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ON FOURIER COEFFICIENTS AND FACTORABLE OPERATORS

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Abstract. In this paper we combine the new interpolation method with the well-known results of convolution operator and we give a good estimate for the Fourier coefficients of a periodic fuctions f belonging to $L_{\Phi}(2\pi), L_{\Phi q}(2\pi),\ W^q_{\Phi}(2\pi)$ and $B^s_{\Phi q}(2\pi)$. Further,we show that the Fourier coefficients belong to some Lorentz-Orlicz space $l_{\Psi,q}$.

It is one of the aims of this paper to give a discription of the Fourier coefficients with some applications in the convolution operators,we refer to the books written by A.Pietsch [15] and H.König [10]. Also, we give estimates of entropy numbers of diagonal operators acting between Lorentz-Orlicz spaces which is a consequence of the paper by B.Carl [2]. Applying these results to foctorable operators acting between arbitrary Banach spaces, we get information about the distribution of eigenvalues.

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1. Fourier coefficients.

In this section, we introduce the necessary definitions and we collect a number of basic results. In many cases,the optimality of the eigenvalues estimates follows considering convolution operator for which the Fourier coefficients are eigenvalues. The fact that the fourier coefficients of a periodic function f coincid up to the factor 2π with the eigenvalues of the associated convolution operator was realized rather late.The first remark concerning due to Carleman [2].

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Remark 1.1. *Every function $f \in L_1(2\pi)$ induces a convolution operator C_f defined by*

$$C_f g(x) \;=\; \int_0^{2\pi} f(y-x)g(x)dx \tag{1}$$

on $C(2\pi)$ and $L_p(2\pi)$ with $1 \le p \le \infty$ (see [7] and [17]).

Proposition 1.2. *Let $f \in L_{r,w}(2\pi)$. Then the associated convolution operator C_f acts from $L_{r',w'}(2\pi)$ into $L_\infty(2\pi)$.*

Proof.

Given $g \in L_{r',w'}(2\pi)$, by Hölder's inequality,for almost all $x \in \mathbb{R}$ we have

$$|\int_0^{2\pi} f(x-y)g(y)dy| \le |\int_0^{2\pi} f^*(t)g^*(t)dt| \le |\int_0^{2\pi} t^{\frac{1}{r}-\frac{1}{w}} f^*(t) t^{\frac{1}{r'}-\frac{1}{w'}} g^*(t)dt|$$

$$\le \{\int_0^{2\pi} [t^{\frac{1}{r}-\frac{1}{w}} f^*(t)]^w dt\}^{\frac{1}{w}} \{\int_0^{2\pi} [t^{\frac{1}{r'}-\frac{1}{w'}} g^*(t)]^{w'} dt\}^{\frac{1}{w'}}.$$

This prove that $\|C_f g\|_{L_\infty} \le \|f\|_{L_{r,w}} \|g\|_{L_{r',w'}}$.

Definition 1.3. *For every function $f \in L_1(2\pi)$ and $k \in \mathbb{Z}$ the k-th Fourier coefficient is defined by*

$$\nu_k(f) \;:=\; \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx \quad . \tag{2}$$

Proposition 1.4. *(T.Carleman 1918) Let $f \in L_1(2\pi)$.The sequence of eigenvalues of C_f can be obtained as the non-increasing rearrangement of $(2\pi\nu_k(f))$.*

Proof.

Define $e_k(x) := e^{ikx}$ for all $x \in \mathbb{Z}$. Then it follows from

$$\int_0^{2\pi} f(x-y)e_k(y)dy \;=\; \int_0^{2\pi} f(y)e^{ik(x-y)}dy \;=\; 2\pi\nu_k(f)e_k(x)$$

that $2\pi\nu_k(f)$ is an eigenvalue of C_f with the associated eigenfuction e_k. Next,if λ denotes an arbitrary eigenvalue of C_f then for any associated eigenfunction $g \in L_1(2\pi)$ we have

$$\begin{aligned}\lambda(g,e_k) \;=\; (C_f g, e_k) \;&=\; \int_0^{2\pi}\int_0^{2\pi} f(x-y)g(y)e^{-ikx}dydx \\ &= \int_0^{2\pi}\int_0^{2\pi} f(x)g(y)e^{-ik(x+y)}dxdy \\ &= \quad (f,e_k)(g,e_k) \;=\; 2\pi\nu_k(f)(g,e_k).\end{aligned}$$

