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# ON ANALYTIC CONTINUATION OF SUMMABILITY TRANSFORMS

Rohitha Goonatilake

**Abstract.** The question of possibility of extending the definition of an analytic function to a larger domain is the concept of analytic continuation. This will be the subject of this paper for various summability transforms of random variables.

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## I. INTRODUCTION

The question of the possibility of extending the definition of an analytic function to a larger domain is the concept of analytic continuation. This will be the subject of the present paper.

At the outset, we recall that the series $\sum_{k=0}^{\infty} a_k z^k$ with $\{a_k\}_{k\geq 0}$ being constants and the positive radius of convergence, $R > 0$, converges for $|z| < R$, where it represents an analytic function. The series diverges for $|z| \geq R$ and so that the domain $|z| < R$ is the largest possible domain of convergence of the series. Over the domain $|z| < R$, the power series converges in general and hence represents an analytic function. In this connection, we now appeal to:

**Theorem 1 (Markushevich, Theorem 6.1 [1]).** *If $\{g_n(z)\}_{n=0}^{\infty}$ is a sequence of functions, each analytic on a domain $D$, and this sequence converges uniformly on compact subsets of $D$, then the limiting function $g(z)$ is analytic on $D$.*

In the event, the transformation to be chosen is reasonably well-behaved then the limiting function $g(z)$ agrees with $f(z)$ on $\{z : |z - z_0| < R\}$, and hence, $g(z)$ will yields the analytic continuation of $f(z)$ to the domain $D$. As an application of summability to analytic continuation, we consider the following theorem [1]. Before begin this task, we recall the definition of the Borel polygon, which plays a significant importance in the discussion to follow.

Division of Mathematics, Department of Natural Sciences, Texas A&M International University
E-mail address: harag@tamiu.edu

**Definition 1.** Let $f$ be analytic at the origin. If $s$ is a singular point of $f$, then let $l(s)$ be the line perpendicular to the line segment $[0, s]$ passing through the point $s$, and let $H_{l(s)}$ be the open half-plane containing the origin with boundary $l(s)$. The Borel polygon with respect to $f$ is the set

$$B(f) = \cap H_{l(s)},$$

where the intersection is taken over all singular points of $f$.

$B(f)$ is a domain, since it is the intersection of open, arc-wise connected set.

**Example 1.** The Borel polygon of $f(z) = 1/(1-z)$ is the half-plane $\{z : \mathrm{Re}(z) < 1\}$, since $z = 1$ is the only point of singularity of $f$.

## II. SUMMABILITY ANALOGUES

The summability analogues of Theorem 1 for Borel exponential method is described in the following theorem.

**Theorem 2 (Powel & Shah [1]).** *Let $s_n(z) = \sum_{k=0}^{n} a_k z^k, z \neq 1$, where $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence $R > 0$. If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ on $S_R(0)$ and $z \in B(f)$, then*

$$\lim_{x\to\infty} e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} s_k(z) = f(z),$$

*where $S_R(0)$ denotes a disc of radius $R$ centered at $0$ and $B(f)$ is the Borel polygon with respect to $f$.*

The proof of this theorem is given for Borel exponential method. The same proof works for the Borel matrix method. The proofs for the Euler and the Taylor transformations can be obtained with a minor adjustment to the proof as follows;

$$\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} (\frac{z}{t})^k = (\frac{pz}{t} + q)^n = (1 + p[\frac{z}{t} - 1])^n \simeq \exp\{np[\frac{z}{t} - 1]\}.$$

