) \leq \frac{a(\tau_{n-1}) + b(\tau_{n-1})}{1 - b(\tau_{n-1})}\phi(\tau_{n-1}) < \phi(\tau_{n-1}) \quad (2)$$

Since ϕ is incressing, $\{\tau_n\}$ is a decreasing sequence. Let $lim_n \tau_n = \tau$, assume that $\tau > 0$. Since $a + 2b$ has property (P), then for $0 \leq$ s -$\tau < \eta(\tau)$,

$$a(s) + 2b(s) \leq \gamma(\tau) < 1,$$

it follows that for these s

$$\frac{a(s) + b(s)}{1 - b(s)} \leq \frac{\gamma(\tau) - b(s)}{1 - b(s)} \leq \gamma(\tau) \quad (3) \text{ too}$$

Since

$$\tau_n \geq \tau, \quad \lim_n \tau_n = \tau \qquad \text{and} \quad \lim_n \frac{\phi(\tau_n)}{\phi(\tau_{n-1})} = 1,$$

there exists an integer $p \geq 1$ such that

$$0 \leq \tau_{p-1} - \tau < \eta(\tau) \qquad (4) \text{ while} \quad \gamma(\tau) < \frac{\phi(\tau_p)}{\phi(\tau_{p-1})},$$

the last inequality with (2) imply that

$$\gamma(\tau) < \frac{a(\tau_{p-1}) + b(\tau_{p-1})}{1 - b(\tau_{p-1})}$$

and (3) with (4) imply that

$$\frac{a(\tau_{p-1}) + b(\tau_{p-1})}{1 - b(\tau_{p-1})} \leq \gamma(\tau),$$

which is a contradiction. Consequently $\tau = 0$. Now, we wish to show that $\{x_n\}$ is a cauchy sequence since $\tau = 0$ it suffices to prove that $\{x_{2n}\}$ is a cauchy sequence we proceed by contraction. Then there exists $\varepsilon > 0$, and two sequences of natural numbers $\{2m(i)\}, \{2n(i)\}$ such that

$$2n(i) > 2m(i) \geq 2i \qquad \text{and} \quad d(x_{2n(i)}, x_{2m(i)}) > \varepsilon \quad (5)$$

For each even integer $2i$, let $2m(i)$ be the least even a integer exceeding $2n(i)$ and satisfy (5) Then.

$$d(x_{2n(i)}, x_{2m(i)}) > \varepsilon \qquad \text{and} \quad d(x_{2n(i)-2}, x_{2m(i)}) \leq \varepsilon$$

Then we have

$$\begin{aligned} \varepsilon < d_{2i} &= d(x_{2n(i)}, x_{2m(i)}) \\ &\leq d(x_{2n(i)}, x_{2n(i)-1}) + d(x_{2n(i)-1}, x_{2n(i)-2}) + d(x_{2n(i)-2}, x_{m(i)}) \\ &\leq \tau_{2n(i)-1} + \tau_{2n(i)-2} + \varepsilon, \end{aligned}$$

since $\{\tau_n\}$ converges to 0, $d_{2i} \to \varepsilon$. Futhermore, by triangler inequality, it follows that

$$d_{2i} - \tau_{2m(i)} \leq d(x_{2n(i)}, x_{2m(i)+1}) \leq d_{2i} + \tau_{2m(i)}$$

and therefor the sequence $\{d(x_{2n(i)}, x_{2m(i)+1})\}$ converges to ε. Moreover, by triangle inquality we have:

$$d_{2i} - \tau_{2n(i)-1} \leq d(x_{2n(i)-1}, x_{2m(i)}) \leq d_{2i} + \tau_{2n(i)-1}$$

Thus

$$d(x_{2n(i)-1}, x_{2m(i)}) \to \varepsilon \qquad \text{as} \quad i \to \infty.$$

From (B. 2), we also deduce :

$$\begin{aligned} \phi(d(x_{2n(i)}, x_{2m(i)+1})) &= \phi(d(Sx_{2n(i)-1}, Tx_{2m(i)})) \\ &\leq C_1 + C_2 + C_3 \\ &\leq C_4(D_j) + C_5(D_j) + C_6(D_j) \quad (6.j) \quad j = 1, 2 \end{aligned}$$

