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A Probabilistic Description of the Three Theorems of Hardy

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Abstract

The purpose of this paper will be to transpose three of Hardy's theorems to the context of unimodality, large deviation theory, and local limits theorems. In his own work, these three theorems were strategic and were used to prove among other things, certain Tauberian theorems for the Euler-Borel class. But where Hardy's results are for particular sums. Our generalizations apply to various general methods of summability; including for example, random-walk methods, and the convolution method.

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I. INTRODUCTION

Let $X_1, X_2, \cdots$ be a sequence of random variables either dependent or independent, and let $\frac{S_n}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i$. The main object of study in probability theory has been the quentity $\frac{S_n}{n}$. The three main branches of classical probability deal with (i). the almost sure behavior of $\frac{S_n}{n}$, (ii). the weak convergence of the appropriate normalized form of $\frac{S_n}{n}$ and (iii). the large deviation results.

From the summability point of view, $\frac{S_n}{n}$ is simply the Cesáro transform of $X_1, X_2, \cdots$. It is natural to ask analogous questions concerning other well known summability transforms of $X_1, X_2, \cdots$. For example, various authors have considered Borel, Euler and Abel summability transforms [2], [9] and [3] for this purpose. It seems that most of the convergence results hold for these summability analogs, under appropriate moment conditions.

The results presented in this paper provide some new results in the summability theory using probabilistic tools.

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II. SOME SUMMABILITY METHODS

In this section, we collect relevant summability methods that are used in this paper. A natural extension of the Euler method of summability is the random-walk family defined as follows.

Definition 1 Random-walk method of summability
Let $\{X_i\}_{i\geq 1}$ be an independent sequence of random variables taking values over nonnegative integers and let

$$S_n = \sum_{k=1}^{n} X_k, \text{ with } S_0 = 0.$$

Then the matrix entries associated with $a_{n,k} = P(S_n = k)$, such that $A = \{a_{n,k}\}_{n,k\geq 0}$ is called the random-walk method.

Let $f(k)$ be a sequence so that $\sum_{k=0}^{\infty} f(k)P(S_n = k)$ exists for each n. If

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} f(k)P(S_n = k) = s$$

then we say that $f(n)$ converges to s in the sense of the random-walk method and write $f(n) \to s$ (P). However, one should note that if $E|f(S_n)| < \infty$ for each n, then the above requirement is equivalent to asking that $E(f(S_n)) \to s$ to conclude that $f(n) \to s$ (P).

Even though the above family of random-walk methods contains the Euler and Borel matrix families, it does not include the Taylor family. The following generalization contains Taylor and other similar methods of summability [8].

Definition 2 Convolution method of summability Let $(p_k)_{k\geq 0}$ and $(q_k)_{k\geq 0}$ be two sequences of nonnegative numbers with $\sum_{k=0}^{\infty} p_k = 1$ and $\sum_{k=0}^{\infty} q_k = 1$. Define a summability matrix $C = \{C_{n,k}\}$, the entries of which are given as follows. Take $C_{0,k} = q_k$ and the $(n+1)$th row, $n \geq 0$ is defined as

$$C_{n+1,k} := (C_{n,\cdot} * p)_k = \sum_{j=0}^{k} p_j C_{n,k-j};\ k = 0, 1, 2, \cdots.$$

We will call C to be a convolution summability method generated by (p, q) and denote it by $C(p, q)$.

Remark 1 Khan [8] showed that $C(p, q)$ is regular and takes l^1 sequences into l^1 sequences if and only if $p_0 < 1$.

The probability description for this matrix summability method is as follows.

Let $Y, X_1, X_2, \cdots$ be a sequence of independent random variables taking nonnegative integer values. Define $S_n = Y + X_1 + X_2 + \cdots + X_n$ for $n \geq 1$ and for $n = 0$ take $S_0 = Y$. Assume that the random variables $X_1, X_2, \cdots$ are identically distributed with probability distribution p and Y has the probability distribution q. The nth row of the matrix $C(p, q)$ is the probability distribution of the random variable S_n. We now consider the conditions on p and q that lead the convolution summability method to Euler, Borel, Taylor and Meyer-König Methods.

Example 1 In the following examples, we assume $0 < r \leq 1$ and $\lambda > 0$.

1. Euler method: Take $p = (1-r, r, 0, 0, \cdots)$, $q = (1, 0, 0, 0, \cdots)$. It is seen that this choice of p and q generates the Euler summability method E_r.

2. Borel matrix method: Take $p = q = (e^{-\lambda}, e^{-\lambda}\lambda, e^{-\lambda}\lambda^2/2!, e^{-\lambda}\lambda^3/3!, \cdots)$. This generates Borel matrix summability method B_λ.

3. Taylor method: Take $p = (0, 1-r, (1-r)r, (1-r)r^2, \cdots)$, $q = (1-r, (1-r)r, (1-r)r^2, \cdots)$. That is, Y is the geometric random variable with parameter $(1-r)$ and X_1 is the shifted (or Pascal) random variable with parameter $(1-r)$. This gives rise to the Taylor summability method T_r.

4. Meyer-König method: Take $p = q = (1-r, (1-r)r, (1-r)r^2, \cdots)$. This gives rise to Meyer-König method of summability denoted by MK_r.

We now discuss two well-known methods of summability called Valiron and circle methods of summability [1].

Definition 3 Valiron and Circle methods of summability
For $a > 0$, the sequence $\{s_n\}_{n\geq 0}$ is said to converge to s in the sense of the Valiron method of summability with parameter a if

$$\sqrt{\frac{a}{2\pi n}} \sum_{i=0}^{\infty} s_i \exp\{-\frac{1}{2}a(i-n)^2/n\} \to s \text{ as } n \to \infty;$$

and we write $s_n \to s$ (V_a).

On the other hand the circle family of summability methods contains the Valiron method and the following.

1. Borel exponential method B, for which $s_n \to s$ (B) means

$$\sum_{i=0}^{\infty} \frac{e^{-x}x^i}{i!} s_i \to s \text{ as } n \to \infty.$$

2. Euler method E_r, $r \in (0,1)$, for which $s_n \to s$ (E_r) means

$$\sum_{i=0}^{n} \binom{n}{i} r^i (1-r)^{n-i} s_i \to s \text{ as } n \to \infty.$$

3. Meyer-König method S_r (or MK_r), $r \in (0,1)$, for which $s_n \to s$ (S_r) means

$$(1-r)^{n+1} \sum_{i=0}^{\infty} \binom{n+i}{i} r^i s_i \to s \text{ as } n \to \infty.$$

4. Taylor method T_r, $r \in (0,1)$, for which $s_n \to s$ (T_r) means

$$(1-r)^{n+1} \sum_{i=0}^{\infty} \binom{n+i}{i} r^i s_{n+i} \to s \text{ as } n \to \infty.$$

Note that the last three members of the circle family are also members of the convolution summability method.

III. THREE THEOREMS OF HARDY

Much of the second half of Hardy's text [6] uses three important theorems, namely, Theorems 137, 138, and 139, pp. 200 - 201, [6]. These theorems have direct probabilistic analogs. These theorems will be extended in the next few sections to the random-walk method and, when appropriate, to the convolution summability method. Here we provide the original theorems of Hardy for the sake of easy referencing. The proposed generalization serves a variety of purposes. The first purpose is to view the results from the probability point of view. The second purpose is to extend these results for the convolution summability method. It is anticipated that the resulting extensions may be useful for extending Hardy's Tauberian theorems. The first of them deals with the Borel exponential method of summability.

Theorem 1 *(Theorem 137, p. 200, [6]) Suppose that* $x > 0$ *and*

$$u_m = u_m(x) = e^{-x}\frac{x^m}{m!} \quad (m = 0, 1, 2, \cdots),$$

so that $\sum u_m = 1$. *Then*

(1). the largest u_m *is* u_M, *where* $M = [x]$, *two terms,* u_{M-1} *and* u_M, *being equal when* x *is an integer;*

(2). if $m = M + h$ *and* $0 < \delta < 1$, *then*

$$\sum_{|h|>\delta x} u_m = O\Big(e^{-\gamma x}\Big), \text{ where } \gamma = \frac{1}{3}\delta^2;$$

(3). if $\frac{1}{2} < \xi < \frac{2}{3}$, *then*

$$\sum_{|h|>x^{\xi}} u_m = O\Big(e^{-x^{\eta}}\Big),$$

where η *is any number less than* $2\xi - 1$;

(4). if $\lambda > 0$, *then*

$$\sum_{|h|>\lambda x^{\frac{1}{2}}} u_m < \epsilon,$$

for $x > x_0(\epsilon),\ \lambda > \lambda_0(\epsilon)$;

(5). if $|h| \le x^{\xi}$, *then*

$$u_m = \sqrt{\Big(\frac{c}{\pi M}\Big)}e^{-ch^2/M}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\},$$

where $c = \frac{1}{2}$;

(6). the estimates (2) and (3) remain valid if u_m *is multiplied by any fixed power of* m.

The next result is about the Euler method.

Theorem 2 *(Theorem 138, p. 201, [6]) Suppose that* $q > 0$ *and*

$$u_m = u_m(n) = (q+1)^{-n} \binom{n}{m} q^{n-m} \quad (0 \leq m \leq n),$$

so that $\sum u_m = (q+1)^{-n} \sum \binom{n}{m} q^{n-m} = 1$. *Then*

(1). the largest u_m *is* u_M, *where* $M = \left[\frac{n+1}{q+1}\right]$, *two terms,* u_{M-1} *and* u_M, *being equal when* $(n+1)/(q+1)$ *is an integer;*

(2). clauses (2) - (6) of Theorem 1 hold with $c = \frac{q+1}{2q}$, *some positive* $\gamma = \gamma(q, \delta)$ *in clause (2)*[1] *and* n *in the place of* x.

The next theorem is the Taylor method.

Theorem 3 *(Theorem 139, p. 201, [6]) Suppose that* $0 < k < 1$ *and*

$$u_m = u_m(n) = k^{n+1} \binom{m}{n} (1-k)^{m-n} \quad (m \leq n),$$

so that $\sum u_m = k^{n+1}\Big\{1 + (n+1)(1-k) + \frac{(n+1)(n+2)}{2!}(1-k)^2 + \cdots\Big\} = 1$. *Then*

(1). the largest u_m *is* u_M, *where* $M = [n/k]$, *two terms,* u_{M-1} *and* u_M, *being equal when* n/k *is an integer;*

(2). clauses (2) - (6) of Theorem 1 hold with $c = \frac{k}{2(1-k)}$, *some positive* $\gamma = \gamma(k, \delta)$ *in clause (2)*[2] *and* n *in the place of* x.

As we shall see later that in the extensions of the results for other summability methods, several restrictions on the parameters are relaxed. In proving our results, we draw attention to three important topics in probability theory; (i) unimodality of a discrete distribution, (ii) large deviation theory, and (ii) local limit theorems. Among them, the theory of large deviations (LDT) is one of the most active areas in probability today, having applications to many areas including statistics, communication networks, information theory, statistical mechanics, risk sensitive control, and queuing systems [13]. The richness of the theory of large deviations derives, in part, from the fact that it is nonlinear, and this in turn contributes to much of the difficulty of large deviation analysis.

IV. LARGE DEVIATION THEORY (LDT)

We will discuss the relatively simple case of the theorem of Cramer in order to introduce the main ideas of the approach. In the context of a summability method, we will explore connections between the representation formulas and computations. This is an aspect of LDT that has not been extensively studied in summability context. Although the general theory of large deviations is well developed, in the summability context there is still a great deal of work to be done. In the end, we will discuss the use of the representation formula to obtain some results.

[1] a definite value to γ is not assigned in this case.
[2] a definite value to γ is not assigned in this case.

