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A SAMPLING SCHEME FOR RELIABILITY ESTIMATION

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Abstract. A Three-stage sampling scheme for estimating the reliability of a series system when the component reliabilities are modeled as beta random variables is proposed. The sampling scheme specifies how a total of T units can be allocated to two independent components. It is shown that the proposed sampling scheme is nearly optimal.

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1. Introduction

Suppose that a system is composed of two independent components connected in series and R_1 and R_2 are the corresponding reliabilities. Then the product R_1R_2 is the reliability of the system, and the problem of estimating this reliability is the problem of estimating the product of means of two independent Bernoulli populations.

We take a Bayesian approach to this problem, assuming that prior information of some sort is available and can be modeled in the form of beta priors on R_1 and R_2.

Although selecting the form of an optimal estimator of the reliability of the system is critical, the real difficulty lies in determining an optimal sampling scheme that specifies how to allocate a total number of T units to two independent components. It is assumed that the total number of units to be tested is fixed, but that the number of units of the first component and of the second component to be tested may be random.

In [1], a frequentist approach was used to examine fixed and sequential sampling schemes for estimating the reliability of a series system. A fully sequential sampling scheme was proposed and shown to perform much better than the balanced sampling scheme. In [2], a Bayesian approach was used to examine the efficiency

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of the myopic sampling scheme. In [3], a two-stage sampling scheme was proposed and shown to be nearly optimal. In this article, we propose a three-stage sampling scheme to [3]. Although the focus here is on estimating the product of means instead of the difference between means. In next section, we define the framework used to formulate the problem.

2. Bayes Risk

Since each component either fails or functions, a Bernoulli formulation of this problem is necessary. Let X_{1j} denote the outcome of the j^{th} unit of the first component and X_{2j} denote the outcome of the j^{th} unit of the second component. Then $X_{1j} = 0$ with probability $1 - R_1$, if the j^{th} unit of type one component fails, and $X_{1j} = 1$ with probability R_1, otherwise. Similarly, let $X_{2j} = 0$ with probability $1 - R_2$, if the j^{th} unit of type two component fails, and $X_{2j} = 1$ with probability R_2, otherwise.

Let E_T denote the data generated by the random variables

$$X_{11}, \ldots X_{1M}, X_{21}, \ldots X_{2N},$$

where $M + N = T$. Let the component reliabilities R_1 an R_2 be independent beta distributed with parameters (a_1, b_1) and (a_2, b_2).

The Bayes estimator of the system reliability is the posterior mean

$$\hat{R}(M,N) = \left[\frac{a_1 + \sum_{j=1}^{M} X_{1j}}{M + r_1}\right]\left[\frac{a_2 + \sum_{j=1}^{N} X_{2j}}{N + r_2}\right], \tag{1}$$

where $r_1 = a_1 + b_1$ and $r_2 = a_2 + b_2$. The Bayes risk is given by

$$\begin{aligned} B(M,N) &= E[(R_1)^2]E[(R_2)^2] \\ &\quad - E\left[[R_1]^2 - \frac{R_1(1-R_1)}{M+r_1}\right] E\left[[R_2]^2 - \frac{R_2(1-R_2)}{N+r_2}\right]. \end{aligned}$$

3. Three-Stage Sampling Scheme

For random sampling schemes, M, N are random positive integers. The Bayes risk can be approximated by

$$E\left[\frac{R_1(1-R_1)}{M+r_1}[R_2]^2\right] + E\left[\frac{R_2(1-R_2)}{N+r_2}[R_1]^2\right]$$

which can also be written as

$$E\left(\frac{U_M}{M+r_1} + \frac{V_N}{N+r_2}\right),$$

where

$$U_M = E\left\{R_1(1-R_1)R_2^2 \middle| E_T\right\}$$

and

$$V_N = E\left\{ R_2(1-R_2)R_1^2 \middle| E_T \right\}.$$

The approximated Bayes risk is minimized by choosing M such that

$$\frac{M+r_1}{T+r_1+r_2} = \frac{\sqrt{U_M}}{\sqrt{U_M}+\sqrt{V_N}} \tag{2}$$

and $N = T - M$.

Simple calculations yield

$$U_M = \frac{a_{1M}b_{1M}}{(M+r_1)(M+r_1+1)} \frac{a_{2N}(a_{2N}+1)}{(N+r_2)(N+r_2+1)}$$

and

$$V_N = \frac{a_{2N}b_{2N}}{(N+r_2)(N+r_2+1)} \frac{a_{1M}(a_{1M}+1)}{(M+r_1)(M+r_1+1)},$$

where

$$a_{1M} = a_1 + \sum_{j=1}^{M} X_{1j}, \quad b_{1M} = b_1 + M - \sum_{j=1}^{M} X_{1j},$$

$$a_{2N} = a_2 + \sum_{j=1}^{N} X_{2j} \quad \text{and} \quad b_N = b_2 + N - \sum_{j=1}^{N} X_{2j}.$$

There are many sampling schemes that may satisfy (2) for large values of T. To describe the three-stage sampling scheme, let $r = r_T$ and $s = s_T$ be chosen such that

$$r + s < \frac{T}{2},$$

and

$$\frac{r+s}{T} \to 0,$$

as $T \to \infty$.

