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Boundary feedback stabilization of a hybrid PDE-ODE system

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Abstract

This paper deals with boundary feedback stabilization of a variant of the SCOLE model. Namely, a flexible beam clamped at one end and free at the other end where a rigid body is attached. Using the frequency domain method, we first prove that the considered system is uniformly stabilizable when control force and control moment are simultaneously applied at the free end of the beam. Secondly when only a control force is applied, we give a sufficient condition on the physical parameters of the system to guarantee the uniform stabilization. Finally it is shown that the presence of boundary control force is necessary to achieve uniform stabilization. To verify these theoretical developments, an asymptotic analysis and numerical study of the spectrum are conducted.

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1 Introduction

The purpose of this paper is to study the boundary feedback stabilization problem of the system presented in the figure 1. It consists of an elastic beam (P) clamped at one end and linked, at the other end, to a rigid body (A). Such systems arise in the study of large scale flexible space structures and well-known as SCOLE model [11]. The system is governed by one partial

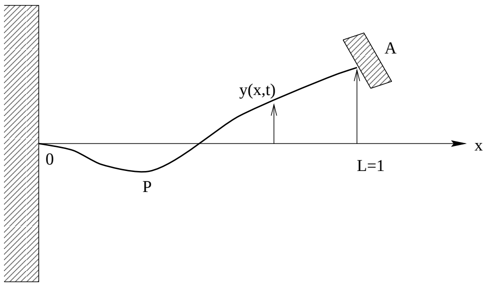

Figure 1: SCOLE model

differential equation (Euler-Bernoulli beam equation for the vibrations of the beam) coupled with two ordinary differential equations (Newton-Euler equations for the oscillations of the rigid body) and often called in literature hybrid system:

$$\begin{cases} \rho y_{tt} + EIy_{xxxx} = 0, \\ y(0,t) = y_x(0,t) = 0, \\ Jy_{xtt}(1,t) + EIy_{xx}(1,t) = \Theta_1(t), \\ my_{tt}(1,t) - EIy_{xxx}(1,t) = \Theta_2(t), \end{cases} \tag{1.1}$$

where the positive constants EI, ρ are respectively the flexual rigidity and the mass per unit length of the beam; J is the moment of inertia and m is the mass of the rigid body (A) attached at the free end of the beam. $\Theta_1(t)$ and $\Theta_2(t)$ denote respectively the linear moment control and force control acting on the free end of the beam.

The objective of the paper is to study the stabilization problem of (1.1) in the case where the **dynamical term $\boldsymbol{my_{tt}(1,t)}$ is neglected**. More precisely, the goal is to find simple feedback laws $\Theta_1(t)$ and $\Theta_2(t)$ such that the solutions of the following system

$$\begin{cases} \rho y_{tt} + EIy_{xxxx} = 0, \\ y(0,t) = y_x(0,t) = 0, \\ Jy_{xtt}(1,t) + EIy_{xx}(1,t) = \Theta_1(t), \\ -EIy_{xxx}(1,t) = \Theta_2(t), \end{cases} \tag{1.2}$$

decay uniformly to zero in some functional space.

The stabilization problem of model (1.1) has been the subject of many studies (see [11], [17], [14], [15], [5] and [10]). Indeed, it is known that the feedback law $\Theta_1(t) = -y_{xt}(1,t)$ and $\Theta_2(t) = -y_t(1,t)$ is sufficient for obtaining the strong stability of (1.1) but not sufficient for obtaining the uniform one [11]. In the case where $m = J = 1$, Slemrod [17] proved the strong stabilization with a priori bounded moment control. This result has been improved by Rao [15] in the sense that the strong stability can be obtained for any length of beam $L < 3$. Furthermore, the strong stabilization has been obtained by a priori bounded feedback force control [15]. Using a method based on compact perturbations due to Russell [16], Rao showed the lack of uniform stabilization of (1.1) with usual feedbacks $\Theta_i(t) = -\delta_{i1}y(1,t) - \delta_{i2}y_x(1,t) - \nu_{i1}y_t(1,t) - \nu_{i2}y_{xt}(1,t)$, whatever real numbers δ_{ij} and ν_{ij} are for $i,j = 1,2$ [14]. This generalizes the result of Littman-Markus [11] insofar as the feedback laws proposed in [11] is obtained by taking $\delta_{i1} = \delta_{i2} = -\nu_{ii} = 0$, for $i,j = 1,2$ and $\nu_{12} = \nu_{21} = 1$. Nevertheless, by means of higher order feedback such as $y_{xxt}(1,t)$ and $y_{xxxt}(1,t)$, uniform stabilization was obtained in [14]. By neglecting the moment of inertia J, it was shown in [5] that the system (1.1) could be exponentially stabilized by $\Theta_1(t) = 0$ and $\Theta_2(t) = -\alpha y_t(1,t) + \beta y_{xxxt}(1,t)$ (α and β are positive constants). Recently, the same result has been obtained in [10] but by using only a simple boundary control moment.

Our contribution in this article is to show that the system (1.2) is exponentially stabilizable provided that both of the two boundary controls (force and moment) are present at the free end of the beam. The proof is based on Huang's theorem (Theorem 3, p.51, [8]) which corresponds to the frequency domain method. We have tried to use the multiplier technique without much success. Moreover, if only the control force is present, we shall show uniform stability of the closed-loop system but under a condition on the physical parameters EI, J and ρ. The proof is based on compact perturbation result (see [6] and [18]). However, it is shown that the considered system can not be uniformly stabilized by only the control moment. Finally, we give asymptotic values of eigenvalues of the controlled operator. The spectral analysis permits us to prove that the condition on the physical parameters EI, J and ρ is necessary. This is illustrated by some numerical results.

Now let us briefly outline the content of this paper. In Section 2, we propose a feedback law; then we formulate the closed loop system as a standard form of evolution equation and prove some preliminary results like well-posedness and strong stability. Section 3, containing the essential part of the paper, is devoted to establishing the uniform stability of the system. Lack of uniform stabilization is also proved when the feedback laws are composed of only control moment. By using an asymptotic analysis, the real part of the spectrum is estimated in section 4 and then we prove that the condition on the physical parameters of the system is impossible to circumvent for the stability. We also give some numerical results of the spectrum.

2 Notations and preliminary results

First of all it is easy to see that, with a suitable change of physical parameters, the system (1.2) can be written as follows:

$$\begin{cases} \rho y_{tt} + EI y_{xxxx} = 0, \\ y(0,t) = y_x(0,t) = 0, \\ J y_{xtt}(1,t) + y_{xx}(1,t) = \Theta_1(t), \\ -y_{xxx}(1,t) = \Theta_2(t). \end{cases} \tag{2.1}$$

We propose the following linear feedback laws

$$\Theta_1(t) = -\alpha y_{xt}(1,t), \quad \Theta_2(t) = -\beta y_t(1,t), \tag{2.2}$$

where α and β are positive constants. For the state space of the system, we take the space

$$\mathcal{H} = H_0^2(0,1) \times L^2(0,1) \times \mathbb{R},$$

equipped with the inner product

$$\langle (f,g,\xi), (\tilde{f},\tilde{g},\tilde{\xi}) \rangle_{\mathcal{H}} = \int_0^1 \Big(\rho g(x)\tilde{g}(x) + EI f_{xx}\tilde{f}_{xx} \Big)\, dx + JEI\xi\tilde{\xi}, \tag{2.3}$$

where

$$H_0^n(0,1) = \Big\{ f \in H^n(0,1);, f(0) = f_x(0) = 0 \Big\}, \quad \text{for } n = 2,3,...$$

Notice that the norm induced by the scalar product (2.3) is equivalent to the usual one of the space $H^2(0,1) \times L^2(0,1) \times \mathbb{R}$ and thus the space $\mathcal{H}$ is a Hilbert space. Then, the closed loop system (2.1)-(2.2) can be written on $\mathcal{H}$ in a standard form of evolution equation

$$\dot{u}(t) = Au(t), \tag{2.4}$$

where $u(t) = \Big(y(\cdot,t), y_t(\cdot,t), y_{xt}(1,t)\Big)$ and A is an operator defined by

$$\mathcal{D}(A) = \Big\{(f,g,\xi) \in H_0^4(0,1)\times H_0^2(0,1)\times\mathbb{R}; -f_{xxx}(1)+\beta g(1) = 0, \xi = g_x(1)\Big\} \tag{2.5}$$

and

$$A(f,g,\xi) = \left(g, -\frac{EI}{\rho} f_{xxxx}, -\frac{1}{J}\Big[f_{xx}(1) + \alpha\xi\Big]\right). \tag{2.6}$$

The following lemma can be proved by using the classic semigroups theory (see [2] and [13]).

Lemma 1 . *i) The operator A defined by (2.5) and (2.6) generates a continuous semigroup of contractions $S(t)$ on $\mathcal{H}$.*
ii) For any initial data $u_0 = (y_0, z_0, \xi_0) \in \mathcal{D}(A)$, the equation (2.4) admits a unique strong solution $u(t) = (y(t), z(t), \xi(t)) = S(t)(y_0, z_0, \xi_0) \in \mathcal{D}(A)$ for all $t \geq 0$ such that $u(\cdot) \in C^1(\mathbb{R}_+;\mathcal{H}) \cap C(\mathbb{R}_+;\mathcal{D}(A))$. Moreover, the function $t \longmapsto \|Au(t)\|_{\mathcal{D}(A)}$ is decreasing.
iii) For any initial data $u_0 = (y_0, z_0, \xi_0) \in \mathcal{H}$, there exists a unique weak solution $u(t) = (y(t), z(t), \xi(t)) = S(t)(y_0, z_0, \xi_0) \in \mathcal{H}$ of (2.4) such that $u(\cdot) \in C(\mathbb{R}_+;\mathcal{H})$.

It is not difficult to show the existence of a constant $C > 0$ such that

$$\|y\|^2_{H^4(0,1)} + \|z\|^2_{H^2(0,1)} + |\xi|^2 \leq C\|(y,z,\xi)\|^2_{\mathcal{D}(A)}, \qquad \forall (y,z,\xi) \in \mathcal{D}(A).$$

Then by Sobolev embedding [1], one can establish the following elementary lemma.

Lemma 2 . *The canonical embedding $i : \mathcal{D}(A) \to \mathcal{H}$ is compact, where $\mathcal{D}(A)$ is equipped with the graph norm.*

Now, using LaSalle's invariance principle [8], we will show that the semigroup $S(t)$ is strongly stable in $\mathcal{H}$.