The weak laws of large numbers (WLLN) for coin tossing says:

$$P\Big\{\Big|\frac{S_n}{n}-\frac{1}{2}\Big|\geq\epsilon\Big\}\to 0 \text{ as } n\to\infty$$

for all $\epsilon>0$, where S_n represents the number of heads out of n tossings of a fair coin. Of course, the rate at which it tends to 0 and how it depends on $\epsilon>0$ is given by the Chebychev theorem. Using the second moment, we get

$$P\Big\{\Big|\frac{S_n}{n}-\frac{1}{2}\Big|\geq\epsilon\Big\}\leq\frac{1}{4n\epsilon^2}.$$

Using the fourth moments, we obtain

$$P\Big\{\Big|\frac{S_n}{n}-\frac{1}{2}\Big|\geq\epsilon\Big\}\leq\frac{C}{n^2\epsilon^4},$$

etc.. Using higher moments, one can show that the rate of decrease of the left side is even faster. The key result of the LDT is the following theorem of Cramer [13], which shows that the rate of decrease is in fact exponential.

Theorem 4 *(Cramer [13]) Assume $X_1, X_2, \cdots, X_n$ are i. i. d. random variables with common distribution function $F(x)$ with the following properties:*

1. *$\int e^{\theta x}dF(x)=m(\theta)<\infty$, for all θ;*
2. *$0<F(x)<1$, for all $-\infty<x<\infty$.*

Then there exists a function $h(x)$ on $\mathbb{R}$ such that

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log P\Big[\frac{S_n}{n}\in A\Big] &\leq -\inf_{x\in\bar{A}}h(x) \text{ and}\\ \liminf_{n\to\infty}\frac{1}{n}\log P\Big[\frac{S_n}{n}\in A\Big] &\geq -\inf_{x\in A^0}h(x),\end{aligned}$$

where $h(x)=\sup_\theta\{\theta x-\log m(\theta)\}$.

However, this theorem is not applicable to our example, since the coin tossing example is precluded by the second condition above. Nevertheless, the result still holds due to the following version of Cramer's theorem [4].

Theorem 5 *(Cramer [4]) Let $\{X_i\}_{i\geq 1}$ be a sequence of independent and identically distributed random variables and let $S_n=\sum_{i=1}^n X_i$. Suppose in addition to*

1. *$\varphi(\theta)=E\exp(\theta X_1)<\infty$ for some $\theta>0$ and*
2. *the distribution F is not a point mass at μ,*

that for a given $a>\mu=EX_1$, there is a $\theta_a\in(0,\theta_+)$ so that $a=\varphi'(\theta_a)/\varphi(\theta_a)$ and $\theta_+=\sup\{\theta:\varphi(\theta)<\infty\}$. Then as $n\to\infty$,

$$n^{-1}\log P\{S_n\geq na\}\to -a\theta_a+\log\varphi(\theta_a).$$

The purpose of this paper will be to transpose three of Hardy's theorems [6, pp. 200 - 201] to the context of unimodality, large deviation theory, and local limits theorems. In his own work, these three theorems were strategic and were used to prove among other things, certain Tauberian theorems for the Euler-Borel class. But where Hardy's results are for particular sums. Our generalizations apply to various general methods of summability; including for example, random-walk methods, and the convolution method. Corresponding generalizations of Tauberian theorems will be the subject of later work.

Regarding notation, since the general summability methods do not depend as such on [x] (the integer part of x), we simply use x in our formulation.

V. UNIMODALITY FOR RANDOM-WALK METHOD

The following discussion will shed some light on the theorem in extending the first assertion of Hardy's theorem for random-work method. This has to do with the unimodality of the entries of the nth row of the three summability methods (Borel matrix, Euler and Taylor). A probability distribution $\{p_j\}_{j\geq 0}$ is called unimodal if the sequence $\{p_{j+1}-p_j\}_{j=-\infty}^{\infty}$ has exactly one change of sign. The unimodality of the distribution function has been of substantial interest, especially in connection with the question of whether distribution functions of the $n-$fold convolution are unimodal [10].

It was conjectured [**?**] that something stronger ought to hold for a discrete distribution $\{p_j\}$ on the integers, namely that for each such distribution there ought to be an integer n_0 such that the $n-$fold convolution $\{p_j\}^{\star n}$ is unimodal for all $n \geq n_0$. There are counter examples to show that it is hard to guarantee unimodality even for high convolution of a discrete distribution when the distribution has integer support.

A more restrictive concept than that of unimodality is that of strong unimodality; a discrete distribution $\{p_j\}$ is strongly unimodal if $\{p_j\} \star \{q_j\}$ is unimodal for any unimodal discrete distribution $\{q_j\}$. A strongly unimodal distribution is unimodal, but not necessarily conversely. A discrete distribution $\{p_j\}$ is strongly unimodal if and only if it is log concave; $p_j^2 \geq p_{j-1}p_{j+1}$ for all integer values of j [7].

In view of these results, the conclusions of Hardy's theorem will not hold in general for an arbitrary summability method unless the conditions for unimodality are satisfied. Fortunately, the distributions that are addressed by Hardy's theorem obey this property. If a distribution is unimodal then there are two possibilities.

1. There exists an integer m such that $p_j < p_{j+1}$ for $j < m$ and $p_j > p_{j+1}$ for $j \geq m$ or
2. There exist integers m and n, $m < n$, such that $p_j < p_{j+1}$ for $j < m-1$, $p_j = p_m$ for $j = m, m+1, \cdots, n$ and $p_j > p_{j+1}$ for $j > n$.

The following shows that not only Hardy's result holds but they possess the more restrictive property of strong unimodality.

1. For the Borel matrix method $B = \{b_{n,k}\}$, where $b_{n,k} = \frac{e^{-n}n^k}{n!}$; for $k = 0, 1, 2, \cdots$ and $n = 1, 2, 3, \cdots$. The condition for log concavity of the nth row requires that

$$b_{n,j}{}^2 \geq b_{n,j-1}b_{n,j+1}, \text{ for } j = 0, 1, 2, \cdots.$$

Since n is fixed for the time being, let $p_j = b_{n,j}$, with $p_{-1} = 0$. We have

$$\frac{e^{-2n}n^{2j}}{(j!)^2} \geq \frac{e^{-n}n^{j-1}}{(j-1)!}\frac{e^{-n}n^{j+1}}{(j+1)!}$$

$$1 \geq \frac{j}{j+1} \quad \Leftrightarrow \quad j+1 \geq j, \text{ for } j = 0, 1, 2, \cdots.$$

Hence each row of the Borel matrix method is log concave.

2. For the Euler method $E_p = \{e_{nk}\}$, where $e_{n,k} = \binom{n}{j} p^j q^{n-j}$. The condition for log concavity of the nth row requires that

$$\begin{aligned}
{e_{n,j}}^2 &\geq e_{n,j-1}e_{n,j+1} \text{ for } j = 0, 1, 2, \cdots \\
\binom{n}{j}^2 p^{2j} q^{2n-2j} &\geq \binom{n}{j-1} p^{j-1} q^{n-j+1} \binom{n}{j+1} p^{j+1} q^{n-j-1} \\
\binom{n}{j}^2 &\geq \binom{n}{j-1}\binom{n}{j+1} \\
1 &\geq \Big(\frac{j}{j+1}\Big)\Big(\frac{n-j}{n-j+1}\Big) \text{ for } j = 0, 1, 2, \cdots, n.
\end{aligned}$$

Since, this is always true, we have the log concavity.

3. For the Taylor method $T_k = \{t_{n,j}\}$, where $t_{n,k} = k^{n+1} \binom{j}{n} (1-k)^{j-n}$ for $j \geq n$. The condition for log concavity of the nth row requires that

$${t_{n,j}}^2 \geq t_{n,j-1}t_{n,j+1}$$

gives

$$\begin{aligned}
\binom{j}{j+1}^2 &\geq \binom{j-1}{n}\binom{j+1}{n} \text{ for } j \geq n \\
1 &\geq \Big(\frac{j+1}{j}\Big)\Big(\frac{j-n}{j-n+1}\Big) \text{ for } j \geq n.
\end{aligned}$$

Since this is a triviality, we have once again the log concavity.

The following is a direct consequence of a theorem of [10] and gives a partial (but essentially best possible) result for arbitrary regular random-walk methods. We omit its straightforward proof.

Theorem 6 *For a sequence of random variables $\{X_i\}_{i\geq 0}$ having a probability distribution $\{p_j\}$ on integers with $p_j = 0$ for $j < 0$ and $j > d$ while $p_0 > 0,\ p_1 > 0,\ p_{d-1} > 0,\ p_d > 0$, then there exists an integer n_0 such that for $n > n_0$, the entries of the nth rows of the random-walk method generated by the above sequence are strongly unimodal.*

VI. LARGE DEVIATIONS FOR RANDOM-WALK METHOD

We now turn our attention towards the large deviation aspects of Hardy's three theorems. As we will see later, these results are modified to both establish the probabilistic theory and to obtain the exact closed form of the constants as well. Since the Borel exponential method is not a matrix method, translating the same to a probability language will undergo a separate treatment in dealing with the related results. The result for the Euler method follows without much difficulty, where as that of the Taylor method will be carried out by transforming to a random-walk method.

Theorem 7 *Let $X_1, X_2, \cdots$ be independent and identically distributed random variables having a finite moment generating function and positive variance. Let $S_n = X_1 + X_2 + \cdots + X_n$ and let*

$$u_m = P(S_n = m); \; m = 0, 1, 2, \cdots.$$

If $m = M + h$, where $M = nE(X_1)$ and $\delta > 0$, then

$$\sum_{|h| > \delta n} u_m = O\Big(e^{-n\gamma}\Big), \text{ where } \gamma = \gamma(\delta) > 0.$$

Proof.

Let $\mu = E(X_1)$. Note that

$$\begin{aligned} \sum_{|h|>\delta n} u_m &= P(|S_n - n\mu| > \delta n) \\ &= P(S_n - n\mu > \delta n) + P(S_n - n\mu < -\delta n). \end{aligned}$$

Note also that we need only prove the result for $\delta > 0$ in a neighborhood of zero. We consider only the first term, since the second term can be analyzed similarly by considering $Y_i = -X_i$ for $i \geq 1$. Now

$$P(S_n - n\mu > \delta n) = P(S_n > n(\mu + \delta)),$$

and Theorem 5 will be applicable once we verify its conditions. Let $\phi(\theta) = E\Big(e^{\theta X_1}\Big)$ and $\theta_+ = \sup\{\theta : \phi(\theta) < \infty\}$. Our assumption concerning the existence of the moment generating function gives that $\theta_+ > 0$. And $\mathrm{Var}(X_1) > 0$ implies that the random variable X_1 is not degenerate at a point $a = \mu + \delta$. We need to show the existence of $\theta_a \in (0, \theta_+)$, so that

$$a = \frac{\phi'(\theta_a)}{\phi(\theta_a)}.$$

First note that $\frac{\phi'(0)}{\phi(0)} = \mu > 0$. Let $g(\theta) = \frac{\phi'(\theta)}{\phi(\theta)}$, and note that

$$\frac{d}{d\theta} g(\theta) = \frac{\phi(\theta)\phi''(\theta) - (\phi'(\theta))^2}{(\phi(\theta))^2}.$$

This gives that

$$g'(0) = \frac{d}{d\theta} g(\theta)\Big|_{\theta=0} = E(X_1^2) - (E(X_1))^2 = \mathrm{Var}(X_1) > 0.$$

Therefore, $g(\theta)$ is a strictly increasing function of θ in a neighborhood of 0. This gives that for any $\delta > 0$, in a neighborhood of 0, and $a = \mu + \delta$, there exists a θ_a such that $P(S_n - n\mu > \delta n) = O\Big(e^{-n\gamma}\Big)$, where $\gamma = \gamma(\delta) > 0$ is given by Theorem 5. The case $P(S_n - n\mu < -n\delta)$ is similarly taken care of by using $Y_i = -X_i$ for $i \geq 1$. This finishes the proof.