Hence $\lambda \ne 2\pi\nu_k(f)$ implies $(g,e_k) = 0$.Since $g \ne 0$,the eigenvalue must coincide with at least one of the coefficients $2\pi\nu_k(f)$.

Theorem 1.5. *(Hausdorff-Young inequality)[1],[17]. Let* $2 \leq p < \infty$ *and* $f \in L_{p'}(2\pi)$*.Then the Fourier coefficients*

$$(\nu_n(f))_{n\in\mathbb{Z}} \in l_p(\mathbb{Z}) \quad with \quad |(\nu_n(f))\|_{l_p} \leq \|f\|_{L_{p'}}.$$

Remark 1.6. *[10] The proof of the Haosdorff-Young inequality is usually accomplished by using the complex interpolation method (cf. [17]).The proof in König (cf.3a.4 [10]) does not use any interpolation theory.Using real interpolation theory one can slightly improve the following corollary:*

Let $2 < p < \infty$ *and* $f \in L_{p'}(2\pi)$*.Then the Fourier coefficients* $(\nu_n(f))_{n\in\mathbb{Z}} \in l_{p,p'}(\mathbb{Z})$ *and with some constant* $c_p > 0$ *depending only on* p *we have*

$$\|(\nu_n(f))\|_{l_{p,p'}} \leq c_p\|f\|_{L_{p'}}.$$

Note that $l_{p,p'}$ *is a slightly smaller Lorentz sequence space than* $l_p = l_{p,p}$ *since* $p' < 2 < p$.

Theorem 1.7. *(W.H.Young 1912 ; F.Hausdorff 1923 ; G.Hardy , J.E. Littlewood 1931) Let* $1 < r < 2$ *and* $1 \leq w \leq \infty$ *. Then*

$$f \in L_{r,w}(2\pi) \quad implies \quad (\nu_n(f)) \in l_{r',w}(\mathbb{Z}). \tag{3}$$

Theorem 1.8. *(A.Pietsch 1980) Let* $\frac{1}{s} = \lambda + \frac{1}{r^+}$ *where* $\frac{1}{r^+} = \frac{1}{\max(r',2)}$*. Then*

$$f \in B^{\lambda}_{rw}(2\pi) \quad implies \quad (\nu_n(f)) \in l_{s,w}(\mathbb{Z}). \tag{4}$$

Theorem 1.9. *[16] Let* $1 < r < \infty, \frac{1}{s} = l + \frac{1}{r^+}$ *and* $w := \min(2, r)$ *. Then*

$$f \in W^l_r(2\pi) \quad implies \quad (\nu_n(f)) \in l_{s,w}(\mathbb{Z}). \tag{5}$$

2. Entropy numbers of diagonal operators .

In the last year several results on the beheviour of entropy numbers of diagonal operators acting between l_p-spaces as well as embedding maps between Besov spaces have been derived. In the following ,we deal with diagonal operators $D \in L(l_p, l_q)$ defined by $D(\xi_i) := (\sigma_i\xi_i)$ and generated by a sequence (σ_i) belongs to Lorentz space $l_{r,t}$.