Lemma 3 . *For any initial data* $u_0 = (y_0, z_0, \xi_0) \in \mathcal{H}$, *the corresponding solution*
$u(t) = S(t)(y_0, z_0, \xi_0)$ *of (2.4) tends strongly to zero in* $\mathcal{H}$ *as* $t \longrightarrow +\infty$.

Proof of Lemma 3. Notice that by a standard argument of density of $\mathcal{D}(A)$ in $\mathcal{H}$ and the contraction of the semigroup $S(t)$, it suffices to prove Lemma 3 for any initial data $u_0 = (y_0, z_0, \xi_0) \in \mathcal{D}(A)$. Let $u(t) = (y(t), z(t), \xi(t)) = S(t)u_0$ and just take the Lyapunov function candidate

$$V(u) = \frac{1}{2}\left\{\int_0^1 \left(\rho y_t^2 + EIy_{xx}^2\right) dx + JEIy_{xt}^2(1,t)\right\}. \tag{2.7}$$

Differentiating V following the trajectories of (2.4) leads us to

$$V(u(t)) - V(u_0) = -EI\int_0^t \left(\alpha y_{x\tau}^2(1,\tau) + \beta y_\tau^2(1,\tau)\right) d\tau. \tag{2.8}$$

By lemma 1 (i), we know that the positive orbit of $S(t)$ is a graph norm bounded and thus precompact by virtue of of Lemma 2. Moreover, the operator A being the generator of semigroup of contractions $S(t)$ on $\mathcal{H}$, one can prove that the ω-limit set $\omega(u_0)$ is also in $\mathcal{D}(A)$. Applying LaSalle's principle, we deduce that $\omega(u_0)$ is non empty, compact, invariant under the semigroup $S(t)$ and in addition $S(t)u_0 \longrightarrow \omega(u_0)$ as $t \to +\infty$ [7].

In order to prove the strong stability, it is sufficient to show that $\omega(u_0)$ reduces to zero. To this end, let $\tilde{u}_0 = (\tilde{y}_0, \tilde{z}_0, \tilde{\xi}_0) \in \omega(u_0)$ and $\tilde{u}(t) = (\tilde{y}(\cdot,t), \tilde{y}_t(\cdot,t), \tilde{y}_{xt}(1,t)) = S(t)\tilde{u}_0 \in \omega(\tilde{u}_0)$ the unique strong solution of (2.4). It is well-known that $V(\tilde{u}(t))$ is constant and thus $\frac{dV}{dt}(\tilde{u}(t)) = \tilde{y}_{xt}(1,t) = \tilde{y}_t(1,t) = 0$ [8]. Hence $\tilde{y}$ is a solution of the following system

$$\begin{cases} \rho\tilde{y}_{tt} + EI\tilde{y}_{xxxx} = 0, \\ \tilde{y}(0,t) = \tilde{y}_x(0,t) = 0, \\ \tilde{y}_{xt}(1,t) = \tilde{y}_{xx}(1,t) = 0, \\ \tilde{y}_{xxx}(1,t) = \tilde{y}_t(1,t) = 0, \\ (\tilde{y}(\cdot,0), \tilde{y}_t(\cdot,0)) \in H_0^4 \times H_0^2. \end{cases}$$

It has been proved in [4] that $\tilde{y} = 0$ is the unique solution of this system and thus $\tilde{u}(t) = S(t)\tilde{u}_0 = 0$. This completes the proof of Lemma 3. $\square$

Remark 1 . *One can check that A^{-1} exists and is a compact operator on $\mathcal{H}$. Consequently, the spectrum $\sigma(A)$ of A consists of only isolated eigenvalues with finite multiplicity [12]. This, together with Lemma 3, implies that the resolvent set $\rho(A)$ of the operator A contains the set $\{i\omega;\ \omega \in \mathbb{R}\}$. Indeed, suppose the contrary to be true. Necessarily, there exists $\omega_0 \in \mathbb{R}$ such that $i\omega_0 \notin \rho(A)$ which is equivalent to saying that $i\omega_0$ is an element of the point spectrum $\sigma(A)$. This means that there exists a nontrivial element $u_0 \in \mathcal{D}(A)$ such that $(i\omega_0 I - A)u_0 = 0$. Setting $u(t) = e^{i\omega_0 t}u_0$, one can verify that $u(t)$ is the solution of (2.4) but $\|u(t)\| \to \|u_0\| \neq 0$ as $t \to +\infty$, which is inconsistent with the strong stability. We thus conclude that $\rho(A) \supset \{i\omega;\ \omega \in \mathbb{R}\}$.*

3 Main results

3.1 Uniform stabilization

Theorem 1 . *The semigroup $S(t)$ generated by A is exponentially stable in $\mathcal{H}$.*

Proof of Theorem 1. By means of Remark 1 and the contraction of $S(t)$, we only need to prove that (see Theorem 3 p. 51, [8])

$$\sup\left\{\|(i\omega - A)^{-1}\|;\ \omega \in \mathbb{R}\right\} < \infty. \tag{H}$$

For simplicity and without loss of generality, we choose $EI = \rho = 1$. Consider then the resolvent equation, i.e, given $\lambda = i\omega,\ \omega \in \mathbb{R}$ and $(f, g, \gamma) \in \mathcal{H}$, find $(y, z, \xi) \in \mathcal{D}(A)$ such that

$$(i\omega I - A)\,(y, z, \xi) = (f, g, \gamma). \tag{3.1}$$

In order to obtain (H), we shall first prove the existence of a positive constant M, independent of ω, such that

$$\int_0^1 \left(y_{xx}^2 + z^2\right) dx + J\xi^2 \leq M\left(\|f_{xx}\|^2 + \|g\|^2 + J\gamma^2\right), \tag{I}$$

which clearly gives the desired property (H). Furthermore, because the function $\lambda \longmapsto \|(\lambda - A)^{-1}\|$ is continuous for any $\lambda \in \rho(A)$ (see [13]), it follows that it is sufficient to show the estimation (I) for $|\omega|$ large. The proof of estimation (I), inspired by the work of Chen et al. [3], is divided into two steps.

First step- The goal is to obtain the following estimation

$$\|y_{xx}\|^2 \le M_1 \left(\|f_{xx}\|^2 + \|g\|^2 + J\gamma^2\right). \tag{I_1}$$

To accomplish this, let $\lambda = i\omega = i\eta^2$, $\eta \in \mathbb{R}$ (the estimates for $\omega = -\eta^2$ are similar). Thus the equation (3.1) is equivalent to

$$\begin{cases} y_{xxxx} - \eta^4 y = i\eta^2 f + g, \\ y_{xx}(1) + (i\alpha\eta^2 - J\eta^4) y_x(1) = (iJ\eta^2 + \alpha) f_x(1) + \gamma J, \\ i\beta\eta^2 y(1) - y_{xxx}(1) = \beta f(1), \\ y(0) = y_x(0) = 0, \\ z = i\eta^2 y - f, \\ \xi = i\eta^2 y_x(1) - f_x(1). \end{cases} \tag{3.2}$$

As in [3], we consider the following two systems of linear differential equations:

$$\begin{cases} \hat{y}_{xxxx} - \eta^4 \hat{y} = i\eta^2 f + g, \\ \hat{y}(0) = \hat{y}_x(0) = \hat{y}_{xx}(0) = \hat{y}_{xxx}(0) = 0, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \tilde{y}_{xxxx} - \eta^4 \tilde{y} = 0, \\ \tilde{y}_{xx}(1) + \eta^2(i\alpha - J\eta^2)\tilde{y}_x(1) = l_1, \\ i\beta\eta^2 \tilde{y}(1) - \tilde{y}_{xxx}(1) = l_2, \\ \tilde{y}(0) = \tilde{y}_x(0) = 0, \end{cases} \tag{3.4}$$

where

$$\begin{cases} l_1 = -\eta^2(i\alpha - J\eta^2)\hat{y}_x(1) - \hat{y}_{xx}(1) + (iJ\eta^2 + \alpha) f_x(1) + \gamma J, \\ l_2 = -i\beta\eta^2 \hat{y}(1) + \hat{y}_{xxx}(1) + \beta f(1). \end{cases} \tag{3.5}$$

Let us observe that if $\hat{y}(x)$ and $\tilde{y}(x)$ are the solutions of (3.3) and (3.4) respectively, then the element $y(x) = \hat{y}(x) + \tilde{y}(x)$ satisfies (3.2). Furthermore, it is a simple task to verify that the unique solution of (3.3) is

$$\hat{y}(x) = \frac{1}{2}\int_0^x \eta^{-3}\Big[\sinh(\eta(x-\tau)) - \sin(\eta(x-\tau))\Big]\left(i\eta^2 f(\tau) + g(\tau)\right) d\tau, \tag{3.6}$$

In order to get the unique solution of (3.4), we set the general solution of the first equation in (3.4) as

$$\tilde{y}(x) = We^{\eta x} + Xe^{i\eta x} + Ye^{-\eta x} + Ze^{-i\eta x}, \tag{3.7}$$

where, W, X, Y and Z are to be determined to satisfy the four boundary conditions of (3.4). Indeed, these boundary conditions lead us to the following linear system

$$M\begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ l_1 \\ l_2 \end{pmatrix},$$

where $M = (m_{ij})_{1\leq i,j\leq 4}$ is a matrix whose elements are

$\mathbf{m_{11}} = 1,\ \mathbf{m_{12}} = 1,\ \mathbf{m_{13}} = 1,\ \mathbf{m_{14}} = 1,$

$\mathbf{m_{21}} = 1,\ \mathbf{m_{22}} = i,\ \mathbf{m_{23}} = -1,\ \mathbf{m_{24}} = -i,$

$\mathbf{m_{31}} = (\eta^2 + i\alpha\eta^3 - J\eta^5)e^{\eta},\ \mathbf{m_{32}} = (-\eta^2 - \alpha\eta^3 - iJ\eta^5)e^{i\eta},$

$\mathbf{m_{33}} = (\eta^2 - i\alpha\eta^3 + J\eta^5)e^{-\eta},\ \mathbf{m_{34}} = (-\eta^2 + \alpha\eta^3 + iJ\eta^5)e^{-i\eta},\ \mathbf{m_{41}} = (i\beta\eta^2 - \eta^3)e^{\eta}$

$\mathbf{m_{42}} = i(\beta\eta^2 + \eta^3)e^{i\eta},\ \mathbf{m_{43}} = (i\beta\eta^2 + \eta^3)e^{-\eta},\ \mathbf{m_{44}} = i(\beta\eta^2 - \eta^3)e^{-i\eta}.$