∎

The following corollary deals with what is known as the Poisson distribution. A random variable X, taking on values $0, 1, 2, \cdots$ is said to be a Poisson random variable with parameter λ, if for $\lambda > 0$,

$$P(X = m) = \frac{e^{-\lambda}\lambda^m}{m!}, \; m = 0, 1, 2, \cdots.$$

We will denote this by $X \sim \text{Pois}(\lambda)$. The following corollary is a Borel matrix (instead of Borel exponential method) version of Theorem 137(2), p. 200, [6].

Corollary 1 *Suppose that*

$$u_m = u_m(x) = \frac{e^{-x}x^m}{m!};\ m = 0, 1, 2, \cdots,$$

where $x = 1, 2, \cdots,$ *so that* $\sum_m u_m = 1$. *If* $m = M + h,$ *where* $M = x$ *and* $\delta > 0$, *then*

$$\sum_{|h|>\delta x} u_m = O(e^{-\gamma x}),\ \textit{where } \gamma = \gamma(\delta) > 0.$$

Proof.

Let $S_n = X_1 + X_2 + \cdots + X_n \overset{\text{i.i.d.}}{\sim} \text{Pois}(1)$ in Theorem 7. We evaluate the function $\gamma(\delta)$ more explicitly as follows. The moment generating function $\varphi(t)$ is given by

$$\varphi(t) = \exp\{(e^t - 1)\} \ \Rightarrow\ \varphi'(t) = e^t \exp\{(e^t - 1)\}.$$

This gives that $a = \frac{\varphi'(t)}{\varphi(t)} = e^t > \mu$. Let us now solve for $a = \frac{\varphi'(t)}{\varphi(t)} \iff 1 + \delta = e^{\theta_a} \iff \theta_a = \log(1 + \delta)$. Then by the large deviation result, Theorem 5,

$$n^{-1} \log P\{S_n > n(1 + \delta)\} \sim -(1 + \delta) \log(1 + \delta) + (1 + \delta) - 1 = -\gamma_1(\delta).$$

Note also that

$$\begin{aligned} \gamma_1(\delta) &= -\delta + (1 + \delta) \log(1 + \delta) \text{ and } \gamma(0) = 0 \\ \gamma_1'(\delta) &= -1 + \log(1 + \delta) + \frac{1 + \delta}{1 + \delta} \\ &= \log(1 + \delta) > 0 \text{ for all } \delta > 0. \end{aligned}$$

Hence, $\gamma_1'(\delta) > 0$ for any $\delta > 0$. We will identify the rate function $\gamma_1(\delta)$ in this case as

$$-\delta + (1 + \delta) \log(1 + \delta) \text{ for } \delta > 0 \text{ and for large values of } x.$$

And note that $\gamma_1(\delta)$ is a strictly increasing function of δ.

∎

The following corollary deals with what is known as the binomial distribution. Suppose that n independent trials are to be performed; each trial results in a "success" with probability $\frac{1}{1+q}$, for $q > 0$, and in a "failure" with probability $\frac{q}{1+q}$. If X represents the number of successes that occur in n trials, then X is said to be a binomial random variable with parameter $(n, \frac{1}{1+q})$. The probability mass function of a binomial random variable, having parameters $(n, \frac{1}{1+q})$ is given by

$$P(X = m) = (q + 1)^{-n} \binom{n}{m} q^{n-m},\ (0 \le m \le n).$$

Theorem 138(2), p. 201, [6] will be modified to accommodate the probabilistic nature of the problem using x instead of $[x]$, which will also make the proof simpler.

Corollary 2 *Suppose that* $q > 0$ *and*

$$u_m = u_m(n) = (q+1)^{-n} \binom{n}{m} q^{n-m}, \ (0 \le m \le n),$$

so that $\sum_m u_m = 1$. *If* $m = M + h$, *where* $M = x = \frac{n+1}{q+1}$ *and* $\delta > 0$, *then*

$$\sum_{|h|>\delta x} u_m = O(e^{-\gamma x}), \ \textit{where } \gamma = \gamma(\delta, q) > 0.$$

Proof.

Let $S_n = X_1 + X_2 + \cdots + X_n$. We now consider the random variables defined as

$$X_i = \begin{cases} 1 & \text{with probability} \quad 1/(q+1), \ q > 0 \\ 0 & \text{with probability} \quad q/(q+1), \end{cases}$$

for $i = 1, 2, \cdots n$. Then $EX_1 = \frac{1}{q+1}$ and $E(X_1)^2 = \frac{1}{q+1} \Rightarrow \mathrm{Var}(X_1) = \frac{q}{(q+1)^2}$. The moment generating function $\varphi(t) = Ee^{tX_1} = \frac{e^t+q}{q+1}$ and $\varphi'(t) = \frac{e^t}{q+1} \Rightarrow \frac{\varphi'(t)}{\varphi(t)} = \frac{e^t}{e^t+q}$. Now we have

$$\begin{aligned} \sum_{\left[\left|\frac{m-x}{x}\right|>\delta\right]} P\{S_n = m\} &= P\Big\{\Big|S_n - \frac{n+1}{q+1}\Big| > \delta\Big(\frac{n+1}{q+1}\Big)\Big\} \\ &= P\Big\{S_n - \frac{n+1}{q+1} > \Big(\frac{n+1}{q+1}\Big)\delta\Big\} + P\Big\{S_n - \frac{n+1}{q+1} < -\Big(\frac{n+1}{q+1}\Big)\delta\Big\}. \end{aligned}$$

As before, we only proceed with the first term. The second term has a similar bound. Since $\frac{\varphi'(\theta_a)}{\varphi(\theta_a)} = \frac{e^{\theta_a}}{e^{\theta_a}+q} = \frac{(1+\delta)}{q+1}$, we obtain

$$\Rightarrow qe^{\theta_a} + e^{\theta_a} = e^{\theta_a} + q + \delta e^{\theta_a} + \delta q \Rightarrow e^{\theta_a}(q-\delta) = (\delta+1)q \Rightarrow e^{\theta_a} = \frac{(\delta+1)q}{q-\delta}.$$

This gives us $\theta_a = \log\Big(\frac{(\delta+1)q}{q-\delta}\Big) = \log\Big(1 + \frac{\delta(1+q)}{q-\delta}\Big)$, which is well defined for $q - \delta > 0$. Otherwise, the probability of the event is zero as $S_n \le n$ with probability 1. Then by the large deviation result, Theorem 5,

$$\begin{aligned} \frac{1}{n}\log P\Big\{S_n > \frac{n+1}{q+1}(1+\delta)\Big\} &\le \frac{1}{n}\log P\Big\{S_n > \frac{n}{q+1}(1+\delta)\Big\} \\ &= -\frac{(1+\delta)}{1+q}\theta_a + \log\varphi(\theta_a) \\ &= -\frac{1+\delta}{1+q}\log\Big\{\frac{q(1+\delta)}{q-\delta}\Big\} + \log\Big\{\Big(\frac{q(1+\delta)}{q-\delta} + q\Big)/(q+1)\Big\} \\ &= -\frac{1+\delta}{1+q}\log\Big\{\frac{q(1+\delta)}{q-\delta}\Big\} + \log\Big\{\frac{q}{q-\delta}\Big\} \\ &= -\frac{1+\delta}{1+q}\Big\{\log\frac{q}{q-\delta} + \log(1+\delta)\Big\} + \log\Big\{\frac{q}{q-\delta}\Big\} \\ &= \Big(1 - \frac{1+\delta}{1+q}\Big)\log\frac{q}{q-\delta} - \frac{1+\delta}{1+q}\log(1+\delta) \\ &= \frac{q-\delta}{q+1}\log\frac{q}{q-\delta} - \frac{1+\delta}{1+q}\log(1+\delta) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{q}{1+q}\Big(\frac{q-\delta}{q}\Big)\log\frac{q-\delta}{q} - \frac{1+\delta}{1+q}\log(1+\delta) \\
&= -\frac{q}{1+q}\Big(1-\frac{\delta}{q}\Big)\log\Big(1-\frac{\delta}{q}\Big) - \frac{1}{q+1}(1+\delta)\log(1+\delta) \\
&= -\frac{q}{1+q}(1-s)\log(1-s) - \frac{1}{q+1}(1+\delta)\log(1+\delta),
\end{aligned}
$$

where $sq = \delta$. We will identify the rate function $\gamma_2(\delta, q)$ in this case as

$$
\gamma_2(\delta, q) = \frac{q}{1+q}\Big(1-\frac{\delta}{q}\Big)\log\Big(1-\frac{\delta}{q}\Big) + \frac{1+\delta}{1+q}\log(1+\delta) \text{ for } 0 < \delta < q.
$$

Lemma 9.4, p. 71 of [4] guarantees that $\gamma_2(\delta, q) > 0$ for $a > \mu$ and small θ. For the required assertion; the relation,

$$
P\Big\{S_n > \frac{n+1}{q+1}(1+\delta)\Big\} = O(e^{-x\gamma}), \text{ where } \gamma = \gamma_2(\delta, q) > 0,
$$

completes the proof.

∎

The next corollary is about the negative binomial distribution. A coin having a probability k of coming up heads is successively flipped until the nth head appears. If Y is the number of flips required to obtain n heads, then,

$$
P(Y = m) = \binom{m-1}{n-1} k^n (1-k)^{m-n}, \; m \geq n.
$$

We call Y a negative binomial random variable having parameter $k \in (0, 1)$. When $n = 1$, Y is called a geometric random variable.

Corollary 3 *(Theorem 139(2), p. 200, [6]) Suppose that* $0 < k < 1$ *is fixed and*

$$
u_m = u_m(n) = k^{n+1}\binom{m}{n}(1-k)^{m-n}; \; (m \geq n),
$$

so that $\sum_m u_m = 1$. *If* $m = M + h$, *where* $M = [x] = [\frac{n}{k}]$ *and* $\delta > 0$, *then*

$$
\sum_{|h| > \delta x} u_m = O(e^{-\gamma x}), \text{ where } \gamma = \gamma(\delta, k) > 0.
$$

Proof.