Definition 2.1. *Let* S *be a bounded linear operator acting from* E *into* F *(i.e.* $S \in L(E, F)$*. Then the n-th entropy number* $e_n(S)$ *is defined to be the infimum of all* $\varepsilon \geq 0$ *such that there are* $y_1, y_2, \dots, y_{2^{n-1}} \in F$ *for which*

$$S(U_E) \subseteq \bigcup_{i=1}^{2^{n-1}} \{y_i + \varepsilon U_F\} \qquad holds.$$

The sets U_E *and* U_F *are the closed unit balls of* E *and* F*,respectively. Put*

$$L^{(e)}_{s,t} := \{S \in L : (e_n(S)) \in l_{s,t}\} \text{ and } \|S\|_{L^{(e)}_{s,t}} := k_{s,t}\|(e_n(S))\|_{l_{s,t}}, \tag{6}$$

where $k_{s,t}$ *is some norming constant.*

In the following ,we deal with diagonal operators $D \in L(l_p, l_q)$ defined by $D(\xi_i) := (\sigma_i\xi_i)$ and generated by a sequence (σ_i) belongs to Lorentz space $l_{r,t}$.

Theorem 2.2. *[2] Let* $1 \leq p, q \leq \infty$ $\frac{1}{r} > \max(\frac{1}{q} - \frac{1}{p}, 0)$ $, 0 < t \leq \infty$ *and* $\frac{1}{s} = \frac{1}{r} + \frac{1}{p} - \frac{1}{q}$*. Then*

$$D \in L^{(e)}_{s,t}(l_p, l_q) \qquad if and only if \qquad (\sigma_i) \in l_{r,t}. \tag{7}$$

Remark 2.3.

(*i) The statement of the above theorem extend those of Marcus [13] and Oloff [14] to the case* $1 \leq p, q \leq \infty$ *which possesses interesting applications to eigenvalue problems of operators acting on Banach spaces.*

(*ii) In the sequel we extend the statement of the preceding theotem to diagonal operators acting from* $l_{p,u}$ *into* $l_{q,v}, 1 < p, q < \infty, 1 \leq u, v \leq \infty$*. It turns that the characterization of diagonal operators belonging to the entropy ideals* $L^{(e)}_{s,t}(l_{p,u}, l_{q,v})$ *does not depend on the indices u and v of the underlying Lorents sequence spaces.*

Theorem 2.4. *[2] Let* $1 < p, q < \infty$ $, 1 \leq u, v \leq \infty$ $, \frac{1}{r} > \max(\frac{1}{q} - \frac{1}{p}, 0)$ $, 0 < t \leq \infty$ *and* $\frac{1}{s} = \frac{1}{r} + \frac{1}{p} - \frac{1}{q}$*. Then*

$$D \in L^{(e)}_{s,t}(l_{p,u}, l_{q,v}) \qquad if and only if \qquad (\sigma_i) \in l_{r,t}. \tag{8}$$

Definition 2.5. *An operator* $S \in L(E, F)$ *is called an* $(r, t; p, q)$*-factorable operator,*$0 < r < \infty, 0 < t \leq \infty, 1 < p, q < \infty$*, iff there exists a commutative diagram*

$$S : E \xrightarrow{A} l_{p,\infty} \xrightarrow{D} l_{q,1} \xrightarrow{Y} F$$

such that $D \in L(l_{p,\infty}, l_q, 1)$ *is a diagonal operator of the form* $D(\xi_i) = (\sigma_i \xi_i)$ *with* $(\sigma_i) \in l_{r,t}$ $, \frac{1}{r} > \max(\frac{1}{q} - \frac{1}{p}, 0)$*,* $A \in (E, l_{p,\infty})$ *and* $Y \in (l_{q,1}, F)$*.*

The class of those operators is denoted by $\mathbb{Z}_{(r,t;p,q)}$.

Applying the preceding results to eigenvalue problems of factorable operators acting on arbitrary Banach spaces.For $1 < p \leq q < \infty$

B.Carl [2] proved that the sequence of eigenvalues $(\lambda_n(S))$ of operators $S \in \mathbb{Z}_{(r,t;p,q)}(E, E)$ belongs to certain Lorentz sequence spaces $l_{s,t}$. The following results complement those obtained in [3,11,12].For nuclear operators (cf also [4,6]).