Where

$$\begin{aligned} C_1 &= a(d(x_{2n(i)-1}, x_{2m(i)})\phi(d(x_{2n(i)-1}, x_{2m(i)}), \\ C_2 &= b(d(x_{2n(i)-1}, x_{2m(i)}))[\phi(\tau_{2n(i)-1}) + \phi(\tau_{2m(i)})], \\ C_3 &= c(d(x_{2n(i)-1}, x_{2m(i)}))min\{\phi(d(x_{2n(i)}, x_{2m(i)}), \phi(d(x_{2n(i)-1}, x_{2m(i)+1}))\}, \\ C_4(D_1) &= a(d(x_{2n(i)-1}, x_{2m(i)})\phi(d_{2i} + \tau_{2n(i)-1}), \\ C_5(D_1) &= [\phi(\tau_{2n(i)-1}) + \phi(\tau_{2m(i)})], \\ C_6(D_1) &= c(d(x_{2n(i)-1}, x_{2m(i)})\phi(d_{2i} + \tau_{2n(i)-1}), \end{aligned}$$

and

$$\begin{aligned} C_4(D_2) &= a(d(x_{2n(i)-1}, x_{2m(i)})\phi(d_{2i} + \tau_{2n(i)-1} + \tau_{2m(i)}), \\ C_5(D_2) &= 2b(d(x_{2n(i)-1}, x_{2m(i)})\phi(d_{2i} + \tau_{2n(i)-1} + \tau_{2m(i)}), \\ C_6(D_2) &= c(d(x_{2n(i)-1}, x_{2m(i)})(\phi(d_{2i} + \tau_{2n(i)-1}) + \tau_{2m(i)}). \end{aligned}$$

Since $d(x_{2n(i)-1}, x_{2m(i)}) \to \varepsilon$ as $i \to \infty$, there exists an integer $q \geq 1$ such that

$$d(x_{2n(i)-1}, x_{2m(i)}) > \frac{\varepsilon}{2} \quad (7).$$

for all $i \geq q$.
From (D_1), (6. 1) and (7) we have:

$$\phi(d(x_{2n(i)}, x_{2m(i)+1})) \leq [a(\frac{\varepsilon}{2}) + c(\frac{\varepsilon}{2})]\phi(d_{2i} + \tau_{2n(i)-1}) + [\phi(\tau_{2n(i)-1}) + \phi(\tau_{2m(i)})]$$

for all $i \geq q$. As $i \to \infty$, we have:

$$\phi(\varepsilon) \leq [a(\frac{\varepsilon}{2}) + c(\frac{\varepsilon}{2})]\phi(\varepsilon) < \phi(\varepsilon)$$

This is a contradiction.
From (D_2), (6. 2) and (7) we have:

$$\phi(d(x_{2n(i)}, x_{2m(i)+1})) \leq [a(\frac{\varepsilon}{2}) + c(\frac{\varepsilon}{2}) + 2b(\frac{\varepsilon}{2})]\phi(d_{2i} + \tau_{2n(i)-1} + \tau_{2m(i)})$$

for all $i \geq q$. As $i \to \infty$, we have:

$$\phi(\varepsilon) \leq [a(\frac{\varepsilon}{2}) + c(\frac{\varepsilon}{2}) + 2b(\frac{\varepsilon}{2})]\phi(\varepsilon) < \phi(\varepsilon)$$

This is a contradiction.
Hence $\{x_n\}$ is a cauchy sequence in a complete metric space X, then there exists a point $x \in X$ such that

$$x_n \to x \qquad \text{as} \quad n \to \infty.$$

Now we shall show that $Sx = Tx = x$. From (B. 2), we have:

$$\begin{aligned}\phi(d(x_{2n+2}, Tx)) &= \phi(d(Sx_{2n+1}, Tx)) \\ &\leq a(d(x_{2n+1}, x))\phi((d(x_{2n+1}, x)) \\ &+ b((d(x_{2n+1}, x)))[\phi(d(x_{2n+1}, x_{2n+2}) + \phi(d(Tx, x))] \\ &+ c((d(x_{2n+1}, x)))min\{\phi((d(x_{2n+1}, Tx)), \phi(d(x_{2n+2}, x))\} \quad (8)\end{aligned}$$

Using $b < \frac{1}{2}$ and letting $n \to \infty$ in (8), we have

$$\phi(d(x, Tx)) \leq \frac{1}{2}\phi(d(x, Tx)).$$

That is $\phi(d(x, Tx)) = 0$ and therefore $d(x, Tx) = 0$, i.e. $x = Tx$. Similarly, it can be shown that $Sx = x$. Hence S and T have a common fixed point $x \in X$. We claim that x is the unique common fixed point of S and T. For this suppose that y $(x \neq y$) is another fixed point of T and S. Then

$$\begin{aligned}\phi(d(x, y)) = \phi(\delta(Sx, Ty)) &\leq a\phi(d(x, y)) + b\Big[\phi(\delta(x, Sx)) + \phi(\delta(y, Ty))\Big] + \\ &+ c\min\Big\{\phi(d(x, Ty)), \phi(d(y, Sx))\Big\} \\ &\leq (a + c)\phi(d(x, y)) \\ &< \phi(d(x, y)),\end{aligned}$$

a contradiction. This completes the proof of the theorem.

Remark 2.1. *We first note that Theorem 1.2 is a special case of each results of Theorem 2.1. Also, it is obvious that in case* (D_1) *of Theorem 2.1 the function* $a + 2b + c$ *can take values larger than 1.*

References

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