Also note that for η large $\det M \neq 0$ (see (3.18)) and thus after a careful computation we get

$$\begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = (i+1)\eta^2(\det M)^{-1}\begin{pmatrix} \times & \times & \mu_{13} & -\mu_{14} \\ \times & \times & -\mu_{23} & -\mu_{24} \\ \times & \times & -\mu_{33} & \mu_{34} \\ \times & \times & -\mu_{43} & -\mu_{44} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ l_1 \\ l_2 \end{pmatrix}, \qquad (3.8)$$

where $\times$ denotes the unnecessary elements for subsequent calculus and

$\mu_{13} = (\beta + \eta)e^{i\eta} + i(\eta - \beta)e^{-i\eta} + [(i-1)\beta + (i+1)\eta]e^{-\eta},$
$\mu_{14} = (i + i\alpha\eta - J\eta^3)e^{i\eta} + (1 - \alpha\eta - iJ\eta^3)e^{-i\eta} + [(i+1)(1 + J\eta^3) + (1-i)\alpha\eta]e^{-\eta},$
$\mu_{23} = (\beta + i\eta)e^{\eta} + (i+1)(\eta - \beta)e^{-i\eta} + (i\beta + \eta)e^{-\eta},$
$\mu_{24} = (i - \alpha\eta - iJ\eta^3)e^{\eta} + [(i-1)(1 - \alpha\eta) + (i+1)J\eta^3]e^{-i\eta} - (1 - i\alpha\eta + J\eta^3)e^{-\eta},$
$\mu_{33} = -i(\beta + \eta)e^{i\eta} + [(i-1)\beta - (i+1)\eta]e^{\eta} + (\beta - \eta)e^{-i\eta},$
$\mu_{34} = (1 + \alpha\eta + iJ\eta^3)e^{i\eta} + [(i+1)(1 - J\eta^3) + (i-1)\alpha\eta]e^{\eta} + (i - i\alpha\eta + J\eta^3)e^{-i\eta},$
$\mu_{43} = (i+1)(\beta + \eta)e^{i\eta} + (\eta - i\beta)e^{\eta} + (i\eta - \beta)e^{-\eta},$
$\mu_{44} = [(1-i)(1 + \alpha\eta) + (i+1)J\eta^3]e^{i\eta} + (1 + i\alpha\eta - J\eta^3)e^{\eta} - (i + \alpha\eta + iJ\eta^3)e^{-\eta}.$

¿From (3.6) and integrating twice by parts, we find

$$\hat{y}_{xx}(x) = \frac{\eta^{-1}e^{\eta x}}{4}\int_0^1 e^{-\eta\tau}\left(if_{\tau\tau}(\tau) + g(\tau)\right)d\tau + \mathcal{O}\left(\eta^{-1}[\|f_{xx}\| + \|g\|]\right). \quad (3.9)$$

Similarly we have the following estimations

$$\begin{cases} \hat{y}(x) = -\dfrac{i}{\eta^2} f(x) + \dfrac{\eta^{-3}}{4} \displaystyle\int_0^1 e^{\eta(x-\tau)} \left(i f_{\tau\tau}(\tau) + g(\tau)\right) d\tau + \mathcal{O}\left(\eta^{-3}[\|f_{xx}\| + \|g\|]\right), \\ \hat{y}_x(x) = -\dfrac{i}{\eta^2} f_x(x) + \dfrac{\eta^{-2}}{4} \displaystyle\int_0^1 e^{\eta(x-\tau)} \left(i f_{\tau\tau}(\tau) + g(\tau)\right) d\tau + \mathcal{O}\left(\eta^{-2}[\|f_{xx}\| + \|g\|]\right), \\ \hat{y}_{xxx}(x) = \dfrac{1}{4} \displaystyle\int_0^1 e^{\eta(x-\tau)} \left(i f_{\tau\tau}(\tau) + g(\tau)\right) d\tau + \mathcal{O}\left(\|f_{xx}\| + \|g\|\right). \end{cases} \tag{3.10}$$

Combining (3.5) and (3.10), we obtain after a straightforward computation

$$\begin{aligned} l_1 &= \gamma J + \frac{e^{\eta}}{4}(J\eta^2 - i\alpha - \eta^{-1}) \int_0^1 e^{-\eta\tau}(i f_{\tau\tau}(\tau) + g(\tau)) d\tau \\ &+ \mathcal{O}\left(\eta^2[\|f_{xx}\| + \|g\|]\right), \qquad (3.11) \\ l_2 &= \frac{e^{\eta}}{4}(1 - i\beta\eta^{-1}) \int_0^1 e^{-\eta\tau}(i f_{\tau\tau}(\tau) + g(\tau)) d\tau + \mathcal{O}\left(\|f_{xx}\| + \|g\|\right) \qquad (3.12) \end{aligned}$$

Now let us define an element W^* by

$$W^* = (i+1)\eta^2 \left[\left(J\eta^2 - i\alpha - \eta^{-1}\right) \mu_{13} - \left(1 - i\beta\eta^{-1}\right) \mu_{14} \right]. \tag{3.13}$$

Substituting μ_{13} and μ_{14} by their expressions in the above identity, we get

$$\begin{aligned} W^* = (i+1)\Big\{ & e^{-i\eta}[2iJ\eta^5 + \beta J((1-i)\eta^4 + 2\alpha\eta^3 - \eta^2(i+1)(1+\alpha\beta) + 2i\beta\eta] \\ & + e^{i\eta}[2J\eta^5 + \beta J((1-i)\eta^4 - 2i\alpha\eta^3 - \eta^2(i+1)(1+\alpha\beta) - 2\beta\eta] \\ & + e^{-\eta}[2(1-i)\beta J\eta^4 + 2(\alpha\beta - 1)(i+1)\eta^2] \Big\}. \end{aligned} \tag{3.14}$$

Clearly, the dominant term of W^* is η^5 and $|W^*|$ is greater than $\mathcal{O}(\eta^4)$. Namely,

$$|W^*| = \mathcal{O}(\eta^5), \ | W^* | \geq \sqrt{2}\beta J\eta^4. \tag{3.15}$$

On the other hand, we get from the definition of the matrix M that its determinant is given by

$$\begin{aligned}\det M = & -2\eta^4 e^{(1-i)\eta}[(i-1)J\eta^4 + \beta J\eta^3 + (i+1)\alpha\eta^2 - i(1+\alpha\beta)\eta + (i-1)\beta] \\ & -2\eta^4 e^{(i-1)\eta}[(1-i)J\eta^4 + \beta J\eta^3 - (i+1)\alpha\eta^2 - i(1+\alpha\beta)\eta + (1-i)\beta] \\ & -2\eta^4 e^{(i+1)\eta}[(1+i)J\eta^4 + \beta J\eta^3 + (-i+1)\alpha\eta^2 - i(1+\alpha\beta)\eta - (i+1)\beta] \\ & -2\eta^4 e^{-(i+1)\eta}[-(1+i)J\eta^4 + \beta J\eta^3 + (i-1)\alpha\eta^2 - i(1+\alpha\beta)\eta + (i+1)\beta] \\ & +8\eta^4[\beta J\eta^3 + i(1-\alpha\beta)\eta].\end{aligned} \tag{3.16}$$

Factoring the first and the third line of (3.16) by $-\eta^3 e^\eta$ and using (3.14), we obtain

$$\begin{aligned}\det M = & -\eta^3 e^\eta \Big(W^* - e^{-\eta}[2(1-i)\beta J\eta^4 + 2(\alpha\beta-1)(i+1)\eta^2]\Big) \\ & -2\eta^4 e^{(i-1)\eta}[(1-i)J\eta^4 + \beta J\eta^3 - (i+1)\alpha\eta^2 - i(1+\alpha\beta)\eta + (1-i)\beta] \\ & -2\eta^4 e^{-(i+1)\eta}[-(1+i)J\eta^4 + \beta J\eta^3 + (i-1)\alpha\eta^2 - i(1+\alpha\beta)\eta + (i+1)\beta] \\ & +8\eta^4[\beta J\eta^3 + i(1-\alpha\beta)\eta].\end{aligned} \tag{3.17}$$

Except the first term $-\eta^3 e^\eta W^*$ of (3.17), all the others terms are bounded by $\mathcal{O}(\eta^7)$. Then, the relation between W^* and $\det M$ is

$$\det M = -\eta^3 e^\eta W^* + \mathcal{O}(\eta^7), \tag{3.18}$$

which implies, thanks to (3.15) and (3.18) that

$$(\det M)^{-1} = -\eta^{-3} e^{-\eta} (A^*)^{-1} + \mathcal{O}(\eta^{-7} e^{-2\eta}) = \mathcal{O}(\eta^{-7} e^{-\eta}). \tag{3.19}$$

Estimation of W:

On one hand, we have from (3.8)

$$W = (i+1)\eta^2 (\det M)^{-1} \left(\mu_{13} l_1 - \mu_{14} l_2\right). \tag{3.20}$$

On the other hand, the estimations (3.11) and (3.12) yield

$$\begin{aligned}\mu_{13} l_1 - \mu_{14} l_2 = & \left(J\eta^2 - i\alpha - \eta^{-1}\right)\mu_{13} - \left(1 - i\beta\eta^{-1}\right)\mu_{14} \times \\ & \times \frac{e^\eta}{4}\int_0^1 e^{-\eta\tau}\left(i f_{\tau\tau}(\tau) + g(\tau)\right) d\tau \\ & + \mu_{13}\Big(\gamma J + \mathcal{O}(\eta^2[\|f_{xx}\| + \|g\|])\Big) - \mu_{14}\mathcal{O}\left(\|f_{xx}\| + \|g\|\right).\end{aligned}$$

Because of (3.13), the above identity can be rewritten as follows

$$\begin{aligned}\mu_{13}l_1 - \mu_{14}l_2 = \frac{W^*}{(i+1)\eta^2}\frac{e^\eta}{4}\int_0^1 e^{-\eta\tau}\left(if_{\tau\tau}(\tau)+g(\tau)\right)d\tau \\ +\mu_{13}\Big(\gamma J + \mathcal{O}(\eta^2[\|f_{xx}\|+\|g\|])\Big) - \mu_{14}\mathcal{O}\left(\|f_{xx}\|+\|g\|\right).\end{aligned} \tag{3.21}$$

Inserting (3.21) into (3.20) and using the first estimation of (3.19), we deduce that

$$\begin{aligned}W = \frac{-1}{4\eta^3}\int_0^1 e^{-\eta\tau}[if_{\tau\tau}(\tau)+g(\tau)]d\tau + \frac{e^\eta}{4}W^*\mathcal{O}(\eta^{-7}e^{-2\eta})\int_0^1 e^{-\eta\tau}[(if_{\tau\tau}(\tau)+g(\tau)]d\tau \\ +(i+1)\eta^2(\det M)^{-1}\Big[\mu_{13}\left(\gamma J + \mathcal{O}(\eta^2[\|f_{xx}\|+\|g\|])\right) - \mu_{14}\mathcal{O}\left(\|f_{xx}\|+\|g\|\right)\Big].\end{aligned} \tag{3.22}$$