Theorem 2.6. *[2] Let* $1 < p \leq q < \infty$ $, 0 < r < \infty$ $, 0 < t \leq \infty$ *and* $\frac{1}{s} = \frac{1}{r} + \frac{1}{p} - \frac{1}{q}$*. Then,for any Banach space E*

$$(\lambda_n(S)) \in l_{s,t} \qquad for any \qquad S \in \mathbb{Z}_{(r,t;p,q)}(E, E) \quad .$$

3. Interpolation spaces.

The following theorem shows that reflexive Orlicz spaces can be characterized as interpolation spaces between L_p-spaces. This is a converse of Gustavsson-Peetre's theorem ([18] theorem (7.3)).

By an Orlicz function we mean a convex and continuous function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$.

Theorem 3.1. *[9] Let Φ be an Orlicz function and let $L_\Phi(E)$ be the corresponding E-valued Orlicz space on a σ-finite measure space, where E is a U.M.D. Banach space. Suppose that Φ and the conjugate Φ^* satisfy the Δ_2-condition. Then there exist p_0 and p_1, with $1 < p_1 < p_0 < \infty$,*

$$\langle L_{p_0}(E), L_{p_1}(E)\rangle_{\rho,p} \;=\; L_\Phi(E) \quad ,\textit{where} \tag{9}$$

$$\rho(t) = \begin{cases} t^{\frac{p_1}{p_1-p_0}} \quad \Phi^{-1}(t^{\frac{p_0p_1}{p_0-p_1}}) & t>0, \\ 0 & t=0 \end{cases}. \tag{10}$$

and $1 < p < \infty$.

Definition 3.2. *Let Φ be an Orlicz function as before. We define for $1 \le q < \infty$, the Lorentz-Orlicz spaces $l_{\Phi q}$ as follows*

$$l_{\Phi q} = \left\{ x = (x_n)_n : \|x\|_{l_{\Phi q}} := \inf\left(c : (\sum_{n=1}^{\infty} \Phi(\frac{|x_n|}{c})^q)^{1/q} \le 1 \right) < \infty \right\}, \tag{11}$$

and,

$$l_{\Phi\infty} = \left\{ x = (x_n)_n \;: \|x\|_{l_{\Phi\infty}} := \inf\left(c : \sup_n \Phi(\frac{|x_n|}{c}) \le 1 \right) < \infty \right\}. \tag{12}$$

Definition 3.3. *Let Φ be an Orlicz function . We define for $1 \le q \le \infty$, the Schatten-Orlicz classes $S^s_{\Phi q}$ as follows*

$$S^s_{\Phi q}(X) \;:=\; \{T \in L(X) \quad : (s_n(T))_n \in l_{\Phi q}\}. \tag{13}$$

Here, $s_n(T)$ be the n-th s-number of T.

Theorem 3.4. *[8] Let $1 < p_0, p_1 < \infty$. Then we have*

$$\langle S^s_{p_0 q}(X), S^s_{p_1 q}(X)\rangle_{\rho,p} \subset S^s_{\Phi q}(X), \tag{14}$$

where ρ is given by (10).

Definition 3.5. *Let Φ be an Orlicz function . We define for $1 \le t \le \infty$, the classes $L^e_{\Phi t}$ as follows*

$$L^e_{\Phi t} \;:=\; \{T \in L \quad : (e_n(T))_n \in l_{\Phi t}\}. \tag{15}$$

Corollary 3.6. *Let $1 < p_0, p_1 < \infty$. Then we have*

$$\langle L^e_{p_0 t}, L^e_{p_1 t}\rangle_{\rho,p} \subset L^e_{\Phi t}, \tag{16}$$

where ρ is given by (10).