Much of the discussion will now revolve around the following types of summability methods. Let $Y, X_1, X_2, \cdots$ be a sequence of independent non-negative integer valued random variables such that $Y$ has probability function $q$ and the $X_i's$ are identically distributed with probability function $p$. Let $S_0 = Y$ and $S_n = Y + X_1 + \cdots + X_n$ for $n \geq 1$. Let $\{p_j\}_{j\geq 0}$ and $\{q_j\}_{j\geq 0}$ be the probability distributions of $X_1, X_2, \cdots$ and $Y$ respectively. The $n^{\text{th}}$ row $k^{\text{th}}$ column entry of the convolution summability matrix $C$ is the probability $C_{n,k} = P(S_n = k)$. The method $C$ is regular if and only if $P(X_1 = 0) < 1$. Some classical summability methods are examples of the method $C$. For instance, when $Y = 0$ and $X_1 \sim \mathrm{Binomial}(1, 1)$, then $C$ becomes the Euler method denoted by $E_r$. When $Y \sim X_1 \sim \mathrm{Poisson}(1)$ we get the Borel matrix method. When $Y \sim \mathrm{Geometric}(1 - r)$ and $X_1 \sim Y + 1$, then we get the Taylor method. And when $Y \sim X_1 \sim \mathrm{Geometric}(1 - r)$ we get the Meyer-Kónig method. We shall call $C$ a convolution method and when $Y = 0$ with probability 1, it is called the random-walk method. The method $C$ can be extended easily to non-identically distributed random variables; however, it will serve our purpose adequately, as it is. Hence, it is suspected that the regular convolution summability matrix transformation $\{C_{n,k}\}_{n,k\geq 0}$ obeys this theorem. The regular convolution summability matrix referred to everywhere in this paper has the above construction with appropriate moment conditions.

**Theorem 3.** *Let $s_n(z) = \sum_{k=0}^{n} a_k z^k$, where $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence $R > 0$. If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ on $S_R(0)$, $z \in B(f)$, and $\{p_j\}_{j\geq 0}$ has the property that $\sum_{j=0}^{\infty} p_j \theta^j < \infty$ for $0 \leq \theta < R < \infty$, then*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} C_{n,k} s_k(z) = f(z).$$

*Proof.*

We proceed as in [1]. Let $z \in B(f)$. If $z = 0$, the result follows immediately. It is therefore assumed that $z \neq 0$. By lemma 6.4 of [1], we have that $f$ is analytic inside and on the circle $C$ having diameter $[0, z]$. Let $\Gamma_1$ and $\Gamma$ be two circles, each concentric to $C$, where $f$ is analytic on $\operatorname{int}(\Gamma_1)$, $C \subseteq \operatorname{int}(\Gamma)$, and $\Gamma \subseteq \operatorname{int}(\Gamma_1)$. Now

$$\begin{aligned}
s_n(z) &= \sum_{k=0}^{n} a_k z^k \\
&= \sum_{k=0}^{n} \Big( \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t^{k+1}} dt \Big) z^k \\
&= \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t} \Big( \sum_{k=0}^{n} \Big(\frac{z}{t}\Big)^k \Big) dt \\
&= \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t} \Big[ \frac{1-(z/t)^{n+1}}{1-(z/t)} \Big] dt \\
&= \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} \Big( 1 - \Big(\frac{z}{t}\Big)^{n+1} \Big) dt \\
&= \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} dt - \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} \Big(\frac{z}{t}\Big)^{n+1} dt \\
&= f(z) - \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} \Big(\frac{z}{t}\Big)^{n+1} dt.
\end{aligned}$$

Taking the above into consideration, we consider

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=0}^{\infty} C_{n,k} s_k(z) &= \lim_{n\to\infty} \Big\{ \sum_{k=0}^{\infty} C_{n,k} f(z) - \sum_{k=0}^{\infty} C_{n,k} \Big( \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} \Big(\frac{z}{t}\Big)^{k+1} dt \Big) \Big\} \\
&= f(z) - \lim_{n\to\infty} \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} \sum_{k=0}^{\infty} C_{n,k} \Big(\frac{z}{t}\Big)^{k+1} dt.
\end{aligned}$$

Thus in order to obtain the required assertion, we will show that

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} P\{S_n = k\} (\frac{z}{t})^{k+1} = 0.$$

This will be done in the following manner.

$$\sum_{k=0}^{\infty} P\{S_n = k\} (\frac{z}{t})^{k+1} = (\frac{z}{t}) \sum_{k=0}^{\infty} P\{S_n = k\} (\frac{z}{t})^{k}.$$

The assumption $\sum_{j=0}^{\infty} p_j\theta^j < \infty$ for $0 \le \theta < R < \infty$, guarantees the convergence of the series.