It is not difficult to show that μ_{13} and μ_{14} (see (3.8)) are bounded by $\mathcal{O}(\eta)$ and $\mathcal{O}(\eta^3)$ respectively. Moreover, we know from (3.19) that $(\det M)^{-1} = \mathcal{O}(\eta^{-7}e^{-\eta})$ and from (3.15) that $|W^*| = \mathcal{O}(\eta^5)$. By virtue of these arguments, the estimation (3.22) becomes

$$W = -\frac{1}{4\eta^3}\int_0^1 e^{-\eta\tau}(if_{\tau\tau}(\tau)+g(\tau))d\tau + \mathcal{O}\left(\eta^{-2}e^{-\eta}[\|f_{xx}\|+\|g\|]\right) + \gamma J\mathcal{O}(\eta^{-4}e^{-\eta}). \tag{3.23}$$

Estimation of X, Y and Z :

By (3.8), we have $X = -(i+1)\eta^2(\det M)^{-1}\left(\mu_{23}l_1+\mu_{24}l_2\right)$. One can show from (3.11), (3.12) and the expressions of μ_{23}, μ_{24} that the coefficient of the dominant term $e^{2\eta}$, appearing in $\mu_{23}l_1+\mu_{24}l_2$, is zero. Indeed, we have

$$X = (\det M)^{-1}\mathcal{O}\left(\eta^5 e^\eta[\|f_{xx}\|+\|g\|]\right) + \gamma J\mathcal{O}(\eta^3 e^\eta).$$

Using this, it is easy to check from (3.19) that

$$X = \mathcal{O}\left(\eta^{-2}[\|f_{xx}\|+\|g\|]\right) + \gamma J\mathcal{O}(\eta^{-4}). \tag{3.24}$$

Similarly, we can obtain

$$\begin{aligned} Y &= \mathcal{O}\left(\eta^{-2}[\|f_{xx}\|+\|g\|]\right) + \gamma J\mathcal{O}(\eta^{-4}), && (3.25)\\ Z &= \mathcal{O}\left(\eta^{-2}[\|f_{xx}\|+\|g\|]\right) + \gamma J\mathcal{O}(\eta^{-4}). && (3.26)\end{aligned}$$

On the other hand, it follows from (3.7) and (3.9) that

$$
\begin{aligned}
y_{xx}(x) &= \hat{y}_{xx}(x) + \tilde{y}_{xx}(x) \\
&= \frac{1}{4\eta} e^{\eta x} \int_0^1 e^{-\eta\tau}(i f_{\tau\tau}(\tau) + g(\tau)) d\tau + \mathcal{O}\left(\eta^{-1}[\|f_{xx}\| + \|g\|]\right) \\
&+ W\eta^2 e^{\eta x} - X\eta^2 e^{i\eta x} + Y\eta^2 e^{-\eta x} - Z\eta^2 e^{-i\eta x}.
\end{aligned}
$$

This implies together with (3.23)-(3.26) that

$$
\begin{aligned}
\|y_{xx}\| \mid &\leq \left\{\mathcal{O}\left(e^{-\eta}[\|f_{xx}\| + \|g\|]\right) + \gamma J \mathcal{O}(\eta^{-2} e^{-\eta})\right\} \left(\int_0^1 e^{2\eta x} dx\right)^{1/2} \\
&+ \left\{\mathcal{O}\left(\|f_{xx}\| + \|g\|\right) + \gamma J \mathcal{O}(\eta^{-2})\right\} \left(\int_0^1 e^{-2\eta x} dx\right)^{1/2} \\
&+ \mathcal{O}\left(\|f_{xx}\| + \|g\|\right) + \gamma J \mathcal{O}(\eta^{-2}). \qquad (3.27)
\end{aligned}
$$

Since

$$
\left(\int_0^1 e^{2\eta x} dx\right)^{\frac{1}{2}} = \mathcal{O}\left(\eta^{-\frac{1}{2}} e^{\eta}\right) \text{ and } \left(\int_0^1 e^{-2\eta x} dx\right)^{\frac{1}{2}} = \mathcal{O}\left(\eta^{-\frac{1}{2}}\right),
$$

the estimate (3.27) implies

$$
\|y_{xx}\| = \mathcal{O}\left(\|f_{xx}\| + \|g\| + J \mid \gamma \mid\right). \qquad (3.28)
$$

This proves the estimation (I_1).

Final step- Now we shall prove that

$$
\|z\|^2 + J\xi^2 \leq M_2 \left(\|f_{xx}\|^2 + \|g\|^2 + J\gamma^2\right).
$$

Since $z = i\eta^2 y - f$ and $\xi = z_x(1) = i\eta^2 y_x(1) - f_x(1)$, it is clear from (3.28) that this estimation holds if we can find a positive constant M_2 such that

$$
\eta^4 \left(\|y\|^2 + J \mid y_x(1) \mid^2\right) \leq M_2 \left(\|f_{xx}\|^2 + \|g\|^2 + J\gamma^2\right). \qquad (I_2)
$$

As in [3], let us multiply (3.2) by $\bar{y}$ (conjugate of y) and integrate by parts twice. We obtain after using the boundary conditions in (3.2) that

$$
\begin{aligned}
&\eta^4 \left(\|y\|^2 + J \mid y_x(1) \mid^2\right) = \|y_{xx}\|^2 \\
&\quad + \Re e \left\{-\int_0^1 (g\bar{y} + i\eta^2 f\bar{y}) dx - \beta f(1)\bar{y}(1) - (iJ\eta^2 + \alpha) f_x(1)\bar{y}_x(1) - \gamma J \bar{y}_x(1)\right\}.
\end{aligned}
$$

Applying Cauchy-Schwartz inequality to all terms in braces, the above equality gives

$$\frac{\eta^4}{2}\left(\|y\|^2 + J \mid y_x(1) \mid^2\right) \leq \frac{1}{2}\left(3+J+\alpha+\beta\right)\|y_{xx}\|^2 + \frac{1}{2}(1+J+\beta+\alpha)\|f_{xx}\|^2 + \frac{1}{2}\|g\|^2 + \frac{J}{2}\gamma^2.$$

which implies together with (3.28) that

$$\eta^4\left(\|y\|^2 + J \mid y_x(1) \mid^2\right) \leq M_2\left(\|f_{xx}\|^2 + \|g\|^2 + J\gamma^2\right),$$

with a positive constant M_2. Thus the estimation (I_2) is proved. Combining (I_1) and (I_2), we have the estimation (I). This concludes the proof of Theorem 1. □

Let A_0 be the linear operator obtained from (2.6) by taking $\alpha = 0$ in the feedback law (2.2), i.e, there is no control moment. Then it is clear that A_0 generates a continuous semigroup of contractions $S_0(t)$. We then have the following result:

Theorem 2 . *Let $\mu > 0$ be the largest constant such that*

$$\mu\varphi_{xx}^2(1) \leq \int_0^1 \varphi_{xxxxx}^2(x)dx, \qquad (*)$$

for all $\varphi \in \{H^5(0,1); \varphi(0) = \varphi(1) = \varphi_x(0) = \varphi_{xxx}(1) = \varphi_{xxxx}(0) = \varphi_{xxxx}(1) = 0\}$. Assume that $3\mu EIJ^2 > \rho$. Then, the semigroup $S_0(t)$ is exponentially stable in $\mathcal{H}$.

Proof of Theorem 2. First, we write the operator A defined by (2.5)-(2.6) as the sum of a linear unbounded operator A_0 with $\mathcal{D}(A_0) = \mathcal{D}(A)$ and a bounded compact operator B on $\mathcal{H}$. In other words, for all $u = (y, z, \xi) \in \mathcal{D}(A_0) = \mathcal{D}(A)$,

$$A(y, z, \xi) = (A_0 + B)(y, z, \xi),$$

where

$$A_0(y, z, \xi) = \left(z, -\frac{E}{\rho}y_{xxxx}, -\frac{1}{J}y_{xx}(1)\right),$$

and

$$B(y, z, \xi) = \left(0, 0, -\frac{\alpha}{J}\xi\right). \qquad (3.29)$$

Second, since B is a bounded compact operator on $\mathcal{H}$ and since the semigroup $S(t)$ generated by $A = A_0 + B$ is exponentially stable (see Theorem 1), then it is sufficient for us to prove the strong stability of $S_0(t)$ in $\mathcal{H}$ (see [6] and [18]). To this end, we first establish the result for smooth initial data $u_0 = (y_0, z_0, \xi_0) \in \mathcal{D}(A^2)$ and then we can extend the result to $\mathcal{H}$ by denseness. Now let $u(t) = (y(t), z(t), \xi(t))$ be the solution stemmed from $u_0 = (y_0, z_0, \xi_0)$ and consider the Lyapunov function $V(u(t))$ defined in (2.7). Since $\alpha = 0$, we obtain from (2.8) that

$$V(u(t)) - V(u_0) = -EI \int_0^t \beta y_\tau^2(1,\tau)d\tau.$$

Next, repeating the argument used in the proof of Lemma 2, we need to show a uniqueness result, i.e, the only solution of the system below stemmed from $u_0 = (y_0, z_0, \xi_0) \in \mathcal{D}(A^2)$ is the trivial one:

$$\begin{cases} \rho y_{tt} + EIy_{xxxx} = 0, \\ y(0,t) = y_x(0,t) = 0, \\ Jy_{xtt}(1,t) + y_{xx}(1,t) = 0, \\ y_{xxx}(1,t) = y_t(1,t) = 0. \end{cases} \tag{3.30}$$

For this, we will use the multiplier method as developed in [15] (p. 301-303). First multiplying the equation (3.30) by $x^2(x-3)$ and integrating by parts, we obtain:

$$\begin{aligned} 6EI\int_0^T y(1,t)dt &= \rho\int_0^1 (x^3 - 3x^2)\left[y_t(x,T) - y_t(x,0)\right]dx \\ &+ 3EIJ\left[y_{xt}(1,T) - y_{xt}(1,0)\right]. \end{aligned} \tag{3.31}$$