Definition 3.7. *Suppose $s \in \mathbb{Z}$ and let (ϕ_n) be a system of functions (see [19] p. 171). Let X be a Banach space and Φ be an Orlicz function satisfy the Δ_2-condition,we define for $1 \leq q \leq \infty$, the Besov-Orlicz spaces*

$$B^s_{\Phi q}(\mathbb{Z}^N; X) = \{f \in S' \ : (\phi_n * f)_n \in l^s_q(L_\Phi(\mathbb{Z}^N; X))\} \tag{17}$$

We equip the spaces $B^s_{\Phi q}(\mathbb{Z}^N; X)$ with the norm

$$\|f\|_{B^s_{\Phi q}} = \|(\phi_n * f)_n\|_{l^s_q(L_\Phi(\mathbb{Z}^N;X))} = \|(\|\phi_n * f\|_{L_\Phi(X)})_n\|_{l^s_q}, \tag{18}$$

Theorem 3.8. *[8] Let Φ be an Orlicz function which satisfies, together with its conjugate Φ^*, the Δ_2-condition. Then there exist p_0 and p_1, with $1 < p_0, p_1 < \infty$, such that*

$$\langle B^s_{p_0 q}(\mathbb{Z}^N; X), B^s_{p_1 q}(\mathbb{Z}^N; X)\rangle_{\rho,p} \ = \ B^s_{\Phi q}(\mathbb{Z}^N; X), \tag{19}$$

where ρ is given by (10) .

4. Some applications of the interpolation theory.

Now, we derive our main results using the well-known results of Fourier coefficients and the new interpolation results (interpolation of S^s_{pq}-spaces, and B^s_{pq}-spaces). We will prove that the Fourier coefficients of a periodic function $f \in B^s_{\Phi q}(2\pi)$ belong to some Lorentz-Orlicz space $l_{\Psi q}$. Finally,we give some statements which are obtained using results recently proved and developed by the author in [8] and the Carl's results

4.1 Fourier coeficients of some periodic functions.

Theorem 4.1. *Let Φ and Ψ be two Orlicz functions satisfying the Δ_2 condition together with their conjugate functions Φ^* and Ψ^* , let $1 \leq w \leq \infty$. Then the convolution operator C_f of any function $f \in L_{\Phi w}(2\pi)$ acts from $L_{\Psi w'}(2\pi)$ into $L_\infty(2\pi)$ with*

$$\Phi^{-1}(\tau) = \tau \Psi^{-1}(\tau^{-1}).$$

Proof.

By using proposition (1.2) for different values of r ,we get

$$f \in L_{r_i w}(2\pi) \qquad \text{implies} \qquad C_f : L_{r'_i w'}(2\pi) \to L_\infty(2\pi) \qquad , i = 0, 1.$$

Therefore,by the interpolation method we have

$$f \in L_{\Phi w}(2\pi) \qquad \text{implies} \qquad C_f : L_{\Psi w'}(2\pi) \to L_\infty(2\pi) \qquad \text{where}$$

$$\begin{aligned} \rho(t) \ &= \ t^{\frac{r_1}{r_1 - r_0}} \Phi^{-1}(t^{\frac{r_0 r_1}{r_0 - r_1}}) \ = \ t^{\frac{r'_1}{r'_1 - r'_0}} \Psi^{-1}(t^{\frac{r'_0 r'_1}{r'_0 - r'_1}}) \quad \text{i.e.} \\ t^{\frac{-\alpha}{r_0}} \Phi^{-1}(t^\alpha) \ &= \ t^{\frac{\alpha}{r'_0}} \Psi^{-1}(t^{-\alpha}) \qquad , \alpha := \frac{r_0 r_1}{r_0 - r_1} = -\frac{r'_0 r'_1}{r'_0 - r'_1} \end{aligned}$$

Hence,taking $t^\alpha = \tau$,we have

$$\Phi^{-1}(\tau) = \tau \Psi^{-1}(\tau^{-1}).$$

Theorem 4.2. *Let Φ and Ψ be two Orlicz functions satisfying the Δ_2 condition together with their conjugate functions Φ^* and Ψ^* . Then the Fourier coefficients of any function $f \in L_\Phi(2\pi)$ belong to the Lorentz-Orlicz space $l_\Psi(\mathbb{Z})$ with*

$$\Phi^{-1}(\tau) = \tau \Psi^{-1}(\tau^{-1}) \qquad i.e.$$

$$f \in L_\Phi(2\pi) \qquad \textit{implies} \qquad (\nu_n(f))_{n\in\mathbb{Z}} \in l_\Psi(\mathbb{Z}). \tag{20}$$

Proof.