$$\sum_{k=0}^{\infty} P\{S_n = k\}(\frac{z}{t})^{k+1} = (\frac{z}{t})E(\frac{z}{t})^{S_n};\ \text{by the definition of the expectation.}$$

$$= (\frac{z}{t})E\{(\frac{z}{t})^Y(\frac{z}{t})^{\sum_{i=1}^n X_i}\}$$

$$= (\frac{z}{t})\{E(\frac{z}{t})^Y \Pi_{i=1}^n E(\frac{z}{t})^{X_1}\}$$

as r. v.'s are independent and $X_1's$ are i. i. d..

$$= (\frac{z}{t})E\{(\frac{z}{t})^Y(E(\frac{z}{t})^{X_1})^n \to 0,$$

in view of the inclusion;

$$\{z : |z| < t\} \supseteq \{z : |z|^{X_i} < t^{X_1} \text{ a. s.}\} \supseteq \{z : E|\frac{z}{t}|^{X_1} < 1\};$$

which implies that $E(|\frac{z}{t}|^{X_1})^n \to 0$. This completes the proof of the theorem. □

We see that the convolution summability matrix transformation of the sequence of partial sums of $\sum_{k=0}^{\infty} a_k z^k$ gives us the analytical continuation of the power series $\sum_{k=0}^{\infty} a_k z^k$ to the Borel polygon $B(f)$ (since, $B(f)$ is a domain and $S_R(0) \subset B(f)$.) However, we are very much interested in obtaining a theorem in the following setting.

## III. THE OKADA THEOREM

We now examine the above theorem in a more general setting, in particular; we will use a general matrix transformation of the sequence of partial sum of $\sum_{k=1}^{\infty} a_k z^k$, which has a positive radius of convergence.

As shown in [1], the Okada theorem holds in general for a regular infinite matrix provided that the three assumptions therein are satisfied. Condition (iii) of the Silverman-Toeplitz theorem need not hold. We will show that the convolution summability method satisfies the rigorous conditions stipulated in the Okada theorem. This is worth considering, as the convolution summability method includes Borel, Euler, Meyer-König and Taylor transformation methods and much more.

We will denote the complement of a set $A$ of complex numbers by $C(A)$ and that of $cA$ by $cA = \{cz : z \in A\}$.

**Theorem 4 (Okada Theorem [1]).** *Let $A = \{a_{n,k}\}$ be an infinite matrix of complex entries and denote the $k^{th}$ partial sum of the geometric series, $\sum_{j=0}^{k} z^j$, by $s_k(z)$.*

*Let $R' = \{z : \lim_{n\to\infty} \tau_n(z) := \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}s_k(z) = \frac{1}{(1-z)}\}$ and assume that $B' = \{z : |z| < 1\} \subseteq R', R'$ is open and $\lim_{n\to\infty} \tau_n(z) = \frac{1}{(1-z)}$ uniformly on compact subsets of $R'$. Let $R = \{\omega = 1/z : z \in C(R')\} \cup \{0\}$. Let $D$ be a fixed domain, where $0 \in D$ and denote by $\mathcal{F}$, the collection of all functions analytic on $D$. For $f \in \mathcal{F}$ denote the power series representation of $f$ about the origin by $\sum_{k=0}^{\infty} b_k(f)z^k$.*

*If $U(D) = \{z : \lim_{n\to\infty} \sigma_n^f(z) = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}(\sum_{j=0}^{k} b_j(f)z^j) = f(z)$ for all $f \in \mathcal{F}\}$, then*

(1) **(i)** *$U(D) = \cap_{c\notin D} cR' = \{c : cR \subseteq D\}$ and*

(2) **(ii)** *$\lim_{n\to\infty} \sigma_n^f(z)$ converges uniformly on compact subsets of $U(D)$.*

**Note 1.** *The set $U(D)$ is some sense a set of $z's$, where the transformation reproduces the function $f$ under consideration. Thus, the transform matrix behaves as an identity matrix.*

**Theorem 5.** *For the regular convolution summability transformation matrix given by $C = \{C_{n,k}\}_{n,k\geq 0}$ with $\sum_{j=0}^{\infty} p_j j < \infty$, we have*

(1) $B' = \{z : |z| < 1\} \subseteq R'$,
(2) $R'$ *is open, and*
(3) $\lim_{n\to\infty} \tau_n(z) = \sum_{k=0}^{\infty} C_{n,k} s_k(z) = \frac{1}{(1-z)}$ *uniformly on compact subsets of* $R'$, *where* $s_k(z) = \sum_{j=0}^{k} z^j$ *and*
$R' = \{z : \lim_{n\to\infty} \tau_n(z) := \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} s_k(z) = \frac{1}{(1-z)}\}$.