Since $y_t(1,t) = 0$, it follows that $y(1,t)$ is constant. Then applying the Cauchy-Schwartz inequality in (3.31) and using the fact that the semigroup is a contraction, it follows that for all $T > 0$ and $u_0 \in \mathcal{D}(A^2)$, there exists a positive constant C_1 such that $|y(1,t)| \le \frac{C_1}{T}\|u_0\|_{\mathcal{H}}$. Hence $y(1,t) = 0$. Next, multiplying (3.30) by xy_{xxx} and integrating by parts, we get

$$\frac{1}{2}\int_0^T\int_0^1 \left(3\rho y_{xt}^2 + EIy_{xxx}^2\right)dxdt = \frac{\rho}{2}\int_0^T y_{xt}^2(1,t)dt + \rho\int_0^1 \left[xy_ty_{xxx}\right]_0^T dx. \tag{3.32}$$

Since $y_{xxx}(1,t) = 0$, it follows that

$$\begin{aligned}
\int_0^1 xy_t y_{xxx} dx &= \int_0^1 xy_t \int_1^x y_{\xi\xi\xi\xi} d\xi dx \\
&\leq C_2 \int_0^1 \left(y_t^2 + y_{xxxx}^2 \right) dx \\
&\leq C_2 \|u_0\|^2_{\mathcal{D}(A)}. \qquad (3.33)
\end{aligned}$$

Inserting (3.33) into (3.32) yields:

$$\int_0^T \int_0^1 \left(3\rho y_{xt}^2 + EIy_{xxx}^2 \right) dxdt \leq \rho \int_0^T y_{xt}^2(1,t)dt + C_3 \|u_0\|^2_{\mathcal{D}(A)}. \qquad (3.34)$$

Now, replacing y by y_t in (3.34) and using (3.30), we obtain:

$$\begin{aligned}
\int_0^T \int_0^1 \left(3EIy_{xxxxx}^2 + EIy_{xxxt}^2 \right) dxdt &\leq \rho \int_0^T y_{xtt}^2(1,t)dt + C_3 \|u_0\|^2_{\mathcal{D}(A^2)} \\
&\leq \frac{\rho}{J^2} \int_0^T y_{xx}^2(1,t)dt + C_3 \|u_0\|^2_{\mathcal{D}(A^2)} \qquad (3.35)
\end{aligned}$$

Let us observe that $u_0 \in \mathcal{D}(A^2)$ implies that the corresponding solution y of (3.30) satisfies $y(0,t) = y(1,t) = y_x(0,t) = y_{xxx}(1,t) = y_{xxxx}(0,t) = y_{xxxx}(1,t)$. We deduce thus from (3.35) and the hypothesis $(*)$ of Theorem 2 that for $3\mu EIJ^2 > \rho$, we have

$$\int_0^T \int_0^1 y_{xxxt}^2 dt \leq C_3 \|u_0\|^2_{\mathcal{D}(A^2)}. \qquad (3.36)$$

In addition, the conditions $y(0,t) = y_x(0,t) = y(1,t) = 0$ give us:

$$\begin{cases}
y_{xt}(1,t) = \displaystyle\int_0^1 (1-x)xy_{xxxt}dx, \\
y_{xx}(1,t) = \displaystyle\int_0^1 x(2-x)xy_{xxx}dx.
\end{cases}$$

Thus, by means of Cauchy-Schwartz inequality, we have

$$\begin{cases}
y_{xt}^2(1,t) \leq C_4 \displaystyle\int_0^1 y_{xxxt}^2(x,t)dx, \\
y_{xx}^2(1,t) \leq C_5 \displaystyle\int_0^1 y_{xxx}^2(x,t)dx,
\end{cases} \qquad (3.37)$$

which, together with (3.36), implies that

$$\int_0^T y_{xt}^2(1,t)dt \leq C_6\|u_0\|^2_{\mathcal{D}(A^2)}. \tag{3.38}$$

Combining (3.38) and (3.34), we obtain

$$\int_0^T\int_0^1 \left(3\rho y_{xt}^2 + EIy_{xxx}^2\right) dxdt \leq C_7\|u_0\|^2_{\mathcal{D}(A^2)}. \tag{3.39}$$

Because $y(0,t) = 0$, we obtain after using the Cauchy-Schwartz inequality that

$$\int_0^1 y_t^2 dx \leq \int_0^1 y_{xt}^2 dx. \tag{3.40}$$

Moreover, using Cauchy-Schwartz inequality and the second inequality of (3.37), a direct computation shows that

$$\begin{aligned}\int_0^1 y_{xx}^2(x,t)dx &= \int_0^1 \left(\int_1^x y_{\xi\xi\xi}(\xi,t)d\xi + y_{\xi\xi}(1,t)\right)^2 dx \\ &\leq 2|y_{\xi\xi}(1,t)|^2 + 2\int_0^1 y_{xxx}^2(x,t)dx \\ &\leq C_8\int_0^1 y_{xxx}^2(x,t)dx. \end{aligned} \tag{3.41}$$

Finally, putting (3.38), (3.40) and (3.41) into (3.39), we obtain

$$\int_0^T \|u(t)\|^2_{\mathcal{H}}dt = \int_0^T \left(\int_0^1 \left(y_t^2 + y_{xx}^2\right) dx + y_{xt}^2(1,t)\right) dt \leq C_9\|u_0\|^2_{\mathcal{D}(A^2)}.$$

This means, since $\|u(t)\|^2_{\mathcal{H}}$ is constant, that

$$\|u(t)\|^2_{\mathcal{H}} \leq \frac{C_9\|u_0\|^2_{\mathcal{D}(A^2)}}{T}, \text{ for all}\, T > 0,$$

which gives that $y = 0$. We have proved that if $3\mu EIJ^2 > \rho$, then for any $u_0 = (u(\cdot,0), u_t(\cdot,0), u_{xt}(1,0)) \in \mathcal{D}(A_0^2)$, the corresponding solution of (3.30) is trivial. This completes the proof of Theorem 2. $\square$

Remark 2 . *It is easy to verify that the five boundary conditions of Theorem 2 yield*

$$y_{xx}(1,t) = \frac{1}{3}\int_0^1 \left(-\frac{1}{4}x^4 + x^3\right) y_{xxxxx}(x,t)dx.$$

Then, the Cauchy-Schwartz inequality implies that

$$\begin{aligned} y_{xx}^2(1,t) &\leq \frac{1}{9}\int_0^1 x^6\left(-\frac{x}{4}+1\right)^2 dx \int_0^1 y_{xxxxx}^2(x,t)dx \\ &\leq \frac{11}{1134}\int_0^1 y_{xxxxx}^2(x,t)dx. \end{aligned}$$

Hence we can claim that $\mu \geq \dfrac{1134}{11}$ *in our Theorem 2.*

3.2 Lack of uniform stabilization

In order to achieve our investigation concerning stabilization problem of system (2.1), it remains to see whether the system is uniformly stabilizable by control moment. For this, assume that only the moment control is present in the feedback law (2.2), i.e, $\beta = 0$ and $\alpha > 0$. Hence the closed loop system is:

$$\begin{cases} \rho y_{tt} + EIy_{xxxx} = 0, \\ y(0,t) = y_x(0,t) = 0, \\ Jy_{xtt}(1,t) + y_{xx}(1,t) = -\alpha y_{xt}(1,t), \\ y_{xxx}(1,t) = 0, \\ y(\cdot,0) = y_0,\ y_t(\cdot,0) = z_0,\ y_t(1,0) = \xi_0, \end{cases} \tag{3.42}$$

which can be written as follows

$$\begin{cases} \dot{u}(t) = (A_1 + B)u(t), \\ u(0) = u_0, \end{cases} \tag{3.43}$$

where $u_0 = (y_0, z_0, \xi_0)$; B is given by (3.29) and A_1 is defined by:

$$\mathcal{D}(A_1) = \{(y,z,\xi) \in H_0^4 \times H_0^2 \times \mathbb{R};\ y_{xxx}(1) = 0, \xi = z_x(1)\}, \tag{3.44}$$

and for all $(y,z,\xi) \in \mathcal{D}(A_1)$,

$$A_1(y,z,\xi) = \left(z, -\frac{EI}{\rho}y_{xxxx}, -\frac{1}{J}y_{xx}(1)\right). \tag{3.45}$$

We first state a lemma which concerns the properties of the operator A_1.

Lemma 4 . *The operator A_1 defined by (3.44)-(3.45) generates an unitary group $S_1(t)$ of isometries.*

Proof of Lemma 4. Obviously, A_1 is maximal and $< A_1 u, u >= 0, \ \forall u \in \mathcal{D}(A_1)$. This implies that $< A_1(u+v), (u+v) >= 0, \ \forall u, \ v \in \mathcal{D}(A_1)$, and $< A_1 u, v >= - < u, A_1 v >, \ \forall u, \ v \in \mathcal{D}(A_1)$. Thus the operator A_1 is anti-symmetric. On the other hand, one can prove that $\mathcal{D}(A_1) = \mathcal{D}(A_1^*)$ and $A_1^* = -A_1$. Hence, A_1 is skew-adjoint. Finally, it follows that A_1 generates an unitary group $S_1(t)$ of isometries [13]. □

Now, we are ready to state a result which highlights the leading role of control force.

Proposition 1 . *The closed-loop system (3.42) is not uniformly stable.*

Proof of Proposition 1. As in [14], the proof is based on the result of compact perturbations [16]. Let $S_{A_1+B}(t)$ and $S_{A_1-B}(t)$ denote the groups generated by $A_1 + B$ and $A_1 - B$ respectively. Let $u(t) = (y, z, \xi)(t)$ be the solution of (3.43) stemmed from $u_0 \in \mathcal{D}(A_1)$, i.e, $u(t) = S_{A_1+B}(t)u_0$. Define $\tilde{u}(t) = (\tilde{y}(t), \tilde{z}(t), \tilde{\xi}(t))$ where

$$\tilde{y}(x,t) = y(x,-t), \ \tilde{z}(x,t) = -z(x,-t), \ \tilde{\xi}(t) = -\xi(t).$$

It can be verified that $\tilde{u}(t) = S_{A_1-B}(t)\tilde{u}(0)$. Since B is compact, Russell's result [16] permits us to claim that there is no positive constants $0 \leq \mu < 1$ and $T > 0$ such that we have simultaneously

$$||S_{A_1+B}(t)|| \leq \mu \quad \text{and} \quad ||S_{A_1-B}(-t)|| \leq \mu, \ \forall t \geq T. \tag{3.46}$$

Since $||\tilde{u}(-t)|| = ||\tilde{u}(t)||$, for all $t \in \mathbb{R}$, we get $||S_{A_1+B}(t)|| = ||S_{A_1-B}(-t)||$. This, together with (3.46), completes the proof of Proposition 1. □

4 Asymptotic analysis of the real part of the spectrum

Let us consider the closed loop system (2.4). Using the results of Langer [9], we can estimate the real part of the spectrum with large module. The following theorem establishes explicitly a relation between the feedback gain β and the real part of the eigenvalues λ_k.