By using Hausdorff-Young theorem (1.5) ,we have

$$f \in L_{p_i}(2\pi) \qquad \text{implies} \qquad (\nu_n(f)) \in l_{p_i'}(\mathbb{Z}) \qquad , i = 0, 1.$$

Therefore, by the interpolation theorem (3.1), we have

$$f \in L_\Phi(2\pi) \qquad \text{implies} \qquad (\nu_n(f))_{n\in\mathbb{Z}} \in l_\Psi(\mathbb{Z}). \qquad \text{where}$$

$$\rho(t) \;=\; t^{\frac{p_1}{p_1-p_0}} \Phi^{-1}(t^{\frac{p_0 p_1}{p_0-p_1}}) \;=\; t^{\frac{p_1'}{p_1'-p_0'}} \Psi^{-1}(t^{\frac{p_0' p_1'}{p_0'-p_1'}}) \quad \text{i.e.}$$

$$t^{\frac{-\alpha}{p_0}} \Phi^{-1}(t^\alpha) \;=\; t^{\frac{\alpha}{p_0'}} \Psi^{-1}(t^{-\alpha}) \qquad , \alpha := \frac{p_0 p_1}{p_0 - p_1} = -\frac{p_0' p_1'}{p_0' - p_1'}$$

Hence,taking $t^\alpha = \tau$,we have

$$\Phi^{-1}(\tau) = \tau \Psi^{-1}(\tau^{-1}).$$

The following main results are aconsequence of theorems (1.7),(1.8) and (1.9) and the interpolation theorems (3.4) and (3.8).

Theorem 4.3. *Let Φ and Ψ be two Orlicz functions satisfying the Δ_2 condition together with their conjugate functions Φ^* and Ψ^* , let $1 \leq w \leq \infty$. Then the Fourier coefficients of any function $f \in L_{\Phi w}(2\pi)$ belong to the Lorentz-Orlicz space $l_{\Psi w}(\mathbb{Z})$ with*

$$\Phi^{-1}(\tau) = \tau \Psi^{-1}(\tau^{-1}) \qquad i.e.$$

$$f \in L_{\Phi w}(2\pi) \qquad \textit{implies} \qquad (\nu_n(f))_{n\in\mathbb{Z}} \in l_{\Psi w'}(\mathbb{Z}). \tag{21}$$

Theorem 4.4. *Let Φ and Ψ be two Orlicz functions satisfying the Δ_2 condition together with their conjugate functions Φ^* and Ψ^* , let $1 \leq w \leq \infty$. Then the Fourier coefficients of any function $f \in B^s_{\Phi q}(2\pi)$ belong to the Lorentz-Orlicz space $l_{\Psi q}(\mathbb{Z})$ with*

$$\Phi^{-1}(\tau) = \tau^{s+1} \Psi^{-1}(\tau) \qquad i.e.$$

$$f \in B^s_{\Phi q}(2\pi) \qquad \textit{implies} \qquad (\nu_n(f))_{n\in\mathbb{Z}} \in l_{\Psi q}(\mathbb{Z}). \tag{22}$$

Theorem 4.5. *Let Φ and Ψ be two Orlicz functions satisfying the Δ_2 condition together with their conjugate functions Φ^* and Ψ^* , let $1 \leq w \leq \infty$. Then the Fourier coefficients of any function $f \in W^l_\Phi(2\pi)$ belong to the Lorentz-Orlicz space $l_\Psi(\mathbb{Z})$ with*

$$\Phi^{-1}(\tau) = \tau^l \Psi^{-1}(\tau) \qquad i.e.$$

$$f \in W^l_\Phi(2\pi) \qquad \textit{implies} \qquad (\nu_n(f))_{n\in\mathbb{Z}} \in l_\Psi(\mathbb{Z}). \tag{23}$$

4.2 Eigenvalue distribution of factorable operators .

The following statements are obtained using results recently proved and developed by the author in [8] and the Carl's results in the second section.