Consider a geometric random variable with parameter $k \in (0, 1)$,

$$
P(X = x) = k(1-k)^{x-1}, \; x = 1, 2, 3, \cdots.
$$

Then $EX = \frac{1}{k}$. The n^{th} row of the Taylor method is $P(Y + \sum_{i=1}^{n} X_i = m)$, where $X_0, X_1, X_2, \cdots \sim X$ and $Y = X_0 - 1$. To verify this, note that

$$
\begin{aligned}
P(Y + \sum_{i=1}^{n} X_i = m) &= P(X_0 - 1 + \sum_{i=1}^{n} X_i = m) \\
&= P(\sum_{i=1}^{n+1} X_i = m+1) \\
&= k^{n+1}\binom{m}{n}(1-k)^{m-n} \text{ for } m \geq n.
\end{aligned}
$$

We now apply the large deviation results. Let $S_{n+1} = \sum_{i=1}^{n+1} X_i$. We need to consider

$$\begin{aligned}\sum_{|h|>\delta x} P\{S_{n+1} = m+1\} &= P\Big\{\Big|S_{n+1} - 1 - \Big[\frac{n}{k}\Big]\Big| > \frac{n}{k}\delta\Big\} \\ &= P\Big\{S_{n+1} > 1 + \Big[\frac{n}{k}\Big] + \frac{n}{k}\delta\Big\} + P\Big\{S_{n+1} < 1 + \Big[\frac{n}{k}\Big] - \frac{n}{k}\delta\Big\}.\end{aligned}$$

As before, we only proceed with the first term since it can be shown that the second term has a similar bound.

$$\begin{aligned} & P\Big\{S_{n+1} > 1 + \Big[\frac{n}{k}\Big] + \frac{n}{k}\delta\Big\} \le P\Big\{S_{n+1} > \frac{n}{k}(1+\delta)\Big\} \\ = \;& P\Big\{S_{n+1} > \Big(\frac{n}{n+1}\Big)\frac{n+1}{k}(1+\delta)\Big\} \le P\Big\{S_{n+1} > \frac{n+1}{k}(1+\delta/2)\Big\}, \text{ for all } n > \frac{2+\delta}{\delta}, \\ = \;& P\Big\{S_{n+1} > (n+1)a\Big\}, \text{ where } a = \frac{1+\delta/2}{k}.\end{aligned}$$

The moment generating function $\varphi(t) = Ee^{tX} = \frac{ke^t}{1-(1-k)e^t}$,
and $\varphi'(t) = \frac{(1-(1-k)e^t)ke^t + ke^t(1-k)e^t}{(1-(1-k)e^t)^2} = \frac{ke^t}{(1-(1-k)e^t)^2} \Rightarrow \frac{\varphi'(t)}{\varphi(t)} = \frac{1}{(1-(1-k)e^t)}$.
Solution of $\frac{\varphi'(\theta_a)}{\varphi(\theta_a)} = \frac{1}{(1-(1-k)e^{\theta_a})} = \frac{1+\delta/2}{k}$ gives $e^{\theta_a} = \frac{1-k+\delta/2}{(1+\delta/2)(1-k)}$. This gives us that $\theta_a = \log\Big(1 + k\frac{\delta/2}{(1+\delta/2)(1-k)}\Big)$, which is well defined. Then by the large deviation result, Theorem 5,

$$\begin{aligned} & \frac{1}{n+1}\log P\{S_{n+1} < (n+1)\frac{1+\delta/2}{k}\} \\ \sim\; & -\frac{(1+\delta/2)}{k}\log\frac{(1-k+\delta/2)}{(1+\delta/2)(1-k)} \\ & + \log\{k\frac{(1-k+\delta/2)}{(1+\delta/2)(1-k)}/(1-(1-k)\frac{(1-k+\delta/2)}{(1+\delta/2)(1-k)})\} \\ \sim\; & -\frac{(1+\delta/2)}{k}\log\frac{(1-k+\delta/2)}{(1+\delta/2)(1-k)} + \log\frac{(1-k+\delta/2)}{1-k} \\ \sim\; & -\frac{(1+\delta/2)}{k}[\log\frac{(1-k+\delta/2)}{1-k} + \log\frac{1}{1+\delta/2}] + \log\frac{(1-k+\delta/2)}{1-k} \\ \sim\; & \frac{(1+\delta/2)}{k}\log(1+\delta/2) + (1 - \frac{1+\delta/2}{k})\log\frac{(1-k+\delta/2)}{1-k} \\ \sim\; & \frac{(1+\delta/2)}{k}\log(1+\delta/2) - \frac{1-k}{k}(\frac{1-k+\delta/2}{1-k})\log\frac{(1-k+\delta/2)}{1-k} \\ \sim\; & \frac{(1+\delta/2)}{k}\log(1+\delta/2) - \frac{1-k}{k}(1 + \frac{\delta/2}{1-k})\log\{1 + \frac{\delta/2}{1-k}\}.\end{aligned}$$

We will once again identify the rate function $\gamma_3(\delta, k)$ in this case as

$$\gamma_3(\delta, k) = -\frac{(1+\delta/2)}{k}\log(1+\delta/2) + \frac{1-k}{k}(1+\frac{\delta/2}{1-k})\log(1+\frac{\delta/2}{1-k}) \text{ for } \delta > 0.$$

Lemma 9.4, p. 71 of [4] guarantees that $\gamma_3(\delta, k) > 0$. For the required assertion; the relation,

$$P\{S_{n+1} > \frac{(1+\delta)n}{k}\} = O(e^{-n\gamma}), \text{ where } \gamma = \gamma_3(\delta, k) > 0,$$

completes the proof.

∎

We now proceed to the remaining parts of Hardy's theorem. We quote the following theorem of Feller (p. 517, [5]) in order to generalize the next assertion.

Theorem 8 *(Feller [5]) Let $X, \{X_i\}_{i\geq 1}$ be a sequence of independent, identically distributed random variables with Ee^{tX} converges for $|t| \leq t_0$ and x varies with n in such a way that $x = o(n^{1/6})$. Let $S_n = X_1 + X_2 + \cdots + X_n$. Then*

$$\frac{1 - F_n(x)}{1 - \Phi(x)} \to 1, \text{ where } F_n(x) = P\Big(\frac{S_n - ES_n}{\sigma\sqrt{n}} \leq x\Big),$$

and $\Phi(x)$ is the standard normal distribution.

Remark 2 We now make some comments about the values of ξ and two related theorems.

1. The upper and the lower bounds of ξ, namely $\xi = \frac{1}{2}$ and $\xi = \frac{2}{3}$ in the next three corollaries are determined by theorems in probability theory.

2. The cases $\xi = \frac{1}{2}$ and $\xi = 1$ correspond to the central limit theorem (CLT) and the strong law of large numbers (SLLN) respectively, under some moment conditions on the sequence of random variable, (namely, finiteness of second moment for the CLT and the existence of mean for the SLLN). The probability distributions considered by these corollaries obviously have these conditions satisfied.

3. As stated in [5], the proof of Feller's Theorem 8 carries through without difficulty, if the condition $x = o(n^{1/6})$ is replaced by $x = o(n^{1/2})$. We require $\frac{1}{6} > \xi - \frac{1}{2} > 0$. This justifies the range of ξ, namely $\frac{2}{3} > \xi > \frac{1}{2}$.

4. According to the CLT, for a sequence of independent and identically distributed non-degenerate random variables $\{X_i\}_{i\geq 1}$ with $EX^2 < \infty$, we have

$$P\Big\{\frac{S_n - ES_n}{\sqrt{\mathrm{Var}(S_n)}} \leq x\Big\} \to \Phi(x), \ x \in \mathbb{R},$$

where $S_n = X_1 + X_2 + \cdots + X_n$, and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$. In other words,

$$P\Big\{\frac{S_n - n\mu}{\sqrt{n\sigma^2}} > x\Big\} \to 1 - \Phi(x).$$

5. SLLN gives

$$P\Big\{\frac{S_n}{n} \to^n \mu\Big\} = 1,$$

provided $\mu = EX_1$ exists.

Various other equivalent forms are available for $1 - \Phi(x)$ [5].

Lemma 1 *(Feller [5]) As $x \to \infty$,*

$$1 - \Phi(x) \sim \frac{1}{(2\pi)^{1/2} x} e^{\frac{-1}{2}x^2};$$

and more precisely, for every $x > 0$ the double inequality

$$\frac{1}{(2\pi)^{1/2}} e^{\frac{-1}{2}x^2} \{\frac{1}{x} - \frac{1}{x^3}\} < 1 - \Phi(x) < \frac{1}{(2\pi)^{1/2}} e^{\frac{-1}{2}x^2} \frac{1}{x} \text{ holds.}$$

We have three immediate corollaries accordingly. Note that Corollary 4 concerns the Borel matrix method instead of the Borel exponential method treated in Theorem 137(3), p. 200, [6].

Corollary 4 *Suppose that u_m are defined as in Corollary 1.*
If $m = M + h$, where $M = x$ and $\frac{1}{2} < \xi < \frac{2}{3}$, then

$$\sum_{|h| > x^\xi} u_m = O(e^{-x^\eta}), \text{ where } \eta \text{ is any number less than } 2\xi - 1.$$

Proof.
Let $S_n = X_1 + X_2 + \cdots + X_n$, where $X_i \sim^{\text{i. i. d.}}$ Pois(1), $x = n$ and $h = m - x$. Note that $\mu = \sigma^2 = 1$ so that $n\mu = x$. We now write

$$\sum_{|h| > x^\xi} u_m = P\{|S_n - x| > x^\xi\} = P\{S_n - x > x^\xi\} + P\{S_n - x < -x^\xi\}.$$

We will deal only with the first term as the second term has a similar bound. Hence

$$P\Big\{\frac{S_n - n\mu}{\sigma\sqrt{n}} > \frac{x - x + x^\xi}{\sqrt{x}}\Big\} \leq 1 - \Phi\Big(\frac{x^\xi}{\sqrt{x}}\Big)$$

by Theorem 8.
By Lemma 1,

$$\begin{aligned} P\Big\{\frac{S_n - n\mu}{\sigma\sqrt{n}} > x^{\xi - \frac{1}{2}}\Big\} &\sim \frac{1}{x'\sqrt{2\pi}} e^{-x'^2/2}, \text{ where } x' = x^{\xi - \frac{1}{2}} \\ &\sim \frac{1}{\sqrt{2\pi}} \Big(\frac{1}{x^{\xi - \frac{1}{2}}}\Big) e^{-\frac{1}{2}(x^{2\xi - 1})}. \end{aligned}$$

Since $\frac{1}{\sqrt{2\pi}}\Big(\frac{1}{x^{\xi - \frac{1}{2}}}\Big)$ is a decreasing function of x, it is bounded. Hence

$$\sum_{h > x^\xi} u_m = O(e^{-\frac{1}{2}x^{2\xi - 1}}).$$

A similar argument takes care of the other term.
∎

The following two corollaries follow similarly.

Corollary 5 *(Theorem 138(3), p. 201, [6]) Suppose that u_m are defined as in Corollary 2. If $m = M + h$, where $M = [x] = [\frac{n+1}{q+1}]$ and $\frac{1}{2} < \xi < \frac{2}{3}$, then*

$$\sum_{|h|>x^{\xi}} u_m = O(e^{-x^{\eta}}), \text{ where } \eta \text{ is any number less than } 2\xi - 1.$$

Corollary 6 *(Theorem 139(3), p. 201, [6]) Suppose that u_m are defined as in Corollary 3. If $m = M + h$, where $M = [x] = [\frac{n}{k}]$ and $\frac{1}{2} < \xi < \frac{2}{3}$, then*

$$\sum_{|h|>x^{\xi}} u_m = O(e^{-x^{\eta}}), \text{ where } \eta \text{ is any number less than } 2\xi - 1.$$

Tust as Theorem 8 carries through without difficulty, if the condition $x = o(n^{1/6})$ is replaced by $x = o(n^{1/2})$, we have:

Theorem 9 *(Feller [5]) Under the assumptions of Theorem 8, if the condition $x = o(n^{1/6})$ is replaced by $x = o(n^{1/2})$, then*

$$1 - F_n(x) \sim 1 - \Phi(x),$$

where $F_n(x) = P\Big(\frac{S_n - ES_n}{\sigma\sqrt{n}} \le x\Big)$ and $\Phi(x)$ is the standard normal distribution.

The following three corollaries give Hardy's results. Once again, we give the result for the Borel matrix method instead of the Borel exponential method as given in Theorem 137(4), p. 200, [6].

Corollary 7 *Suppose that u_m are defined as in Corollary 1. If $m = M + h$, where $M = x$ and for every $\epsilon > 0$, there exist $x_0(\epsilon)$ and $\lambda_0(\epsilon) > 0$ such that*

$$\sum_{|h|>\lambda x^{1/2}} u_m < \epsilon, \text{ for all } x > x_0(\epsilon) \text{ and } \lambda > \lambda_0(\epsilon).$$

Proof.