In the first batch test r units from each component. In the second batch test $m-r$ units from the first component and $n-r$ units from the second component, where

$$m = max\{r, (T-2s)\hat{C}_{r,r}\}$$

where

$$\hat{C}_{r,r} = \frac{\sqrt{U_r}}{\sqrt{U_r}+\sqrt{V_r}},$$

and

$$n = T - 2s - m.$$

In the third batch test $2s$ units of which $M-m$ are from the first component and $N-n$ are from the second component, where

$$M = max\{m, T\hat{C}_{m,n}\}$$

where

$$\hat{C}_{m,n} = \frac{\sqrt{U_m}}{\sqrt{U_m} + \sqrt{V_n}},$$

and

$$N = T - M.$$

Theorem Let R_T denote the Bayes risk incurred by the three-stage sampling scheme. Then

$$\lim_{T\to\infty} MR_T = \frac{a_1 a_2[b_1(a_2+1) + b_2(a_1+1)]}{(a_1+b_1)(a_1+b_1+1)(a_2+b_2)(a_2+b_2+1)} + r_0,$$

where

$$r_0 = \frac{2(a_1+.5)(a_2+.5)\Gamma(a_1+.5)\Gamma(b_1+.5)\Gamma(a_2+.5)\Gamma(b_2+.5)}{(a_1+b_1)(a_1+b_1+1)(a_2+b_2)(a_2+b_2+1)\Gamma(a_1)\Gamma(b_1)\Gamma(a_2)\Gamma(b_2)}.$$

Proof. Let the scaled approximated Bayes risk be denoted by $(T+r+s)R_T$. It is the sum of two positive terms

$$E[(\sqrt{U_M} + \sqrt{V_N})^2]$$

and

$$E\left[\frac{[(N+r_2)\sqrt{U_M} - (M+r_1)\sqrt{V_N}]^2}{(M+r_1)(N+r_2)}\right].$$

The proof will follow by showing that

$$U_M \to R_1(1-R_1)R_2^2 \quad \text{and} \quad V_N \to R_2(1-R_2)R_1^2$$

as $T\to\infty$, and that

$$E\left[\frac{[(N+r_2)\sqrt{U_M} - (M+r)\sqrt{V_N}]^2}{(M+r_1)(N+r_2)}\right] \to 0 \quad \text{as} \quad T\to\infty.$$

Since $M\to\infty$ and $N\to\infty$ and U_M and V_N are uniformly integrable martingale,

$$U_M \to R_1(1-R_1)R_2^2 \quad \text{and} \quad V_N \to R_2(1-R_2)R_1^2$$

by the martingale convergence theorem. Therefore

$$\hat{C}_{M,N} \to C,$$

in probability as $T\to\infty$ where

$$C = \frac{R_2\sqrt{R_1(1-R_1)}}{R_2\sqrt{R_1(1-R_1)} + R_1\sqrt{R_2(1-R_2)}}.$$

From the conditions on r and s it follows that

$$\frac{M}{N} \to \frac{R_2\sqrt{[R_1(1-R_1)]}}{R_1\sqrt{[R_2(1-R_2)]}} \quad \text{almost surely as} \quad T \to \infty.$$

The proof of the theorem follows by showing that

$$\frac{[(N+r_2)\sqrt{U_M}-(M+r)\sqrt{V_N}]^2}{(M+r_1)(N+r_2)} \quad \text{is uniformly integrable.}$$

Since

$$\frac{[(N+r_2)\sqrt{U_M}-(M+r)\sqrt{V_N}]^2}{(M+r_1)(N+r_2)} \leq 2\left[\frac{N+r_2}{M+r_1}U_M + \frac{M+r_1}{N+r_2}V_N\right],$$

uniform integrability follows by showing that

$$\frac{N+r_2}{M+r_1}U_M \quad \text{and} \quad \frac{M+r_1}{N+r_2}V_N \quad \text{are both uniformly integrable.}$$

The rest of the proof is similar to [3]. A sketch of it follows. Using the three-stage sampling scheme,

$$\frac{T}{M} \leq \frac{1}{\hat{C}_{m,n}},$$

therefore

$$\frac{N+r_2}{M+r_1}U_M \leq \frac{U_M}{\hat{C}_{m,n}} + U_M(r_1+r_2).$$

Since U_M is uniformly integrable and

$$\frac{U_M}{\hat{C}_{m,n}} \leq max_{1\leq k\leq T}(U_k + \sqrt{U_k V_k}),$$

uniform integrability is established by applying Doob's inequality for nonnegative submatingales. A completely anlogous argument for $\frac{M+r_1}{N+r_2}V_N$ yields the theorem.

References

[1] Rekab, K.: A Sampling Scheme for Estimating the Reliability of a Series System, IEEE Trans. Reliability, **42**, (1992), 287-291.

[2] Shapiro, C. P.: Allocation Schemes for Estimating the Product of Positive Parameters, Journal of the American Statistical association, **80**, (1985), 449-454

[3] Rekab ,K. and Tahir, M.: A Two-Stage Allocation Scheme For Estimating The Product of Several Means , Stochastic Analysis, **18** , (2000), 289-298

[4] Woodroofe, M. and Hardwick, J.: Sequential Allocation for an Estimation Problem with Ethical Costs. Ann. Statist, **18**, (1990), 1358-1377