Theorem 3 . *Let λ_k be an eigenvalue of system (2.4). Then*

$$\Re e\lambda_k \longrightarrow -\beta\frac{EI}{\rho}, \qquad as\ k \longrightarrow \infty.$$

Proof of Theorem 3. First one can readily show that $\lambda \in \mathbb{C}\setminus\{0\}$ is an eigenvalue of the operator A if and only if there exists a non trivial element $y \in H^4(0,1)$ such that

$$\begin{cases} \dfrac{EI}{\rho}y_{xxxx} + \lambda^2 y = 0, \\ y(0) = y_x(0) = 0, \\ y_{xx}(1) + (\alpha\lambda + J\lambda^2)y_x(1) = 0, \\ \beta\lambda y(1) - y_{xxx}(1) = 0. \end{cases} \tag{4.1}$$

Substituting $\lambda = i\omega^2\sqrt{\dfrac{EI}{\rho}}$, $(\omega \in \mathbb{C}\setminus\{0\})$, $\tilde{\beta} = \beta\sqrt{\dfrac{EI}{\rho}}$, $\tilde{\alpha} = \alpha\sqrt{\dfrac{EI}{\rho}}$ and $\tilde{J} = J\dfrac{EI}{\rho}$ into (4.1), we obtain

$$\begin{cases} y_{xxxx} - \omega^4 y = 0, \\ y(0) = y_x(0) = 0, \\ y_{xx}(1) + (i\tilde{\alpha}\omega^2 - \tilde{J}\omega^4)y_x(1) = 0, \\ i\tilde{\beta}\omega^2 y(1) - y_{xxx}(1) = 0. \end{cases}$$

In the same way as for (3.2), one can verify that the system below has a non trivial solution if and only if ω satisfies

$$\begin{aligned} P(\omega) = {} & e^{(1-i)\omega}[(i-1)\tilde{J}\omega^4 + \tilde{\beta}\tilde{J}\omega^3 + (i+1)\tilde{\alpha}\omega^2 - i(1+\tilde{\alpha}\tilde{\beta})\omega + (i-1)\tilde{\beta}] \\ & +e^{(i-1)\omega}[(1-i)\tilde{J}\omega^4 + \tilde{\beta}\tilde{J}\omega^3 - (i+1)\tilde{\alpha}\omega^2 - i(1+\tilde{\alpha}\tilde{\beta})\omega + (1-i)\tilde{\beta}] \\ & +e^{(i+1)\omega}[(1+i)\tilde{J}\omega^4 + \tilde{\beta}\tilde{J}\omega^3 + (-i+1)\tilde{\alpha}\omega^2 - i(1+\tilde{\alpha}\tilde{\beta})\omega - (i+1)\tilde{\beta}] \\ & +e^{-(i+1)\omega}[-(1+i)\tilde{J}\omega^4 + \tilde{\beta}\tilde{J}\omega^3 + (i-1)\tilde{\alpha}\omega^2 - i(1+\tilde{\alpha}\tilde{\beta})\omega + (i+1)\tilde{\beta}] \\ & -4(\tilde{\beta}\tilde{J}\omega^3 + i(1-\tilde{\alpha}\tilde{\beta})\omega) = 0. \end{aligned} \tag{4.2}$$

A straightforward computation shows that $P(-\omega) = -P(\omega)$ and $iP(\omega) = \overline{P(i\bar{\omega})}$. Thus the solution set of (4.2) admits the two bisectors of the complex plan as symmetrical axes and the origin as a symmetrical center. It is sufficient then to seek the solutions of (4.2) in a sector S_1 defined by $\arg(\omega) = \theta$ with $\epsilon < \theta < \pi/2 + \epsilon$ for small ϵ. Next, following the same step as in [9] , we claim that in the sector S_1, the solutions of (4.2) are asymptotically represented by those of

$$\hat{P}(\omega) = e^{\omega} A_1(\omega) + e^{-\omega} A_2(\omega) = 0, \tag{4.3}$$

where

$$\begin{aligned} A_1(\omega) &= (i-1)\tilde{J}\omega^4 + \tilde{\beta}\tilde{J}\omega^3 + (i+1)\tilde{\alpha}\omega^2 - i(1+\tilde{\alpha}\tilde{\beta})\omega + (i-1)\tilde{\beta}, \\ A_2(\omega) &= -(1+i)\tilde{J}\omega^4 + \tilde{\beta}\tilde{J}\omega^3 + (i-1)\tilde{\alpha}\omega^2 - i(1+\tilde{\alpha}\tilde{\beta})\omega + (i+1)\tilde{\beta}. \end{aligned}$$

Because of $\tilde{J} = J\dfrac{EI}{\rho} > 0$, the solutions of (4.3) are themselves asymptotic to the solutions of

$$(i-1)e^{\omega} - (i+1)e^{-\omega} = 0. \tag{4.4}$$

Obviously, the solutions of (4.4) are given by

$$\omega_k = i\pi\left(\frac{3}{4} + k\right), \quad k \in \mathbb{N}.$$

Now, we seek ω_k as follows

$$\omega_k = i\pi\left(\frac{3}{4} + k\right) + \frac{z_1}{k} + \frac{z_2}{k^2} + \frac{z_3}{k^3} + \frac{z_4}{k^4} + \mathcal{O}\left(\frac{1}{k^5}\right), \tag{4.5}$$

where $z_i,\ i = 1, 2, 3, 4$ are complex constants.
Next, by using (4.5), we calculate the development of e^{ω}, $e^{-\omega}$ and the expression of $\hat{P}(\omega)$ defined in (4.3). We obtain after a careful computation that

$$\begin{aligned} \hat{P}(\omega) &= (-1)^k\frac{\sqrt{2}}{2}\left\{2i\tilde{J}\pi^3(\tilde{\beta} - 2\pi z_1)k^3 + \tilde{J}\pi^3\left[2(\tilde{\beta} - 6i\pi)z_1 - 4i\pi z_2 + \frac{9}{2}i\tilde{\beta}\right]k^2\right. \\ &+ k\Big[(-6\tilde{\beta}\tilde{J} + 4\tilde{\alpha} - 16\pi + i\tilde{\beta}\tilde{J}\pi z_1)\pi^2 z_1 + 2(\tilde{\beta} - 6i\pi)\tilde{J}\pi^3 z_2 - 4i\tilde{J}\pi^4 z_3 \\ &\left. -\frac{9}{8}i\tilde{\beta}\tilde{J}\pi^3 - \frac{27}{2}i\tilde{J}\pi^4 - 2\pi(1+\tilde{\alpha}\tilde{\beta})\Big] + Z + \mathcal{O}\left(\frac{1}{k}\right)\right\}, \end{aligned}$$

where Z is a complex number depending on z_i, $i = 1, 2, 3, 4$. Finally, we choose the complex z_i such that each coefficient of k^{4-i} is zero for $i = 1, 2, 3, 4$. Next, we replace the obtained values in (4.5) to get:

$$\omega_k = i\pi\left(\frac{3}{4} + k\right) + \frac{\tilde{\beta}}{2\pi k} + \frac{\tilde{\beta}}{4i\pi}\left(\frac{\tilde{\beta}}{\pi} - \frac{3}{2}i\right)\frac{1}{k^2} + \cdots.$$

Since $\lambda_k = i\omega^2\sqrt{\frac{EI}{\rho}}$, we conclude that $\Re e\lambda_k = \Re e\left(i\omega^2\sqrt{\frac{EI}{\rho}}\right) \longrightarrow -\beta\frac{EI}{\rho}$, as $k \longrightarrow \infty$. This completes the proof of Theorem 3. $\square$

As mentioned in Remark 2, assume that the condition $(*)$ of Theorem 2 is satisfied with $\mu = \frac{1134}{11}$. Recall that if the feedback law consists of only a control force ($\alpha = 0$) then the hybrid system (2.1) is governed by the operator A_0 which generates a semigroup $S_0(t)$. An important consequence of the previous spectral analysis is the following proposition:

Proposition 2 . *Assume that $3\mu EIJ^2 \leq \rho$. Then, the semigroup $S_0(t)$ is not strongly stable in $\mathcal{H}$.*

Proof of Proposition 2. Consider the following physical parameters:

$$J = \frac{\sinh\pi}{(1 + \cosh\pi)\pi^3}, \; EI = \rho = \beta = 1.$$

Then it is easy to check that $3\mu EIJ^2 < \rho$ where $\mu = \frac{1134}{11}$. Since $EI = \rho = 1$, we can claim from the proof of Theorem 3 that $\lambda_k = i\omega^2$ is an eigenvalue of A_0 of our hybrid system if and only if ω satisfies equation (4.2) with $\tilde{\alpha} = 0$ and $\tilde{\beta} = \beta$, $\tilde{J} = J$ (recall that $\tilde{\beta} = \beta\sqrt{\frac{EI}{\rho}}$, $\tilde{\alpha} = \alpha\sqrt{\frac{EI}{\rho}}$ and $\tilde{J} = J\frac{EI}{\rho}$). This gives after straightforward computation that ω is solution of the following equation:

$$\begin{aligned}
&e^{\omega}\Big[\cos\omega\,(2J\omega^3 - 2) + 2\sin\omega + i\left\{\cos\omega\,(2J\omega^4 - 2\omega) + 2J\omega^4\sin\omega\right\}\Big] \\
&+e^{-\omega}\Big[\cos\omega\,(2J\omega^3 + 2) + 2\sin\omega + i\left\{\cos\omega\,(-2J\omega^4 - 2\omega) + 2J\omega^4\sin\omega\right\}\Big] \\
&-4J\omega^3 - 4i\omega = 0.
\end{aligned} \tag{4.6}$$

Since $J = \frac{\sinh\pi}{(1+\cosh\pi)\pi^3}$, one can check that $\omega = \pi$ is solution of (4.6) and thus $\lambda = i\omega^2 = i\pi^2$ is an eigenvalue of A_0. We conclude that the semigroup $S_0(t)$ is not stable. $\square$

Numerical results: To see how different the eigenvalues are for the values of β and α, we shall do some computer simulations. First we know from the previous spectral study that λ is an eigenvalue if and only if ω is solution of (4.2) where λ and ω are related by $\lambda = i\omega^2\sqrt{\frac{EI}{\rho}}$. Let $\omega = x + iy$, with x and y being real numbers. Substituting ω into $P(\omega) = 0$ (see (4.2)), it follows that (x, y) is solution of the equation $R(x, y) + iI(x, y) = 0$, where R and I are two real **explicit** functions. Obviously, λ is an eigenvalue if and only if $R \equiv I \equiv 0$. Using MAPLE V, we find the points of intersection (x, y) of the graphs of $R = 0$ and $I = 0$. Once the values of $\omega = x + iy$ are obtained, we can recover the eigenvalues λ by using the relationship $\lambda = i\omega^2\sqrt{\frac{EI}{\rho}} = \sqrt{\frac{EI}{\rho}}\Big(- 2xy + i(x^2 - y^2)\Big)$. We consider three cases:

Case 1: $\boldsymbol{\alpha > 0,\ \beta > 0}$.