Let $S_n = X_1 + X_2 + \cdots + X_n$ and let F be the distribution of $X_1 - 1$ with each $X_i \sim$ Pois(1). Let $x = n$ and $h = m - x$. We now write

$$\sum_{|h|>\lambda x^{1/2}} u_m = P\{|S_n - x| > \lambda x^{1/2}\} = P\{S_n - x > \lambda x^{1/2}\} + P\{S_n - x < -\lambda x^{1/2}\}.$$

As before, for the first term of the above

$$P\{S_n - x > \lambda (n)^{\frac{1}{2}}\} \le P\{\frac{S_n - n}{\sqrt{n}} > \lambda\}.$$

Since $\lim_{\lambda\to\infty} 1 - \Phi(\lambda) = 0$, for every $\epsilon > 0$, there exists some $\lambda_0(\epsilon) > 0$ such that $1 - \Phi(\lambda) < \epsilon/2$ for all $\lambda \ge \lambda_0(\epsilon)$. We also have

$$\sum_{h>\lambda x^{1/2}} u_m \le \sum_{h>\lambda_0(\epsilon)x^{1/2}} u_m \text{ for all } \lambda \ge \lambda_0(\epsilon).$$

Now from Theorem 9 for fixed $\lambda_0(\epsilon)$

$$\sum_{h>\lambda_0(\epsilon)x^{1/2}} u_m \sim 1-\Phi(\lambda_0(\epsilon)) \text{ for all large values of } n.$$

We now obtain, for any small $\epsilon>0$ an $N(\epsilon)$ so that

$$\sum_{h>\lambda_0(\epsilon)x^{1/2}} u_m \leq (1+\epsilon/2)\Big\{1-\Phi(\lambda_0(\epsilon))\Big\} < \epsilon$$

for all $n \geq N(\epsilon)$ and $\lambda \geq \lambda_0(\epsilon)$. The above follows from

$$\Big|\frac{1-F_n(x)}{1-\Phi(x)}-1\Big| < \epsilon/2 \;\Rightarrow\; 1-F_n(x) < (1-\Phi(x)) + \frac{(1-\Phi(x))}{2}\epsilon.$$

This yields that for every small $\epsilon>0$, there exist some numbers $x_0(\epsilon)$ and $\lambda_0(\epsilon)>0$ such that

$$\sum_{h>\lambda x^{1/2}} u_m < \epsilon$$

for all $x \geq x_0(\epsilon)$ and $\lambda \geq \lambda_0(\epsilon) > 0$, as required. A similar bound is obtained for the second term to complete the proof.

∎

The two corollaries given below have essentially the same proof.

Corollary 8 *(Theorem 138(4), p. 201, [6]) Suppose that u_m are defined as in Corollary 2. If $m = M+h$, where $M=[x]=[\frac{n+1}{q+1}]$ and for and every $\epsilon>0$, there exist $x_0(\epsilon)$ and $\lambda_0(\epsilon)>0$ such that*

$$\sum_{|h|>\lambda x^{1/2}} u_m < \epsilon, \textit{ for all } x > x_0(\epsilon) \textit{ and } \lambda > \lambda_0(\epsilon).$$

Corollary 9 *(Theorem 139(4), p. 201, [6]) Suppose that u_m are defined as in Corollary 3. If $m=M+h$, where $M=[x]=[\frac{n}{k}]$ and for every $\epsilon>0$, there exist $x_0(\epsilon)$ and $\lambda_0(\epsilon)>0$ such that*

$$\sum_{|h|>\lambda x^{1/2}} u_m < \epsilon, \textit{ for all } x > x_0(\epsilon) \textit{ and } \lambda > \lambda_0(\epsilon).$$

VII. LOCAL BEHAVIOR OF RANDOM-WALK METHODS

The succeeding theorems are some examples of local limit theorems, since they concern densities of the probability distributions, while the previous results provided limit theorems involving distribution functions.

The following approach makes use of two important local limit theorems. One is due to Petrov [11] and is called Petrov's approximation, and the other one is due to Richter [12].

Theorem 10 *(Richter [12]) Let $Y, \{Y_i\}_{i\geq 1}$ be a sequence of independent and identically distributed random variables with mean zero and positive second moment, which take* *values only in a certain arithmetic progression with probabilities* $p_k = P\{Y = a + kl\}$, l *the maximal span of the distribution,* k *an integer,* a *a fixed real number. Let us put*

$$P_n(k) = P\{\sum_{j=1}^{n} Y_j = an + kl\}.$$

Assume that there exists a positive number A *such that for every real number* s, $|s| < A$, Ee^{sY} *is convergent. If we put* $t = (an+kl)/(\sigma\sqrt{n})$, *then for* $t > 1$, $t = o(\sqrt{n})$ *as* $n \to \infty$,

$$\frac{\frac{\sigma\sqrt{n}}{l}P_n(k)}{\phi(t)} = e^{(t^3/\sqrt{n})\lambda((t/\sqrt{n}))}\Big[1 + O\Big(\frac{t}{\sqrt{n}}\Big)\Big],$$

where $\lambda(t)$ *is a power series converging for all sufficiently small values of* $|t|$ *and* $\phi(t)$ *is the standard normal density.*

We now prove Hardy's theorem for the random-walk method.

Theorem 11 *Let* X, $\{X_i\}_{i\geq 1}$ *be a sequence of independent and identically distributed non-negative random variables with positive variance and finite moment generating function. Then for* $|h| = |i - x| \leq x^{\xi}$ *for any fixed* ξ *with* $\frac{1}{2} < \xi < \frac{2}{3}$, *we have*

$$P\{S_n = i\} = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} \text{ as } n \to \infty,$$

where $t = \frac{\sqrt{\mu}}{\sigma}\frac{h}{\sqrt{x}}$, $x = n\mu$, $EX = \mu$ *and* $S_n = X_1 + X_2 + \cdots + X_n$.

Proof.

There are two cases. One is $|t| \leq 1$ and other is $|t| > 1$. We will apply Richter's theorem with $a = -EX$ and $l = 1$ for one of these two cases. First, we will consider the case $|t| \leq 1$. From Petrov's approximation [11] for any positive $k \geq 3$, (where k stands for assumed finiteness of the kth moments) we have

$$P\{S_n = i\} = \frac{\phi(t)}{\sigma\sqrt{n}} + \sum_{j=1}^{k-2}\frac{Q_j(t)\phi(t)}{n^{\frac{1}{2}+\frac{j}{2}}} + \frac{o(1)}{n^{\frac{1}{2}(k-1)}(1+|t|^k)}$$

uniformly in i (or t) as $n \to \infty$, where $Q_j's$ are polynomials of degrees at most $3j$ depending only on the probability law P, (through its moments or cumulants) $t = \frac{i-n\mu}{\sigma\sqrt{n}}$, j being integer and ϕ is the standard normal density.

Since the distribution under consideration has at least third finite moment, substituting $k = 3$, we obtain

$$P\{S_n = i\} = \frac{\phi(t)}{\sigma\sqrt{n}} + \frac{Q_1(t)\phi(t)}{n} + \frac{o(1)}{n(1+|t|^3)}.$$

For the appropriate constants A, B, C and D, this is precisely

$$P\{S_n = i\} = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + \frac{A + Bt + Ct^2 + Dt^3}{\sqrt{n}}\Big\} + \frac{o(1)}{n(1+|t|^3)} \text{ as } n \to \infty.$$

The restriction $|h| = |i - x| \le x^{\xi}$ for the fixed ξ with $\frac{1}{2} < \xi < \frac{2}{3}$ implies that $\frac{h}{\sqrt{x}} = o(x^{1/6})$, so that $h = o(x^{2/3})$. In view of these assumptions, we will show that the rate of $\frac{A}{\sqrt{n}} = o\Big(\frac{1}{x^{1/2}}\Big)$ is faster than that of $\frac{Bt}{\sqrt{n}}$, while the rate of $\frac{Ct^2}{\sqrt{n}}$ is faster than that of $\frac{Dt^3}{\sqrt{n}}$. This will be accomplished as follows;

$$\frac{Bt}{\sqrt{n}} = \frac{B'|h|}{x} = \frac{o(1)}{x^{1/3}} = o\Big(\frac{1}{x^{1/3}}\Big).$$

Hence the term that corresponds to the slower rate $\frac{Bt}{\sqrt{n}}$ is retained in the expression. Similarly

$$\frac{Ct^2}{\sqrt{n}} = \frac{C'|h^2|}{x^{3/2}} = \frac{o(1)}{x^{1/6}} = o\Big(\frac{1}{x^{1/6}}\Big),$$

where as

$$\frac{Dt^3}{\sqrt{n}} = \frac{D'|h|^3}{x^2} = o(1).$$

The contribution due to the term $\frac{Dt^3}{\sqrt{n}}$ will be retained as compared to $\frac{Ct^2}{\sqrt{n}}$. Taking the above into consideration, the expression now takes the form

$$P\{S_n = i\} = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} + \frac{o(1)}{n(1+|t|^3)} \text{ as } n \to \infty.$$

In order to prevent the term $O\Big(\frac{|h|}{x}\Big)$ from vanishing, it will be replaced by $O\Big(\frac{|h|+1}{x}\Big)$. This leads to the assertion

$$P\{S_n = i\} = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} + \frac{o(1)}{n(1+|t|^3)} \text{ as } n \to \infty,$$

where $t = \frac{\sqrt{\mu}}{\sigma}\frac{h}{\sqrt{x}}$, $x = n\mu$ and $EX = \mu$.

Now it remains to show that under condition $|t| \le 1$, the last order condition can be absorbed by one of the other terms in the expansion. For this, we will consider

$$\frac{o(1)}{n(1+|t|^3)} \simeq \frac{o(1)}{x} = \frac{\phi(t)}{\sigma\sqrt{n}} \cdot \frac{\sigma\sqrt{n}}{\phi(t)}\frac{o(1)}{x} = \frac{\phi(t)}{\sigma\sqrt{n}}\frac{o(1)}{\sqrt{x}}$$

as $\phi(t)$ is bounded away from zero for $|t| \le 1$. This order term, namely $\frac{o(1)}{\sqrt{x}}$ tends to zero faster than $\frac{|h|}{x} = \frac{o(1)}{x^{1/3}}$ and will be absorbed by $\frac{|h|+1}{x}$. Hence for $|t| \le 1$, we obtain

$$P\{S_n = i\} = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} \text{ as } n \to \infty$$

as required.

To prove the assertion for the remaining case ($|t| > 1$), we apply Theorem 10 with $a = -EY$ and $l = 1$. The theorem is stated only for $t > 1$. The other case, namely $t < -1$ will be obtained by replacing $\{Y_i\}_{i \ge 1}$ by $\{-Y_i\}_{i \ge 1}$. Let us consider

$$P\{S_n = k\} = P\{\sum_{j=1}^{n} X_j - nEX = -nEX + kl\} = P\{\sum_{j=1}^{n}(X_j - EX) = na + kl\}.$$

The sequence of random variables $\{Y_j = X_j - EX\}_{j\geq 1}$ has mean zero, positive variance and finite moment generating function as required by Theorem 10. The order condition $\frac{h}{\sqrt{x}} = o(x^{1/6})$ will imply the condition $t = o(x^{1/2})$. Hence, by Theorem 10,

$$P\{S_n = k\} = \frac{\phi(t)}{\sigma\sqrt{n}} e^{(t^3/\sqrt{n})\lambda((t/\sqrt{n}))}\Big(1 + O\Big(\frac{t}{\sqrt{n}}\Big)\Big) \text{ as } n \to \infty.$$

Since $\frac{t^3}{\sqrt{n}} = O\Big(\frac{|h|^3}{x^2}\Big) = o(1)$, $\frac{t}{\sqrt{n}} = O\Big(\frac{1}{x^{1/3}}\Big)$ and $\lambda((t/\sqrt{n}))$ is convergent for all large values of n, we obtain

$$\begin{aligned} P\{S_n = k\} &= \frac{\phi(t)}{\sigma\sqrt{n}}\Big(1 + O\Big(\frac{|h|^3}{x^2}\Big)\Big)\Big(1 + O\Big(\frac{t}{\sqrt{n}}\Big)\Big) \\ &= \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} \text{ as } n \to \infty. \end{aligned}$$

In order to prevent the vanishing of the term $O\Big(\frac{|h|}{x}\Big)$, we replaced it by $O\Big(\frac{|h|+1}{x}\Big)$.