*Proof.*
(1). For $B' \subseteq R'$, we ask, if $|z| < 1$, then $z \in R'$ ? We have two cases.
(a). The series

$$\tau_n(z) = \sum_{k=0}^{\infty} C_{n,k} s_k(z) = \sum_{k=0}^{\infty} P\{S_n = k\} \sum_{j=0}^{k} z^j, \text{ where } S_n = Y + \sum_{i=1}^{n} X_i,$$

converges. This will be shown as follows: Since $\sum_{k=0}^{\infty} P\{S_n = k\} = 1$, and for fixed $z \in B', z^{k+1} \to^k 0$, there exists $M > 0$ such that $\left|\frac{1-z^{k+1}}{1-z}\right| \leq M$. Hence,

$$\sum_{k=0}^{\infty} \left| P\{S_n = k\} \frac{1 - z^{k+1}}{1 - z} \right| \leq M \sum_{k=0}^{\infty} P\{S_n = k\} \leq M < \infty.$$

(b). Let $z \in R'$, and consider

$$\tau_n(z) = \sum_{k=0}^{\infty} C_{n,k} s_k(z) = \sum_{k=0}^{\infty} P\{S_n = k\} \sum_{j=0}^{k} z^j, \text{ where } S_n = Y + \sum_{i=1}^{n} X_i.$$

By the formula for the sum of the geometric series

$$\begin{aligned}
\tau_n(z) &= \sum_{k=0}^{\infty} P\{S_n = k\} \frac{1 - z^{k+1}}{1 - z} \\
\lim_{n\to\infty} \tau_n(z) &= \frac{1}{1 - z} \{ \lim_{n\to\infty} [\sum_{k=0}^{\infty} P\{S_n = k\} - \sum_{k=0}^{\infty} P\{S_n = k\} z^{k+1}]\} \\
&= \frac{1}{1 - z} \{1 - z \lim_{n\to\infty} \sum_{k=0}^{\infty} P\{S_n = k\} z^k\} \\
&= \frac{1}{1 - z} \{1 - z \lim_{n\to\infty} E z^{S_n}\} \\
&= \frac{1}{1 - z} \{1 - \lim_{n\to\infty} z E z^Y E z^{\sum_{i=1}^{n} X_1}\}, \text{ since } Y, X_i \text{ are independent.} \\
&= \frac{1}{1 - z} \{1 - z E z^Y \lim_{n\to\infty} \Pi_{i=1}^{n} E z^{X_1}\} \text{ since } X_i \text{ are i.i.d..} \\
&= \frac{1}{1 - z} \{1 - z E z^Y \lim_{n\to\infty} (E z^{X_1})^n\} \\
&= \frac{1}{1 - z} - \frac{z}{1 - z} E z^Y \lim_{n\to\infty} (E z^{X_1})^n.
\end{aligned}$$

Now $\lim_{n\to\infty}(Ez^{X_1})^n \to 0$ if and only if $|Ez^{X_1}| < 1$. Thus, the right hand side tends to $\frac{1}{1-z}$ uniformly on $\{z : |z| < 1\}$, as the following inclusion holds. Since $C$ is regular, $p_0 < 1$ and therefore $E(|z|^{X_1}) = |z|^0 p_0 + |z|p_1 + |z|^2 p_2 + \cdots < p_0 + p_1 + p_2 + \cdots = 1$. If $C$ is not regular, then $E(|z|^{X_1}) = 1$ is possible whenever $|z| < 1$. That is,

$$\{z : |Ez^{X_1}| < 1\} \supseteq \{z : E|z|^{X_1} < 1\} \supseteq \{z : |z| < 1\}.$$