Theorem 4.6. *Let Φ and Ψ be Orlicz functions satisfying the Δ_2-condition together with their conjugate functions ,$1 \leq p, q \leq \infty$,$0 < t \leq \infty$. Then*

$$D \in L^{(e)}_{\Phi,t}(l_p, l_q) \qquad \textit{if and only if} \qquad (\sigma_i) \in l_{\Psi,t} \tag{24}$$

with

$$\Phi^{-1}(\tau) \ = \ \tau^{\frac{1}{p}-\frac{1}{q}} \ \ \Psi^{-1}(\tau).$$

Proof.

Using theorem (2.4), we have

$$D \in \mathrm{L}^e_{s_i,t}(l_p, l_q) \qquad \text{iff} \qquad (\sigma_i) \in l_{r_i,t} \quad \text{with} \ \ \frac{1}{s_i} := \frac{1}{r_i} + \frac{1}{p} - \frac{1}{q}. \tag{25}$$

By corollary (3.6), we have

$$\langle L^e_{s_0,t}(l_p, l_q), L^e_{s_1,t}(l_p, l_q)\rangle_{\rho,q} = L^e_{\Phi,t}(l_p, l_q), \tag{26}$$

with,

$$\rho(\tau) = \tau^{\frac{s_1}{s_1-s_0}} \quad \Phi^{-1}(\tau^{\frac{s_0 s_1}{s_0-s_1}}). \tag{27}$$

Using theorem (3.1),we have,

$$\langle l_{r_0 t}, l_{r_1 t}\rangle_{\rho,q} = l_{\Psi t}, \tag{28}$$

with,

$$\rho(\tau) = \tau^{\frac{r_1}{r_1-r_0}} \quad \Psi^{-1}(\tau^{\frac{r_0 r_1}{r_0-r_1}}). \tag{29}$$

Let $\quad \beta = \frac{s_0 s_1}{s_0-s_1} \Rightarrow \quad \frac{1}{\beta} = \frac{1}{s_1} - \frac{1}{s_0} = \frac{1}{r_1} - \frac{1}{r_0} \Rightarrow \quad \beta = \frac{r_0 r_1}{r_0-r_1}$, Consequently, we have

$$\rho(\tau) = \tau^{\frac{-\beta}{s_0}} \ \Phi^{-1}(\tau^\beta) = \tau^{\frac{-\beta}{r_0}} \ \Psi^{-1}(\tau^\beta). \tag{30}$$

Take $\tau^\beta = \xi$, we find

$$\Phi^{-1}(\xi) = \xi^{\frac{1}{s_0}-\frac{1}{r_0}} \quad \Psi^{-1}(\xi) = \xi^{\frac{1}{p}-\frac{1}{q}} \quad \Psi^{-1}(\xi). \tag{31}$$

Then

$$D \in L^{(e)}_{\Phi,t}(l_p, l_q) \qquad \text{if and only if} \qquad (\sigma_i) \in l_{\Psi,t}$$

with

$$\Phi^{-1}(\tau) \ = \ \tau^{\frac{1}{p}-\frac{1}{q}} \ \ \Psi^{-1}(\tau).$$

Theorem 4.7. *Let Φ and Ψ be Orlicz functions satisfying the Δ_2-condition together with their conjugate functions ,$1 \le p,q \le \infty$,$0 < t \le \infty$, $1 \le u,v \le \infty$. Then*

$$D \in L^{(e)}_{\Phi,t}(l_{p,u}, l_{q,v}) \quad \textit{if and only if} \quad (\sigma_i) \in l_{\Psi,t} \tag{32}$$

with

$$\Phi^{-1}(\tau) \;=\; \tau^{\frac{1}{p}-\frac{1}{q}} \;\; \Psi^{-1}(\tau).$$

Proof.

Replacing $L^e_{s,t}(l_p, l_q)$ by $L^e_{s,t}(l_{p,u}, l_{q,v})$ in the preceding theorem and using the same method, we get directly the desired results.