The assertion now is

$$P\{S_n = i\} = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} \text{ as } n \to \infty.$$

We complete the proof by combining the above two cases.

∎

Substituting for particular moments of the Borel matrix and Euler methods, two corresponding assertions now follow as stated in the corollaries. Since the Taylor method is not a member of the random-walk method, it requires additional work for proof of its corollary.

The Theorem 137(5), p. 200, [6] is in fact stated for the Borel exponential method. We will state it for the Borel Matrix method.

Corollary 10 *Suppose that u_m are defined as in Corollary 1. If $|h| \leq x^{\xi}$, then*

$$u_m = \frac{1}{\sqrt{2\pi x}} e^{-\frac{1}{2}(m-x)^2/x}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\}.$$

Proof.

Let $S_n = X_1 + X_2 + \cdots + X_n$. If each $X_i \sim$ Pois(1). Let $[x] = n$ and $h = m - n$. Note that $\mu = \sigma^2 = 1$ so that $n\mu = n$. We now write $t = \frac{m-n\mu}{\sigma\sqrt{n}}$. From Theorem 11, we have

$$P\{S_n = m\} = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} \text{ as } n \to \infty,$$

where $|h| = |m - n|$ and $[x] = n$.

This concludes the proof of the corollary.

∎

In the following two corollaries, we will use x instead of $[x]$ to address the corresponding probabilistic version of Theorem 138(5), p. 201 and Theorem 139(5), p. 201, [6].

Corollary 11 *Suppose that u_m are defined as in Corollary 2. If $|h| \le x^{\xi}$, then*

$$u_m = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\},$$

where $\sigma^2 = \frac{q}{(q+1)^2}$ and $|h| = |m - \frac{n+1}{q+1}|$.

Proof.

Let $S_n = X_1 + X_2 + \cdots + X_n$. We now consider the random variables defined as

$$X_i = \begin{cases} 1 & \text{with probability} \quad 1/(q+1),\ q > 0 \\ 0 & \text{with probability} \quad q/(q+1), \end{cases}$$

for $i = 1, 2, \cdots n$.

Now substitute $\mu = \frac{1}{(q+1)}$, $\sigma^2 = \frac{q}{(q+1)^2}$, and $t = \frac{m - \frac{n+1}{q+1}}{\sqrt{(n+1)\frac{q}{(q+1)^2}}}$ in Theorem 11 to get

$$u_m = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} \text{ as } n \to \infty,$$

where $|h| = |m - \frac{n+1}{q+1}|$ and $x = \frac{n+1}{q+1}$.

This concludes the proof of the corollary.

∎

Corollary 12 *Suppose that u_m are defined as in Corollary 3. If $|h| \le x^{\xi}$, then*

$$u_m = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\},$$

where $\sigma^2 = \frac{1}{k}$ and $|h| = |m - \frac{n}{k}|$.

Proof.

Let $X, X_0, X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables. Assume that X has a geometric distribution with parameter $0 < k < 1$. With $Y = X_0 - 1$, the nth row of the Taylor method is

$$P(Y + \sum_{i=1}^{n} X_i = m) = P(X_0 - 1 + \sum_{i=1}^{n} X_i = m) = P(\sum_{i=0}^{n} X_i = m + 1).$$

The t as defined in Theorem 11 now takes the form

$$\begin{aligned} t &= \frac{m + 1 - (n+1)\mu}{\sigma\sqrt{n+1}} \\ &= \sqrt{\frac{n}{n+1}}\frac{m - n\mu}{\sigma\sqrt{n}} - \sqrt{\frac{n}{n+1}}\frac{1-\mu}{\sigma\sqrt{n}} \\ &= o\Big(x^{1/6}\Big) - O\Big(\frac{1}{x^{1/2}}\Big) \\ &= o\Big(x^{1/6}\Big). \end{aligned}$$

This t has the right order condition as required by the theorem. Hence substituting $\mu = \frac{1}{k}$, $\sigma^2 = \frac{1-k}{k^2}$, and $t = \frac{m-\frac{n}{k}}{\sqrt{\frac{n(1-k)}{k^2}}}$, we get

$$u_m = \frac{\phi(t)}{\sigma\sqrt{n}}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} \text{ as } n \to \infty,$$

where $|h| = |m - \frac{n}{k}|$ and $x = \frac{n}{k}$.

This concludes the proof of the corollary.

∎

VIII. GENERALIZATIONS FOR CONVOLUTION SUMMABILITY

The theory developed in the preceding sections can be generalized to independent, but not identically distributed of random variables. Indeed, every random-walk method can be obtained from a sequence $\{q\}$ and the convolution method. To start, we need a theorem from (p. 524, [5]).

Theorem 12 *(Feller [5]) Suppose there exists an interval $(-a, a)$ in which all the characteristic functions ω_k are analytic; that $E(|X_n|^3) \le M\sigma_n^2$, where M is independent of n. If x and n vary so that $s_n \to \infty$ and $x = o(\sqrt{s_n})$, then*

$$\frac{1 - F_n(x)}{1 - \Phi(x)} \to 1 \textit{ with an error } O(x^3/s_n) \textit{ uniformly in } x > 1.$$

We now state and prove the theorem, which generalizes Hardy's theorem for convolution summability method.

Theorem 13 *Let $C = \{C_{n,k}\}_{n,k\ge1}$ be a regular convolution summability matrix generated by an independent sequence Y, $\{X_i, i \ge 1\}$ with $E(e^{\theta Y}) < \infty$, and $\{X_i, i \ge 1\}$ of identically distributed integer-valued random variables with $E(e^{\theta X_i}) < \infty$ for all $i \ge 1$ and for some $\theta > 0$. The following assertions hold.*

1. *(a) If $\{q_j\}$ is unimodal and $\{p_j\}$ is strongly unimodal, then the entries of the nth row of the convolution method generated by the sequences $\{q_j\}$ and $\{p_j\}$ is unimodal for each n.*

   (b) If $\{q_j\}$ is unimodal and $p_j = 0$ for $j < 0$ and $j > d$ while $p_0 > 0$, $p_1 > 0$, $p_{d-1} > 0$, $p_d > 0$, then there exists an integer n_0 such that for $n > n_0$, the entries of the nth row of the convolution method generated by the sequences $\{q_j\}$ and $\{p_j\}$ is unimodal, where Y has the probability distribution $\{q_j\}_{i\ge0}$ and $\{X_i\}_{i\ge1}$ have probability distribution $\{p_j\}_{i\ge0}$.

2. *If $k = [\mu_Y + n\mu_X] + h$ and $\delta > 0$, then*

$$\sum_{|h|>\delta x} C_{n,k} = O\Big(e^{-\gamma x}\Big), \textit{ where } \gamma > 0.$$

3. *If $\frac{1}{2} < \xi < \frac{2}{3}$, then*

$$\sum_{|h|>x^{\xi}} C_{n,k} = O\Big(e^{-x^{\eta}}\Big), \textit{ where } \eta \textit{ is any number less than } 2\xi - 1.$$

4. *For every $\epsilon > 0$, there exist $x_0(\epsilon)$ and $\lambda_0(\epsilon) > 0$ such that*
$$\sum_{|h|>\lambda x^{\frac{1}{2}}} C_{n,k} < \epsilon, \text{ for all } x > x_0(\epsilon) \text{ and } \lambda > \lambda_0(\epsilon).$$

5. *If $|h| \le x^{\xi}$ for $1/2 < \xi < 2/3$, then*
$$R_{n,k} = \sqrt{\Big(\frac{C}{\pi x}\Big)} e^{-Ch^2/x}\Big\{1 + O\Big(\frac{|h|+1}{x}\Big) + O\Big(\frac{|h|^3}{x^2}\Big)\Big\} \text{ as } t \to 0,$$
where C is a constant; $t = \frac{\sqrt{\mu}}{\sigma}\frac{h}{\sqrt{x}}$, $x = n\mu$ and $EX = \mu$, where R is the random-walk method generated by $\{X_i\}_{i\ge 1}$ and Y is as defined before. It is also assumed that $E(e^{Yt}) < \infty$ for $|t| \le t_0$.

6. *The estimates (2) and (3) remain valid, if $C_{n,k}$ is multiplied by any fixed power of k.*

Proof.
(1). (a) A discrete distribution $\{p_j\}$ on integers is strongly unimodal if $\{p_j\} \star \{q_j\}$ is unimodal for any unimodal discrete distribution $\{q_j\}$. Hence $\{p_j\}^{\star n} \star \{q_j\}$ is unimodal for all values of n, provided $\{p_j\}$ is strongly unimodal while $\{q_j\}$ is unimodal.

(b) Theorem 1 of [10] guarantees that there exists an integer n_0 such that for all $n > n_0$, the $n-$fold convolution $\{p_j\}^{\star n}$ is strongly unimodal. If in addition $\{q_j\}$ is unimodal, then $\{p_j\}^{\star n} \star \{q_j\}$ is unimodal for all values of $n > n_0$. This completes the proof.
(2).
We begin by considering
$$\sum_{|h|>\delta x} C_{n,k} = P\Big\{\Big|\frac{Y + S_n - [\mu_Y + n\mu_X]}{\mu_Y + n\mu_X}\Big| > \delta\Big\}$$
$$= P\{Y + S_n - [\mu_Y + n\mu_X] \ge \delta(\mu_Y + n\mu_X)\} + P\{Y + S_n - [\mu_Y + n\mu_X] \le -\delta(\mu_Y + n\mu_X)\}.$$
We will obtain the appropriate bound for the first term, and the same will be obtained for the second term using a similar technique. Consider
$$P\{Y + S_n - [\mu_Y + n\mu_X] \ge \delta(\mu_Y + n\mu_X)\} \le P\{Y + S_n \ge \delta(\mu_Y + n\mu_X) + \mu_Y + n\mu_X - 1\}.$$
For some $\theta > 0$, we know that $Ee^{\theta X} < \infty$ and $Ee^{\theta Y} < \infty$. Apply Chebyshev's theorem to get
$$\begin{aligned} P\Big\{e^{\theta(Y+S_n)} \ge e^{\theta\{\delta(\mu_Y+n\mu_X)+\mu_Y+n\mu_X-1\}}\Big\} &\le \frac{Ee^{\theta(Y+S_n)}}{e^{n\theta\mu_X(1+\delta)}e^{\theta(\delta\mu_Y+\mu_Y-1)}} \\ &= \frac{Ee^{\theta Y}\Big(Ee^{\theta X}\Big)^n}{e^{n\theta\mu_X(1+\delta)}e^{\theta(\delta\mu_Y+\mu_Y-1)}}. \end{aligned}$$
Letting $\phi(\theta) = Ee^{\theta X}$, we have
$$\begin{aligned} P\Big\{e^{\theta(Y+S_n)} \ge e^{\theta\{\delta(\mu_Y+n\mu_X)+\mu_Y+n\mu_X-1\}}\Big\} &\le \frac{C(\theta)\Big(\phi(\theta)\Big)^n}{e^{n\theta\mu_X(1+\delta)}} \\ &= \frac{C(\theta)\Big(\phi(\theta)\Big)^n}{e^{na\theta}} \\ &= C(\theta)\exp\{-n(\theta a - \log\phi(\theta))\}, \end{aligned}$$

where $a = \mu_X(1+\delta) > \mu_X$ for $\delta > 0$ and $C(\theta)$ is a constant depending on $\theta > 0$. Lemma 9.4 of [4] proves that if $a > \mu$ and θ is small, then $a\theta - \log\phi(\theta) > 0$.