So, $B' \subseteq R'$.
(2). To show that $R'$ is open, we consider $R' = \{z : |Ez^{X_1}| < 1\}$. In other words, we will show that

$$R' = \Big\{z : \Big|\sum_{k=0}^{\infty} p_k z^k\Big| < 1, \sum_{k=0}^{\infty} p_k = 1, p_k \geq 0 \text{ with } p_0 < 1\Big\} \text{ is open.}$$

$$\Longleftrightarrow$$

The function $g : z \to \sum_{k=0}^{\infty} p_k z^k$ is continuous at all $z \in R'$.
For this, let $g(z) = \sum_{k=0}^{\infty} p_k z^k$. The series converges at $z = 1$. For any other $z, z_0 \in B'$ and $|z - z_0| < \delta$, consider,

$$\begin{aligned}
|g(z) - g(z_0)| &= \Big|\sum_{j=0}^{\infty} p_j z^j - p_j z_0^j\Big| \\
&\leq \sum_{j=0}^{\infty} p_j |z^j - z_0^j| \\
&= \sum_{j=0}^{\infty} p_j |z - z_0||z^{j-1} + z^{j-2} z_0 + \cdots + z_0^{j-1}| \\
&< |z - z_0| \sum_{j=0}^{\infty} p_j j \text{ as } |z|, \text{ and } |z_0| < 1 \\
&= |z - z_0| EX.
\end{aligned}$$

Since $E|X| = \sum_{j=0}^{\infty} p_j j < \infty$, we have $|g(z) - g(z_)| < \epsilon$.
(3). Now, we need to show that $\lim_{n\to\infty} \tau_n(z) = \frac{1}{(1-z)}$ uniformly on a compact subset $\kappa$ of $R'$. Is this convergence at a point $z \in R'$ ? For this, consider

$$\Big|\tau_n(z) - \frac{1}{(1-z)}\Big| = \frac{|z|}{|1-z|}\Big|\sum_{k=0}^{\infty} P\{S_n = k\} z^k\Big|.$$

Suppose $g : R' \to^{\text{into}} B'$ is continuous at every point. Let $\kappa$ be a compact subset of $R'$. Let $z \in \kappa$ and $\epsilon > 0$. Now,

$$\Big|\tau_n(z) - \frac{1}{(1-z)}\Big| = q(z)|Ez^Y|\Big|\sum_{k=0}^{\infty} p_k z^k\Big|^n,$$

where $q(z) = \frac{|z|}{|1-z|}$. We need to show that for $z \in \kappa \subseteq R'$, $g(z) \in B'$. Then, there exists $\delta(z, \epsilon)$ such that $|z_0 - z| < \delta(z, \epsilon)$, $\Rightarrow |g(z) - g(z_0)| < \epsilon$ and $|g(z_0)| < 1$. Let

$\{B(z, \delta(z, \epsilon_z)\}$ is an open covering for $\kappa$. Then there exist $z_1, z_2, \cdots, z_l$ for some $l \in \aleph$ such that

$$\cup_{j=l}^{l} B(z_j, \delta(z_j, \epsilon_{z_j})) \supseteq \kappa.$$

Therefore, there exist $\eta_\kappa < 1$ such that

$$|g(z)| \leq \eta_\kappa < 1 \text{ for all } z \in \kappa.$$

So, $\eta_\kappa$ is fixed number and $0 \leq \eta_\kappa < 1$. Let us assume that there exits $N$ such that $n \geq N \Rightarrow \eta_\kappa^n < \epsilon/g(z)|Ez^Y|$. Therefore, $\forall z \in \kappa$ and $n \geq N$,

$$\begin{aligned}
\left|\tau_n(z) - \frac{1}{(1-z)}\right| &= q(z)|Ez^Y||g(z)|^n \\
&\leq q(z)|Ez^Y|\eta_\kappa^n \\
&< q(z)|Ez^Y|\frac{\epsilon}{g(z)|Ez^Y|} \\
&= \epsilon.
\end{aligned}$$

Which asserts that the function $g : R' \to B'$ converges uniformly on $\kappa$. This together with the above gives that $\lim_{n\to\infty} \tau_n(z) = \frac{1}{1-z}$ uniformly on compact subsets of $R'$, $\{z : |z| < 1\} \subseteq R'$ and $R'$ is open. □

**Remark 1.** As the main conditions of the theorem hold for convolution summability method, the conclusions of the Okada theorem follows immediately.