Theorem 4.8. *Let Φ and Ψ be Orlicz functions satisfying the Δ_2-condition together with their conjugate functions ,$1 \le p,q \le \infty$,$0 < t \le \infty$. Then ,for any Banach space E*

$$(\lambda_n(S)) \in l_{\Phi,t} \quad \textit{for any} \quad S \in \mathbb{Z}_{(\Psi,t;p,q)}(E,E)$$

with

$$\Phi^{-1}(\tau) \;=\; \tau^{\frac{1}{p}-\frac{1}{q}} \;\; \Psi^{-1}(\tau).$$

Proof.

Using theorem (2.6),(4.7) and the interpolation method ,we get directly the desired results.

The following results are obtained using the same techniques recently developed above.

Theorem 4.9. *Let Φ and Ψ be Orlicz functions satisfying the Δ_2-condition together with their conjugate functions ,$1 \le q \le \infty$,$0 < t \le \infty$. Then*

$$D \in L^{(e)}_{s,t}(l_{\Phi}, l_q) \quad \textit{if and only if} \quad (\sigma_i) \in l_{\Psi,t} \tag{33}$$

with

$$\Phi^{-1}(\tau) \;=\; \tau^{\frac{1}{s}+\frac{1}{q}} \;\; \Psi^{-1}(\tau^{-1}).$$

Theorem 4.10. *Let Φ and Ψ be Orlicz functions satisfying the Δ_2-condition together with their conjugate functions ,$1 \le q \le \infty$,$0 < t \le \infty$,$1 \le u,v \le \infty$. Then*

$$D \in L^{(e)}_{s,t}(l_{\Phi,u}, l_{q,v}) \quad \textit{if and only if} \quad (\sigma_i) \in l_{\Psi,t}. \tag{34}$$

with

$$\Phi^{-1}(\tau) \;=\; \tau^{\frac{1}{s}+\frac{1}{q}} \;\; \Psi^{-1}(\tau^{-1}).$$

Theorem 4.11. *Let Φ and Ψ be Orlicz functions satisfying the Δ_2-condition together with their conjugate functions ,$1 \le q \le \infty$,$0 < t \le \infty$. Then ,for any Banach space E*

$$(\lambda_n(S)) \in l_{s,t} \quad \textit{for any} \quad S \in \mathbb{Z}_{(\Psi,t;\Phi,q)}(E,E).$$

with

$$\Phi^{-1}(\tau) \;=\; \tau^{\frac{1}{s}+\frac{1}{q}} \;\; \Psi^{-1}(\tau^{-1}).$$

Example 4.12. *Take $\Phi(\tau) = \tau^s$ in theorem (4.6),then we have*

$$\Phi^{-1}(\tau) \ = \ \tau^s \ = \ \tau^{\frac{1}{p}-\frac{1}{q}} \ \ \Psi^{-1}(\tau).$$

Therefore,

$$\Psi^{-1}(\tau) \ = \ \tau^{\frac{1}{s}-\frac{1}{p}+\frac{1}{q}} \ = \ \tau^{\frac{1}{r}} \qquad : \frac{1}{s} = \frac{1}{r} + \frac{1}{p} - \frac{1}{q}.$$

Hence,

$$D \in L^{(e)}_{s,t}(l_p, l_q) \qquad \textit{if and only if} \qquad (\sigma_i) \in l_{r,t} \qquad : \frac{1}{s} = \frac{1}{r} + \frac{1}{p} - \frac{1}{q}.$$

This means that the Carl's theorem (2.2) is a special case of theorem (4.6)

Example 4.13. *Taking $\Phi(\tau) = \tau^p(\log \tau)$ in theorem (4.9) ,we have*

$$D \in L^{(e)}_{s,t}(l_p(\log l), l_q) \qquad \textit{if and only if} \qquad (\sigma_i) \in l_{r,t}(\log 1/l) \quad \textit{where} \qquad \frac{1}{s} = \frac{1}{r}+\frac{1}{p}-\frac{1}{q}.$$

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