Since $\gamma(\theta) = \theta a - \log\phi(\theta) > 0$, we have

$$\sum_{h>\delta x} C_{n,k} = O\Big(e^{-n\gamma}\Big) \text{ for some } \gamma > 0.$$

To obtain a similar bound for the second term, we write

$$\begin{aligned} P\{Y + S_n - [\mu_Y + n\mu_X] &\le -\delta(\mu_Y + n\mu_X)\} \\ &= P\{-Y - S_n \ge \delta(\mu_Y + n\mu_X) - [\mu_Y + n\mu_X]\} \\ &\le P\{-Y - S_n \ge \delta(\mu_Y + n\mu_X) - (\mu_Y + n\mu_X)\}. \end{aligned}$$

Let $S_n^* = \sum_{i=1}^n (-X_i) = \sum_{i=1}^n X_i^*$ and $Y^* = -Y$, so that $EY^* = -\mu_Y$ and $EX^* = -\mu_X$. By writing in terms of Y^* and X^*, we obtain

$$P\{Y^* + S_n^* \ge \delta(\mu_Y + n\mu_X) - (\mu_Y + n\mu_X)\} = P\{Y^* + S_n^* \ge (\mu_{Y^*} + n\mu_{X^*})(1-\delta)\}.$$

As before, we now have by Chebyshev's theorem for some $\theta > 0$, So this last term is

$$\le \frac{C'(\theta)\Big(\phi'(\theta)\Big)^n}{e^{\theta(1-\delta)(\mu_{Y^*}+n\mu_{X^*})}} = \frac{C'(\theta)\Big(\phi'(\theta)\Big)^n}{e^{na'\theta}} = C'(\theta)\exp\{-n(\theta a' - \log\phi'(\theta))\},$$

where $\phi'(\theta) = Ee^{\theta X^*}$, $a' = \mu_{X^*}(1-\delta) > \mu_{X^*}$ and $C'(\theta)$ is a constant depending on $\theta > 0$. As before, if we let $\gamma(\theta) = a'\theta - \log\phi'(\theta) > 0$ for $a' > \mu_{X^*}$ and θ is small.

Combining these two bounds, the assertion is

$$\sum_{|h|>\delta x} C_{n,k} = O\Big(e^{-n\gamma}\Big) \text{ where for some } \gamma > 0.$$

(3).

As computed before

$$\sum_{|h|>x^\xi} C_{n,k} = P\{|Y + S_n - \mu_Y - n\mu_X| > (\mu_Y + n\mu_X)^\xi\}$$

$$= P\{Y + S_n - \mu_Y - n\mu_X > (\mu_Y + n\mu_X)^\xi\} + P\{Y + S_n - \mu_Y - n\mu_X < -(\mu_Y + n\mu_X)^\xi\}.$$

Now consider the first term:

$$P\{Y + S_n - \mu_Y - n\mu_X > (\mu_Y + n\mu_X)^\xi\} = P\Big\{\frac{Y + S_n - \mu_Y - n\mu_X}{\sqrt{\sigma_Y^2 + n\sigma_X^2}} > \frac{(\mu_Y + n\mu_X)^\xi}{\sqrt{\sigma_Y^2 + n\sigma_X^2}}\Big\}$$

$$= P\Big\{\frac{Y + S_n - \mu_Y - n\mu_X}{\sqrt{\sigma_Y^2 + n\sigma_X^2}} > x'\Big\}, \text{ where } x' = \frac{(\mu_Y + n\mu_X)^\xi}{\sqrt{\sigma_Y^2 + n\sigma_X^2}}.$$

Let $s_n = \sqrt{\sigma_Y^2 + n\sigma_X^2}$ and $s_n \to^n \infty$. Then

$$x'/\sqrt{s_n} = \frac{(\mu_Y + n\mu_X)^\xi}{\sqrt{\sigma_Y^2 + n\sigma_X^2}} \cdot \frac{1}{(\sigma_Y^2 + n\sigma_X^2)^{\frac{1}{4}}} = \frac{(\mu_Y + n\mu_X)^\xi}{(\sigma_Y^2 + n\sigma_X^2)^{\frac{3}{4}}} \to 0 \text{ for } \xi < \frac{3}{4}.$$

We also require that $\xi > \frac{1}{2}$ in order to prevent x' itself tends to 0 or a constant. By Theorem 12, we obtain

$$P\{Y + S_n - \mu_Y - n\mu_X > (\mu_Y + n\mu_X)^{\xi}\} \sim 1 - \Phi(x')$$

with an error $O\Big(x'^3/s_n\Big)$ uniformly in $x' > 1$. That is,

$$\begin{aligned} P\{Y + S_n - \mu_Y - n\mu_X > (\mu_Y + n\mu_X)^{\xi}\} &\sim \frac{1}{x'\sqrt{2\pi}} e^{-\frac{1}{2}x'^2} \\ &= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\sigma_Y^2 + n\sigma_X^2}}{(\mu_Y + n\mu_X)^{\xi}} e^{-\frac{1}{2}\left[\frac{(\mu_Y + n\mu_X)^{\xi}}{\sqrt{\sigma_Y^2 + n\sigma_X^2}}\right]^2} \\ &= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\sigma_Y^2 + n\sigma_X^2}}{(\mu_Y + n\mu_X)^{\xi}} e^{-\frac{1}{2}\frac{n^{2\xi}\left(\frac{\mu_Y}{n} + \mu_X\right)^{2\xi}}{n\left(\frac{\sigma_Y^2}{n} + \sigma_X^2\right)}} \\ &= O\Big(e^{-n^{\eta}}\Big), \end{aligned}$$

where η is any number less than $2\xi - 1$. The second term has the same bound, thus proving (3).

(4).

We first consider, for $\lambda > 0$,

$$\begin{aligned} \sum_{|h| > \lambda x^{\frac{1}{2}}} C_{n,k} &= P\{|Y + S_n - \mu_Y - n\mu_X| > \lambda(\mu_Y + n\mu_X)^{\frac{1}{2}}\} \\ &= P\{Y + S_n - \mu_Y - n\mu_X > \lambda(\mu_Y + n\mu_X)^{\frac{1}{2}}\} \\ &\quad + P\{Y + S_n - \mu_Y - n\mu_X < -\lambda(\mu_Y + n\mu_X)^{\frac{1}{2}}\}. \end{aligned}$$

Begining with the first term, we see that

$$P\{Y + S_n - \mu_Y - n\mu_X > \lambda(\mu_Y + n\mu_X)^{\frac{1}{2}}\} = P\Big\{\frac{Y + S_n - \mu_Y - n\mu_X}{\sqrt{\sigma_Y^2 + n\sigma_X^2}} > \frac{\lambda(\mu_Y + n\mu_X)^{\frac{1}{2}}}{\sqrt{\sigma_Y^2 + n\sigma_X^2}}\Big\}.$$

Let $x' = \frac{(\mu_Y + n\mu_X)^{\frac{1}{2}}}{\sqrt{\sigma_Y^2 + n\sigma_X^2}}$. Now consider the following steps:

Step 1: Since

$$\lim_{n\to\infty} x' = \lim_{n\to\infty} \frac{(\mu_Y + n\mu_X)^{\frac{1}{2}}}{\sqrt{\sigma_Y^2 + n\sigma_X^2}} = \frac{\sqrt{\mu_X}}{\sigma_X} < \infty,$$

let $k = \min_n x'$.

Step 2: Since $1 - \Phi(k\lambda) \to 0$ as $\lambda \to \infty$; for every $\epsilon > 0$, there exists $\lambda_0(\epsilon)$ such that

$$1 - \Phi(k\lambda) < \frac{\epsilon}{2} \text{ for all } \lambda \geq \lambda_0(\epsilon) > 0.$$

In particular, we have $1 - \Phi(k\lambda_0(\epsilon)) < \frac{\epsilon}{2}$.

Step 3:

$$\sum_{h > \lambda x^{\frac{1}{2}}} C_{n,k} \leq \sum_{h > \lambda_0(\epsilon) x^{\frac{1}{2}}} C_{n,k} \text{ for all } \lambda \geq \lambda_0(\epsilon).$$

By the Central Limit Theorem, we now obtain

$$\begin{aligned}\sum_{h>\lambda_0(\epsilon)x^{\frac{1}{2}}} C_{n,k} &\leq 1-\Phi(\lambda_0(\epsilon)x')+\frac{\epsilon}{2} \text{ for all } n\geq N_0(\epsilon)\\ &\leq 1-\Phi(\lambda_0(\epsilon)k)+\frac{\epsilon}{2}<\epsilon \text{ for all } n\geq N_0(\epsilon) \text{ for all } \lambda\geq\lambda_0(\epsilon).\end{aligned}$$

Step 4: Let $x_0(\epsilon)=\mu_Y+N_0(\epsilon)\mu_X$. Now it follows that for every $\epsilon>0$, there exist numbers $x_0(\epsilon)$ and $\lambda_0(\epsilon)>0$ such that

$$\sum_{h>\lambda x^{\frac{1}{2}}} C_{n,k}\leq \sum_{h>\lambda_0(\epsilon)x^{\frac{1}{2}}} C_{n,k}<\epsilon$$

for all $x\geq x_0(\epsilon)$ and for all $\lambda\geq\lambda_0(\epsilon)$ as required. To finish, we carry out the calculations for the second term in a similar manner.
(5).

Let us consider $C_{n,k}=P\{Y+S_n=k\}$. We write

$$P\{Y+S_n=k\}=\sum_{j=0}^{\infty} q_j P\{S_n=k-j\},$$

where $\{q_j\}_{j\geq 1}$ is the probability density of the random variable Y. For $|k-\mu_Y-n\mu_X|\leq(\mu_Y+n\mu_X)^{\xi}$ with $1/2<\xi<2/3$,

$$P\{Y+S_n=k\}=\sum_{\{j:|k-j-n\mu_x|\leq(n\mu_X)^{\xi}\}} q_jP\{S_n=k-j\}+$$

$$\sum_{\{j:|k-j-n\mu_x|>(n\mu_X)^{\xi}\}} q_jP\{S_n=k-j\}.$$

Using the results of the previous section the Random-walk method, the first sum above is equal to

$$\sum_{j=0}^{\infty} q_j\Big[\frac{\phi(t)}{\sigma_X\sqrt{n}}\Big\{1+O\Big(\frac{|h|+1}{x}\Big)+O\Big(\frac{|h|^3}{x^2}\Big)\Big\}\Big], \text{ where } t=\frac{k-j-n\mu_X}{\sigma_X\sqrt{n}} \text{ and } x=n\mu_X.$$

We have $|k-j-n\mu_X|\leq(n\mu_X)^{\xi}$ and $k-j\geq 0$. For $|k-\mu_Y-n\mu_X|\leq(\mu_Y+n\mu_X)^{\xi'}$ for every $k\geq 0$ and $\frac{1}{2}<\xi'<\frac{2}{3}$, we obtain $|j-\mu_Y|\leq|k-j-n\mu_X|+|k-\mu_Y-n\mu_X|\leq(n\mu_X)^{\xi}+(\mu_Y+n\mu_X)^{\xi'}<2(\mu_Y+n\mu_X)^{2/3}$.

Let $h=|k-j-n\mu_X|\leq|k-\mu_Y-n\mu_X|+|j-\mu_Y|$ and $h'=|k-\mu_Y-n\mu_X|$. We then have $|h|<|h'|+|j-\mu_Y|$ for all $j\geq 0$.