We now restate the Okada theorem for the convolution summability transformation matrix.

**Theorem 6.** *Let $C = \{C_{n,k}\}_{n,k\geq 0}$ be a regular convolution summability transformation matrix with $\sum_{j=o}^{\infty} p_j\theta^j < \infty$ for $0 \leq \theta < \infty$, and denote the $k^{th}$ partial sum of the geometric series, $\sum_{j=0}^{k} z^j$, by $s_k(z)$.*

*Let $R' = \{z : \lim_{n\to\infty} \tau_n(z) := \lim_{n\to\infty} \sum_{k=0}^{\infty} Ca_{n,k}s_k(z) = \frac{1}{1-z}\}$, and $R = \{\omega = 1/z : z \in C(R')\} \cup \{0\}$. Let $D$ be a fixed domain, where $0 \in D$ and denote by $\mathcal{F}$, the collection of all functions analytic on $D$. For $f \in \mathcal{F}$ denote the power series representation of $f$ about the origin by $\sum_{k=0}^{\infty} b_k(f)z^k$. If*

$$U(D) = \{z : \lim_{n\to\infty} \sigma_n^f(z) = \lim_{n\to\infty} \sum_{k=0}^{\infty} C_{n,k}(\sum_{j=0}^{k} b_j(f)z^j) = f(z) \text{ for all } f \in \mathcal{F}\},$$

*then*

(1) **(i)** *$U(D) = \cap_{c\notin D} cR' = \{c : cR \subseteq D\}$ and*
(2) **(ii)** *$\lim_{n\to\infty} \sigma_n^f(z)$ converges uniformly on compact subsets of $U(D)$.*

*Proof.*
The proof follows from the Okada theorem as the three assumptions now hold true for a regular Convolution summability transformation matrix. □

In Theorem 5, we have shown that the set $R'$ is open. We will now investigate how does this set $R'$ look like for a few known summability methods. It is also noted that the openedness of $R'$ depends only on the random sequence $\{X_i\}_{i\geq 0}$.

Are there interesting cases ? For this, we consider some known examples of this method.

(1) Borel exponential method with $\lambda > 0$; Consider $Ez^X = \sum_{m=0}^{\infty} \frac{e^{-\lambda}\lambda^m}{m!} z^m = e^{-\lambda} \sum_{m=0}^{\infty} \frac{(z\lambda)^m}{m!} = e^{-\lambda} e^{z\lambda} = e^{-\lambda(1-z)} = e^{-\lambda(1-x-iy)}$. Thus, $|Ez^X| < 1 \Rightarrow x < 1$ as $\lambda > 0$. Hence, $R' = \{z : \mathrm{Re}(z) < 1\}$.

(2) Euler method with $0 < p < 1$; Consider $Ez^X = z^0 p + z^1(1-p) = p + (x+iy)(1-p) = p + x(1-p) + iy(1-p)$.
Since $|Ez^X| = \sqrt{(p + x(1-p))^2 + y^2(1-p)^2}$, we obtain $\left(x + \frac{p}{1-p}\right)^2 + y^2 < \frac{1}{(1-p)^2}$. The set $R'$ is interior of a circle of radius $\frac{1}{1-p}$ and center at $\left(-\frac{p}{1-p}, 0\right)$.

(3) Meyer-König method with $0 < k < 1$; Since $Ez^X = \sum_{m=0}^{\infty} z^m p(1-p)^{m-1} = \left(\frac{p}{1-p}\right) \frac{1}{1-z(1-p)}$ for $|z| < 1$,

$$Ez^X = \left(\frac{p}{1-p}\right) \frac{1}{1-(x+iy)(1-p)} = \left(\frac{p}{1-p}\right) \frac{1 - x(1-p) + iy(1-p)}{[1-x(1-p)]^2 + y^2(1-p)^2}.$$

The condition $|Ez^X| < 1$ gives

$$\left(x - \frac{1}{1-p}\right)^2 + y^2 > \frac{p^2}{(1-p)^4}.$$

Hence $R'$ is the exterior of a circle of radius $\frac{p}{(1-p)^2}$ and center at $\left(\frac{1}{1-p}, 0\right)$.