We observe that the real part of the spectrum approaches the vertical axis $-\beta\frac{EI}{\rho}$ (see fig. 2, fig.3,...fig.9). This confirms the result obtained in Theorem 3. It is also interesting to note that if we fix $\alpha = J = 1,\ EI = 9,\ \rho = 4$ and we vary β then we notice two facts: On one hand, the more β increases, for $\beta > 1$, the more the spectrum moves towards the left. On the other hand, the more β decreases, for $\beta < 1$, the more the spectrum moves towards the right (fig. 2,...fig.6). In return, if we fix $\beta = J = 1,\ EI = 9,\ \rho = 4$ and α vary, then we notice that the spectrum is "stable" (fig. 2, fig.7, fig.8, fig.9). This means that the influence of β is numerically dominant.

Case 2: $\boldsymbol{\alpha = 0,\ \beta > 0}$ and $\boldsymbol{3\mu EIJ^2 > \rho}$.

As mentioned in Remark 2, assume that $\mu = \frac{1134}{11}$. Then, we observe that the vertical axis $-\beta\frac{EI}{\rho}$ is still an asymptote for the real part of the spectrum (see fig. 10, fig.11, fig.12). This illustrates the result of Theorem 3.

Case 3: $\boldsymbol{\alpha = 0,\ \beta > 0}$ and $\boldsymbol{3\mu EIJ^2 < \rho}$.

As in the previous case, let $\mu = \frac{1134}{11}$. Clearly, there exists a pure imaginary eigenvalue (see fig. 13 and fig. 14) which means that the system is not even strongly stable as proved in Proposition 2.

5 Conclusion

In this article, we considered the stabilization problem of a variant of the SCOLE model. We have proposed a simple and usual stabilizing feedback law composed of control force and control moment. Then the system is uniformly stabilized by the proposed feedback law as soon as the boundary control force and control moment are present in the feedback. Nevertheless, if only the control force is applied, then the exponential stability is also obtained provided the physical parameters of the system satisfy a certain condition (see Theorem 2). We have shown that this condition is necessary to obtain uniform stabilization. An asymptotic analysis of the real part of the spectrum was also done. This spectral study establishes explicitly a relationship between the feedback gain of the control force and the eigenvalues. The theoretical results obtained in this paper have been illustrated by several numerical results.

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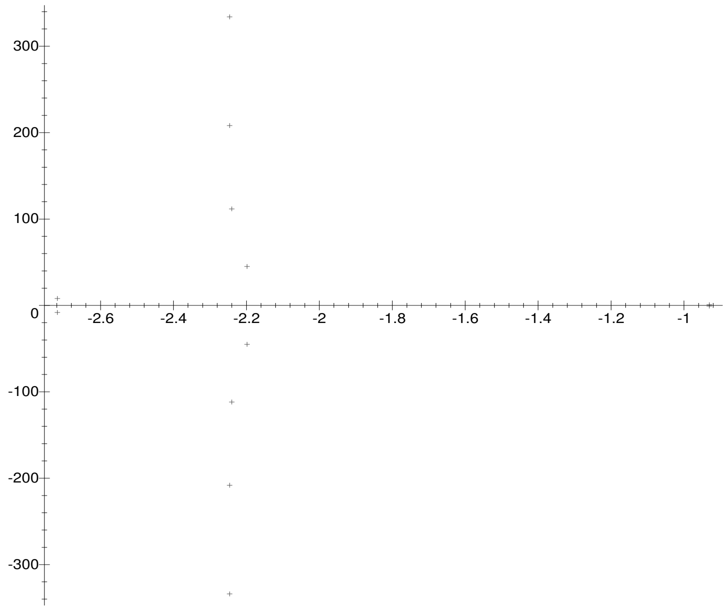