Consider

$$\sum_{j=0}^{\infty} q_j\Big[\frac{\phi(t)}{\sigma_X\sqrt{n}}\Big\{1+O\Big(\frac{|h|+1}{x}\Big)+O\Big(\frac{|h|^3}{x^2}\Big)\Big\}\Big]$$

$$=\sum_{j=0}^{\infty} q_j\Big[\frac{1}{\sqrt{2\pi(\sigma_Y^2+n\sigma_X^2)}}\Big(\frac{\sigma_Y^2+n\sigma_X^2}{n\sigma_X^2}\Big)^{\frac{1}{2}} e^{-\frac{1}{2}\left(\frac{k-j-\mu_Y-n\mu_X+\mu_Y}{\sqrt{\sigma_Y^2+n\mu_X^2}}\right)^2\left(\frac{\sigma_Y^2+n\sigma_X^2}{n\sigma_X^2}\right)}$$

$$\Big\{1+O\Big(\frac{|h'|+|j-\mu_Y|+1}{\mu_Y+n\mu_X}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)$$

$$+O\Big(\frac{|h'|^3+|j-\mu_Y|^3+3|h'|^2|j-\mu_Y|+3|h'||j-\mu_Y|^2}{(\mu_Y+n\mu_X)^2}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)^2\Big\}\Big]$$

$$=\sum_{j=0}^{\infty} q_j\Big[\frac{1}{\sqrt{2\pi(\sigma_Y^2+n\sigma_X^2)}}\Big(\frac{\sigma_Y^2+n\sigma_X^2}{n\sigma_X^2}\Big)^{\frac{1}{2}}$$

$$e^{-\frac{1}{2}\Big(\frac{|k-\mu_Y-n\mu_X|^2+|j-\mu_Y|^2-2|k-\mu_Y-n\mu_X||j-\mu_Y|}{\sigma_Y^2+n\mu_X^2}\Big)\Big(\frac{\sigma_Y^2+n\sigma_X^2}{n\sigma_X^2}\Big)}$$

$$\Big\{1+O\Big(\frac{|h'|+|j-\mu_Y|+1}{\mu_Y+n\mu_X}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)$$

$$+O\Big(\frac{|h'|^3+|j-\mu_Y|^3+3|h'|^2|j-\mu_Y|+3|h'||j-\mu_Y|^2}{(\mu_Y+n\mu_X)^2}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)^2\Big\}\Big].$$

Since $|k-\mu_Y-n\mu_X| < (\mu_Y+n\mu_X)^{\xi'} = (x')^{\xi'}$ and noting $0 \le |j-\mu_Y| \le j+\mu_Y$, we rewrite

$$=\sum_{j=0}^{\infty} q_j\Big[\frac{1}{\sqrt{2\pi(\sigma_Y^2+n\sigma_X^2)}}\Big(\frac{\sigma_Y^2+n\sigma_X^2}{n\sigma_X^2}\Big)^{\frac{1}{2}}$$

$$e^{-\frac{1}{2}\Big(\frac{|h'|^2+|k-\mu_Y-n\mu_X||j-\mu_Y|}{\sigma_Y^2+n\mu_X^2}\Big)\Big(\frac{\sigma_Y^2+n\sigma_X^2}{n\sigma_X^2}\Big)}$$

$$\Big\{1+O\Big(\frac{|h'|+|j-\mu_Y|+1}{\mu_Y+n\mu_X}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)$$

$$+O\Big(\frac{|h'|^3+|j-\mu_Y|^3+3|h'|^2|j-\mu_Y|+3|h'||j-\mu_Y|^2}{(\mu_Y+n\mu_X)^2}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)^2\Big\}\Big].$$

Since $\frac{|k-\mu_Y-n\mu_X||j-\mu_Y|}{\sigma_Y^2+n\mu_X^2} \le \beta j + p$ for some β and $p>0$, we obtain, taking $\sum_{j=0}^{\infty} q_j j^k < \infty$ for all $k \ge 1$ and $\sum_{j=0}^{\infty} q_j e^{\beta j} < \infty$ into consideration that

$$C_{n,k} = \sqrt{\Big(\frac{C}{\pi M}\Big)}e^{-Ch'^2/M}\Big\{1+\Big(\frac{|h'|+1}{x'}\Big)+\Big(\frac{|h'|^3}{x'^2}\Big)\Big\},\ \text{where } C \text{ is a constant.}$$

Note that the omitted terms are of following order, namely

$$O\Big(\frac{|j-\mu_Y|}{\mu_Y+n\mu_X}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big) = o\Big(\frac{1}{x'^{1/3}}\Big),$$

$$O\Big(\frac{|j-\mu_Y|^3}{(\mu_Y+n\mu_X)^2}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)^2 = o(1),$$

$$O\Big(\frac{|h'|^2|j-\mu_Y|}{(\mu_Y+n\mu_X)^2}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)^2 = o(1),$$

$$O\Big(\frac{|h'||j-\mu_Y|^2}{(\mu_Y+n\mu_X)^2}\Big)\Big(\frac{\mu_Y+n\mu_X}{n\mu_X}\Big)^2 = o(1)$$

and are faster than other terms present in the expression.

We will find an appropriate bound for the second sum, namely

$$\sum_{\{j:|k-j-n\mu_x|>(n\mu_X)^{\xi}\ \&\ |k-\mu_Y-n\mu_X|\le(\mu_Y+n\mu_X)^{\xi}\}} q_j P\{S_n = k-j\}.$$

Subject to the finiteness of the second moments of $\{X_i\}_{i\geq 1}$, this is equal to

$$\sum_{\{j:|k-j-n\mu_x|>(n\mu_X)^\xi \ \& \ |k-\mu_Y-n\mu_X|\leq(\mu_Y+n\mu_X)^\xi\}} q_j \frac{1}{\sqrt{2\pi n\sigma_X}} e^{\frac{-1}{2}(k-j-n\mu_X)^2/n\sigma_X^2} + o(1\sqrt{n}).$$

The last term of this is

$$o(1\sqrt{n}) = \frac{\phi(t')}{\sqrt{n\sigma_X^2+\sigma_Y^2}} \frac{\sqrt{n\sigma_X^2+\sigma_Y^2}}{\phi(t')} o(1\sqrt{n}) = \frac{\phi(t')}{\sqrt{n\sigma_X^2+\sigma_Y^2}} \frac{o(1)}{\phi(t')} = \frac{\phi(t')}{\sqrt{n\sigma_X^2+\sigma_Y^2}} o(1),$$

where $t' = \frac{k-n\mu_x\mu_Y}{\mu_Y+n\mu_X}$. We next show that $\phi(t') > 0$. But,

$$\phi(t') = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}(k-n\mu_X\mu_Y)^2/(n\sigma_X^2+\sigma_Y^2)} > \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}n^{2\xi-1}} > \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}n^\epsilon} > 0,$$

where $2\xi - 1 < \epsilon$ and $1/2 < \xi < 2/3$. The first term

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi(n\sigma_X^2+\mu_Y^2)}} \sqrt{\frac{n\sigma_X^2+\mu_Y^2}{n\sigma_X^2}} \sum_{\{j:|k-j-n\mu_x|>(n\mu_X)^\xi \ \& \ |k-\mu_Y-n\mu_X|\leq(\mu_Y+n\mu_X)^\xi\}} q_j \\
\times \ & e^{\frac{-1}{2}\frac{(k-n\mu_X-\mu_Y)^2}{(n\sigma_X^2+\sigma_Y^2)}\left(\frac{n\sigma_X^2+\sigma_Y^2}{n\sigma_X^2}\right)} e^{\frac{-1}{2}(j-\mu_Y)^2/n\sigma_X^2} e^{|k-n\mu_X-\mu_Y||j-\mu_Y|/n\sigma_X^2} \\
\leq \ & \frac{1}{\sqrt{2\pi(n\sigma_X^2+\mu_Y^2)}} e^{\frac{-1}{2}\frac{(k-n\mu_X-\mu_Y)^2}{(n\sigma_X^2+\sigma_Y^2)}} \sum_j q_j e^{(n\mu_X+\mu_Y)^{\xi-1}(j+\mu_Y)} \\
= \ & \frac{1}{\sqrt{2\pi(n\sigma_X^2+\mu_Y^2)}} e^{\frac{-1}{2}\frac{(k-n\mu_X-\mu_Y)^2}{(n\sigma_X^2+\sigma_Y^2)}} \sum_j q_j e^{\theta j},
\end{aligned}$$

where θ is small and $\theta \to 0$ as $n \to \infty$.

$$= \frac{1}{\sqrt{2\pi(n\sigma_X^2+\mu_Y^2)}} e^{\frac{-1}{2}\frac{(k-n\mu_X-\mu_Y)^2}{(n\sigma_X^2+\sigma_Y^2)}} O(1)$$

and this can be absorbed by the first expansion.

This completes the proof of (5).

(6).

For the first part of this assertion, we assume

$$\sum_{|h|>\delta x} C_{n,k} = P\Big\{\Big|\frac{Y+S_n-[\mu_Y+n\mu_X]}{\mu_Y+n\mu_X}\Big| > \delta\Big\} = O(e^{-n\gamma}) \text{ for some } \gamma > 0.$$

We want to show that

$$\sum_{|h|>\delta x} k^l C_{n,k} = O(e^{-n\alpha}) \text{ for fixed number } l \text{ and some } \alpha > 0.$$

Consider

$$E\{Z_n^l \chi_{A_n}\}; \text{ where } A_n = \{|Z_n - [\mu_Y+n\mu_X]| > \delta(\mu_Y+n\mu_X)\} \text{ and } Z_n = Y + S_n.$$

Now by the Cauchy-Schwarz inequality

$$E\{Z_n^l \chi_{A_n}\} \leq \{E(Z_n^{2l})\}^{1/2}\{E(\chi_{A_n}^2)\}^{1/2}.$$

The necessary bounds for $\{E(Z_n^{2l})\}^{1/2}$ and $\{E(\chi_{A_n}^2)\}^{1/2}$ are now computed. As assumed, $\{E(\chi_{A_n}^2)\}^{1/2} = \{P(A_n)\}^{1/2} = O(e^{-n\gamma/2})$. By the strong law of large numbers,

$$\{E(Z_n^{2l})\}^{1/2} = O(n^{l/2}).$$

Therefore

$$E\{Z_n^l \chi_{A_n}\} \leq O(n^l e^{-n\gamma/2}) = O(e^{-n\alpha}) \text{ for some } \alpha > 0.$$

This proves the first part of (6).

For the second part, we assume that

$$\sum_{|h|>x^\xi} C_{n,k} = P\Big\{\Big|Y + S_n - [\mu_Y + n\mu_X]\Big| > (\mu_Y + n\mu_X)^\xi\Big\} = O\Big(e^{-x^\eta}\Big),$$

where η is any number less than $2\xi - 1$. We want to show that

$$\sum_{|h|>x^\xi} k^l C_{n,k} = O(e^{-x^\alpha}) \text{ for fixed number } l,$$

where α is any number less than $2\xi - 1$. Consider

$$E\{Z_n^l \chi_{B_n}\}; \text{ where } B_n = \{|Y + S_n - [\mu_Y + n\mu_X]| > (\mu_Y + n\mu_X)^\xi\}$$

and $Z_n = Y + S_n$. Once again, by the Cauchy-Schwarz inequality

$$E\{Z_n^l \chi_{B_n}\} \leq \{E(Z_n^{2l})\}^{1/2}\{E(\chi_{B_n}^2)\}^{1/2}.$$

The necessary bounds are $E(Z_n^{2l}) = O(n^{2l})$ and $E(\chi_{B_n}^2) = O\Big(e^{-n^\eta}\Big)$ for any $\eta < 2\xi - 1$ by assumption. Therefore, if $\frac{1}{2} < \xi < \frac{2}{3}$, then

$$E\{Z_n^l \chi_{B_n}\} = O\Big(e^{-n^\eta}\Big),$$

for any $\eta < 2\xi - 1$. This completes the proof of second part of (6).
∎

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