(4) Taylor method with $0 < r < 1$; Since $Ez^X = \sum_{m=1}^{\infty} z^m p(1-p)^{m-1} = \frac{p}{(1-p)} \frac{z(1-p)}{1-z(1-p)}$, the condition $|Ez^X| < 1$ gives

$$\left(x - \frac{1-p}{1-2p}\right)^2 + y^2 < \frac{p^2}{(1-2p)^2}.$$

Hence $R'$ is the interior of a circle of radius $\frac{p}{(1-2p)}$ and center at $\left(\frac{1-p}{1-2p}, 0\right)$.

This shows that $R'$ is not necessarily the same set for every member of the convolution method, but the interior of a circle of a positive radius that may be infinite and the center is on the real axis. Based on these observations, the following proposition is appropriate.

**Proposition 1.** *The condition* $\left|Ez^X\right| < 1$ *in the proof of Theorem 5, corresponds to an interior or exterior of a circle of a positive radius r* $> 0$, *possibly infinite and the center is on the real axis. If* $\beta > 1$, *then* $\left|Ez^X\right| < 1$ *is an interior of a circle, and if* $\beta < 1$, *then* $\left|Ez^X\right| < 1$ *is an exterior of a circle, where* $\sum_{k=0}^{\infty} z^k \sum_{j=k}^{\infty} p_j = \beta < \infty$.

*Proof.*

To prove the assertion, we will explore the condition $\left|Ez^X\right| < 1$.

$$\begin{aligned}
\left|Ez^X\right| &= \Big|\sum_{j=0}^{\infty} p_j z^j\Big| \\
&= \Big|\sum_{j=0}^{\infty} p_j \Big\{\sum_{k=0}^{j}(z^k - z^{k-1}) + \frac{1}{z}\Big\}\Big| \\
&= \Big|\sum_{k=0}^{\infty}(z^k - z^{k-1})\sum_{j=k}^{\infty} p_j + \frac{1}{z}\Big| \\
&= \Big|\Big(1-\frac{1}{z}\Big)\sum_{k=0}^{\infty} z^k \sum_{j=k}^{\infty} p_j + \frac{1}{z}\Big|.
\end{aligned}$$

We show that the sum $\sum_{k=0}^{\infty} z^k \sum_{j=k}^{\infty} p_j$, by the ratio test is absolutely convergent. If the sum is $\sum_{k=0}^{\infty} a_k z^k$, then $\left|\frac{a_{k+1}}{a_k}\right| = \Big(\sum_{j=k+1}^{\infty} p_j / \sum_{j=k}^{\infty} p_j\Big)|z| < 1$ as $z$ is chosen to be $|z| < 1$. This means that the series converges for all values of $z$ such that $|z| < 1$. Thus,

$$\begin{aligned}
\left|Ez^X\right| &= \Big|\Big(1-\frac{1}{z}\Big)\beta + \frac{1}{z}\Big| \\
&= \Big|\beta\Big\{1+\frac{\gamma}{z}\Big\}\Big| \text{ where, } \gamma = \Big(\frac{1-\beta}{\beta}\Big) \\
&= \Big|\beta\Big\{\frac{[(x+\gamma)+iy][x-iy]}{x^2+y^2}\Big\}\Big|.
\end{aligned}$$

Now, $\left|Ez^X\right| < 1$, implies

$$\begin{aligned}
\beta^2\Big\{[x(x+\gamma)+y^2]^2 + [xy - y(x+\gamma)]^2\Big\} &< (x^2+y^2)^2 \\
\beta^2\Big\{x^2+y^2+2x\gamma+\gamma^2\Big\} &< (x^2+y^2) \\
\Big(x - \frac{\beta}{1+\beta}\Big)^2 + y^2 &< \frac{1}{(1+\beta)^2}.
\end{aligned}$$

Thus, the latter inequality concludes the assertion as it corresponds to a circle of a radius $\frac{1}{1+\beta}$, centered at $\Big(\frac{\beta}{1+\beta}, 0\Big)$. $\square$

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## References

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