Figure 2: Spectrum for $J = 1,\ \alpha = 1,\ \beta = 1,\ EI = 9,\ \rho = 4$

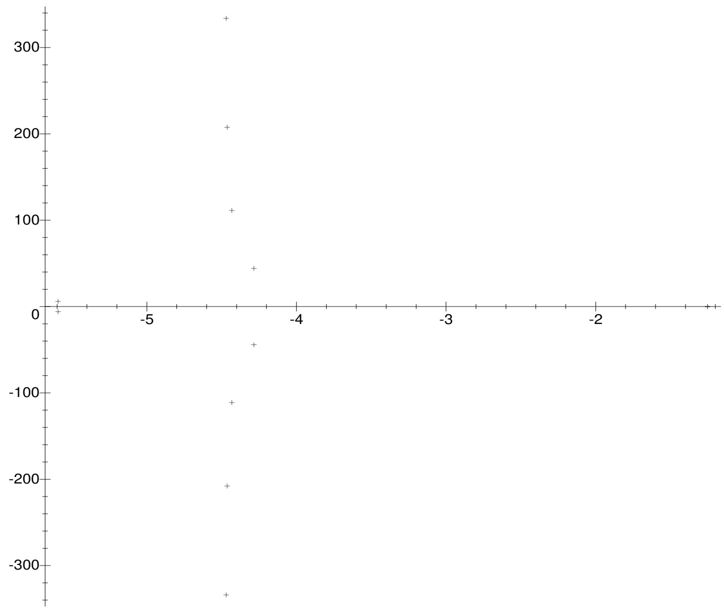

Figure 3: Spectrum for $J = 1,\ \alpha = 1,\ \beta = 2,\ EI = 9,\ \rho = 4$

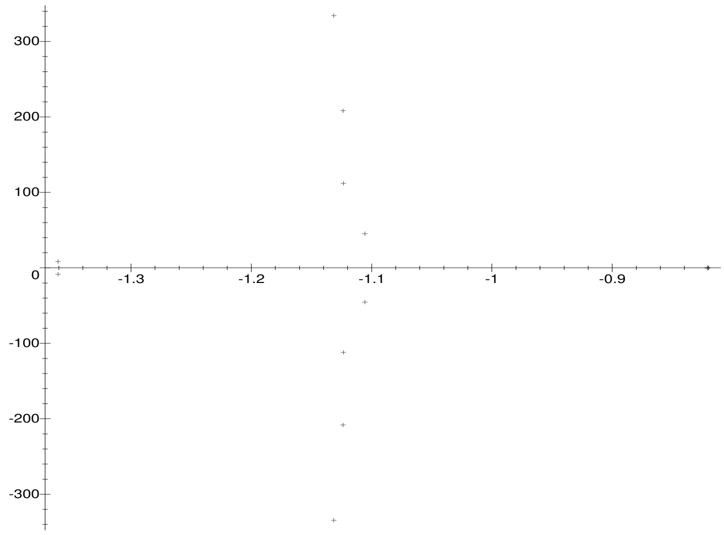

Figure 4: Spectrum for $J = 1,\ \alpha = 1,\ \beta = 0.5,\ EI = 9,\ \rho = 4$

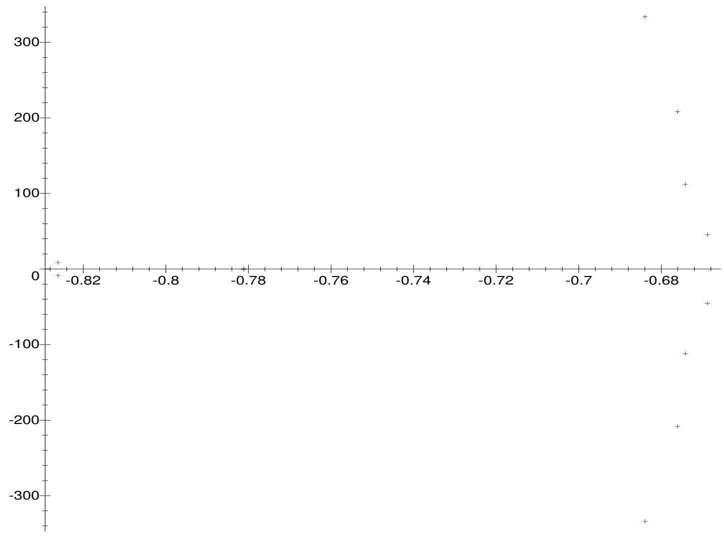

Figure 5: Spectrum for $J = 1,\ \alpha = 1,\ \beta = 0.3,\ EI = 9,\ \rho = 4$

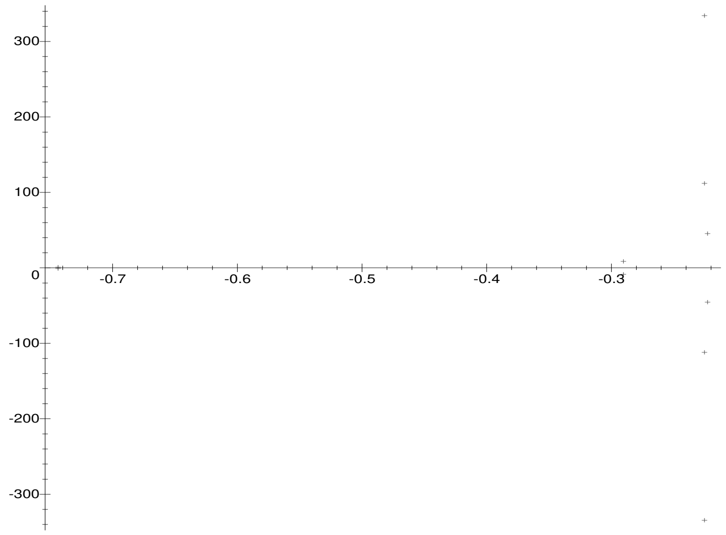

Figure 6: Spectrum for $J = 1,\ \alpha = 1,\ \beta = 0.1,\ EI = 9,\ \rho = 4$

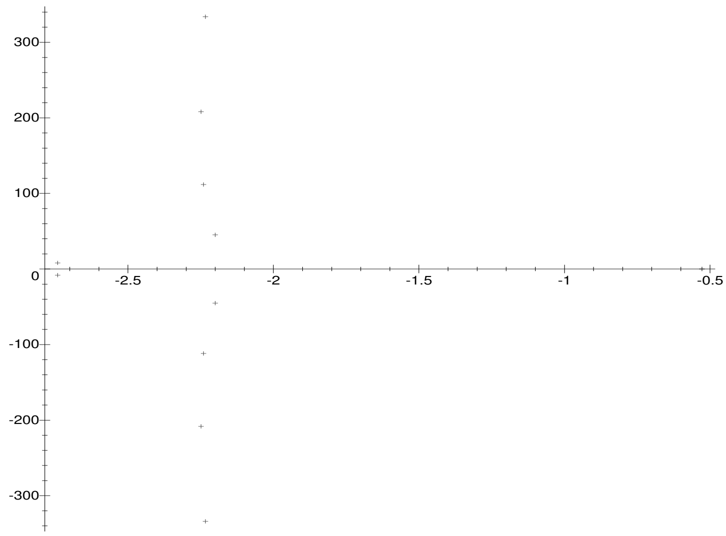

Figure 7: Spectrum for $J = 1,\ \alpha = 2,\ \beta = 1,\ EI = 9,\ \rho = 4$

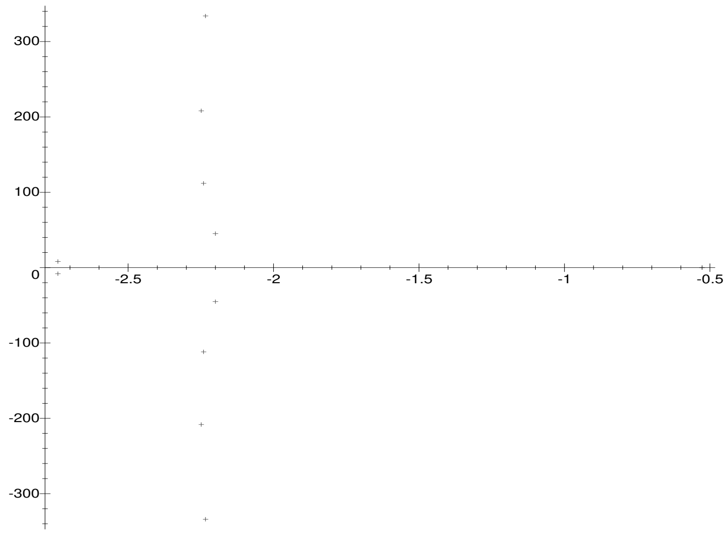

Figure 8: Spectrum for $J = 1,\ \alpha = 1.3,\ \beta = 1,\ EI = 9,\ \rho = 4$

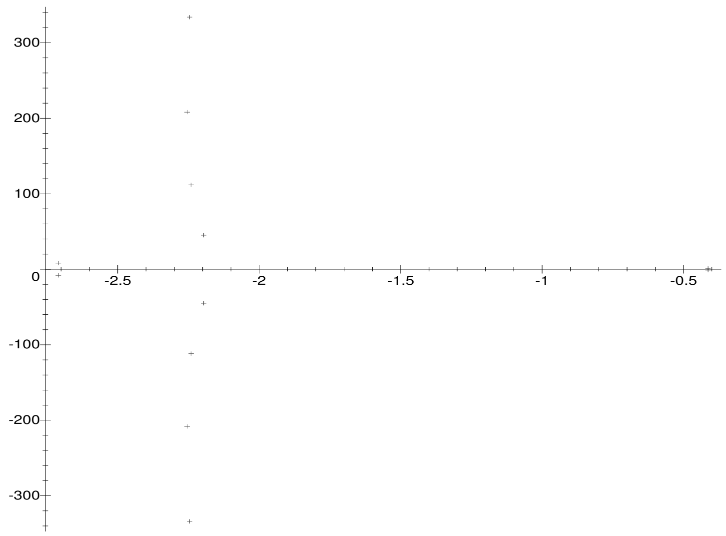

Figure 9: Spectrum for $J = 1,\ \alpha = 0.3,\ \beta = 1,\ EI = 9,\ \rho = 4$

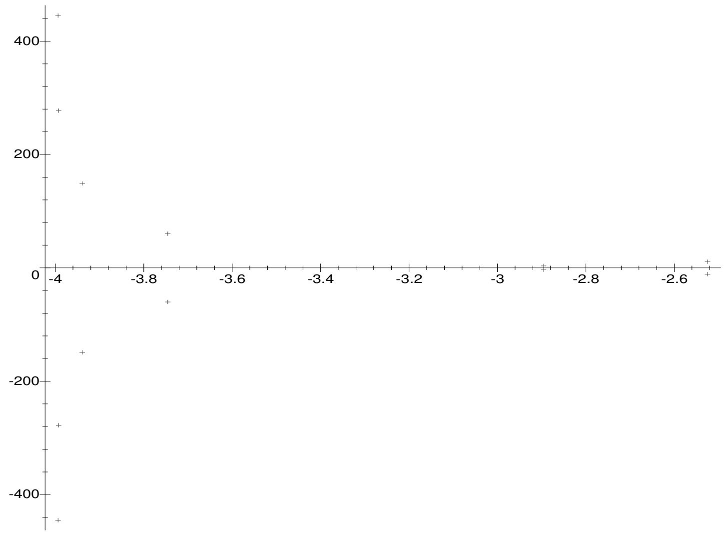

Figure 10: Spectrum for $J = 0.1,\ \alpha = 0,\ \beta = 1,\ EI = 8,\ \rho = 2$

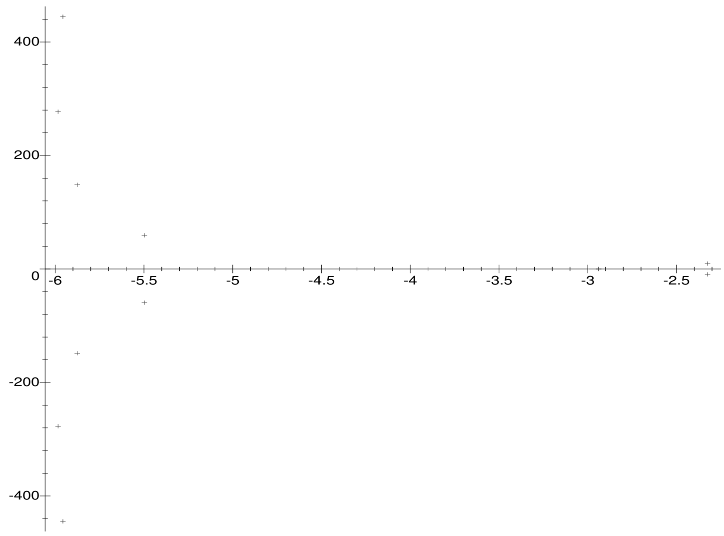

Figure 11: Spectrum for $J = 0.1,\ \alpha = 0,\ \beta = 1.5,\ EI = 8,\ \rho = 2$

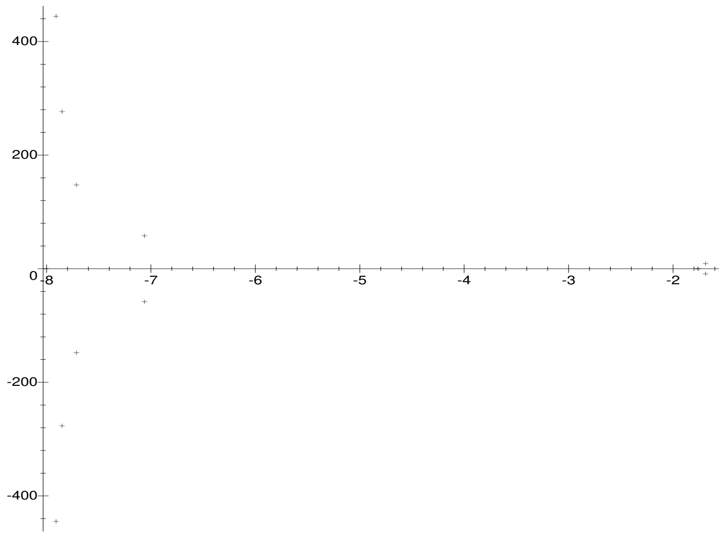

Figure 12: Spectrum for $J = 0.1,\ \alpha = 0,\ \beta = 2,\ EI = 8,\ \rho = 2$

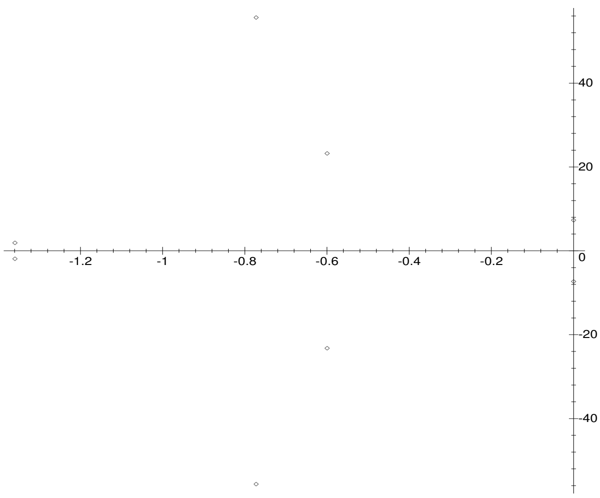

Figure 13: Spectrum for $J = 0.0399,\ \alpha = 0,\ \beta = 1.5,\ EI = 1.1,\ \rho = 2$

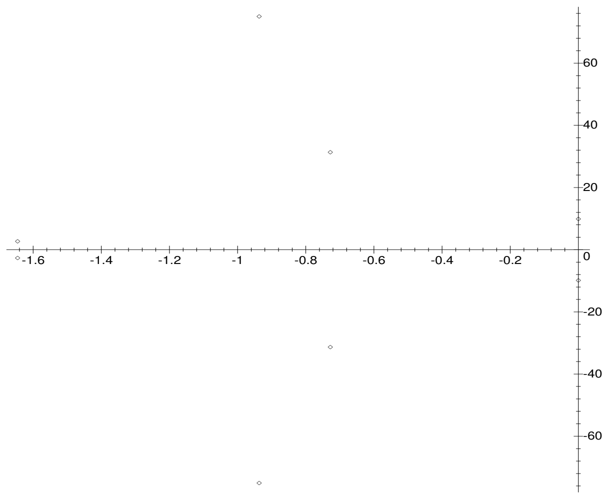

Figure 14: Spectrum for $J = \frac{\sinh\pi}{(1+\cosh\pi)\pi^3},\ \alpha = 0,\ \beta = 1,\ EI = 1,\ \rho = 1$