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THE VANISHING OF THE COHOMOLOGY OF THE ORLIK-SOLOMON ALGEBRAS

Kelly Jeanne Pearson

Abstract The Orlik-Solomon algebra of a hyperplane arrangement first appeared from the Brieskorn and Orlik-Solomon theorems as the cohomology of the complement of this arrangement (if the ground field is complex). Later, it was discovered that this algebra plays an important role in many other problems. In particular, define the cohomology of an Orlik-Solomon algebra as that of the complex formed by its homogeneous components with the differential defined via multiplication by an element of degree one. In this paper, we give necessary and sufficient conditions for the vanishing of the cohomology of the Orlik-Solomon algebra.

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1 Introduction

Let V be an affine space of dimension ℓ over a field F, and let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a nonempty arrangement of hyperplanes of V. Let $L(\mathcal{A})$ be the intersection poset, ordered opposite to inclusion, of $\mathcal{A}$. Consider the complement space $M = V \setminus \cup_{i=1}^{n} H_i$. The problem of expressing $H^*(M, \mathcal{K})$ in terms of generators and relations was first studied by Arnold [2] in the case $\mathcal{A}$ was the braid arrangement and $\mathcal{K} = \mathbb{C}$; that is, $\mathcal{A}$ was the collection of hyperplanes $\{x_i - x_j : 1 \leq i < j \leq \ell\}$. This problem was later studied by Brieskorn [3] for an arbitrary arrangement. For each $H_i \in \mathcal{A}$ fix an affine functional α_i with $\operatorname{Ker} \alpha_i = H_i$. The Orlik-Solomon algebra is the graded F-algebra $= A(\mathcal{A}) = \oplus_p^\ell A_p$ generated by the differential forms $\omega_i = d\alpha_i/\alpha_i \in A_1$. If $F = \mathbb{C}$, then according to Brieskorn's theorem, this algebra is isomorphic under the de Rham map to the cohomology algebra of $M = V \setminus \cup_{i=1}^{n} H_i$. Orlik and Solomon [9] found a purely algebraic characterization of $H^*(M, \mathcal{K})$. These results can be briefly summarized as follows. An algebra $A(\mathcal{A})$ (referred to as the Orlik-Solomon algebra)

Mathematics Department, Murray State University, Murray KY
kelly.pearson@murraystate.edu

over $\mathcal{K}$ is constructed in terms of generators and relations using only $L(\mathcal{A})$. This is a graded algebra with $A(\mathcal{A}) \cong H^*(M, \mathcal{K})$. Hence, $H^*(M, \mathcal{K})$ can be determined by $L(\mathcal{A})$.

The above are important results in that topological invariants of the complement space are expressed in terms of combinatorics. Indeed, a central question in the theory of hyperplane arrangements is the problem of expressing topological invariants of the complement space in terms of combinatorics. In this manner, it is a natural question to consider a generalization of $H^*(M, \mathcal{K})$ to cohomology with local coefficients.

For $a \in A_1(\mathcal{A})$, one can define a local coefficient system $\mathcal{L}(a)$. It turnsout that $H^*(M, \mathcal{L}(a))$ relates closely to the cohomology of the Orlik-Solomon algebra. The connection between $H^*(M, \mathcal{L}(a))$ and the cohomology of the Orlik-Solomon algebra has been studied in many papers, for instance [6].

We construct a cochain complex on the graded linear space $A(\mathcal{A})$ as follows. Let $a \in A_1(\mathcal{A})$ with $a = \sum_{i=1}^{n} \lambda_i a_i$ for $\lambda_i \in \mathcal{K}$. Multiplication by a giving the differential $d_k : A_k(\mathcal{A}) \xrightarrow{a\cdot} A_{k+1}(\mathcal{A})$ forms a complex $(A(\mathcal{A}), a)$. The cohomology of this complex is said to be the cohomology of the Orlik-Solomon algebra and is denoted $H^*(A(\mathcal{A}), a)$.

Recently, there have been many results concerning $\dim H^1(A(\mathcal{A}), a)$. In the case char $\mathcal{K} \neq 2$, it has been shown in [6] that $\dim H^1(A(\mathcal{A}), a)$ can be determined by a particular set of elements from $L(\mathcal{A})$. However, little is known about the higher dimensions $H^p(A(\mathcal{A}), a)$ for $p > 1$ [11], and this is what our work is devoted to.

In this paper, we let $\mathcal{K} = \mathbb{R}$ or $\mathbb{C}$ and establish a necessary and sufficient condition for $H^*(A(\mathcal{A}), a) = 0$. Here is an outline of the paper.

In §2, we employ tools from operator theory to prove the upper semicontinuity of the map $t \mapsto \dim H^p(A(\mathcal{A}), t)$ for any $p \geq 0$ and for any $t \in A_1(\mathcal{A})$. In §3, we analyze tensor products in the category of graded commutative algebras in order to express the cohomology of a reducible arrangement in terms of the cohomology of each factor of the arrangement. In §4, we apply results discussed in §2 and §3 and show $H^*(A(\mathcal{A}), a) = 0$ if and only if $H^\ell(A(\mathcal{A}), a) = 0$, where $\operatorname{rank}(\mathcal{A}) = \ell$. The following theorem, which is the main result of this paper, gives a necessary and sufficient condition for $H^\ell(A(\mathcal{A}), a) = 0$.

Theorem 4.11. *Let $\mathcal{A}$ be an affine ℓ-arrangement. We may write*

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k \times \mathcal{B},$$

where $\mathcal{A}_j$ are each central and $\mathcal{B}$ not central and they contain no proper central factors. Let $a \in A_1(\mathcal{A})$. We have $\dim H^\ell(A(\mathcal{A}), a) \neq 0$ if and only if $\sum_{H_i \in \mathcal{A}_j} \lambda_i = 0$ for all j.

The cohomology of the Orlik-Solomon algebras is a particular case of a more general problem of skew commutative algebras, i.e. studying modules over an exterior algebra E, see [1]. If M is such a module, then $a \in E_1$ is said to be regular on M if and only if

$$H^*(M, a) = \{x \in M;\ ax = 0\}/aM = 0;$$

otherwise, a is said to be singular. The set of all singular elements is called a singular variety of M, denoted $Sing(M)$. So we study $Sing(A(\mathcal{A}))$ as $E(\mathcal{A})$-modules.

2 Upper Semicontinuity

Let $\mathcal{A} = \{H_1, ..., H_n\}$ be a hyperplane arrangement in $V = F^\ell$ for some field F. We recall the definition of the Orlik-Solomon algebra $A(\mathcal{A})$ and the cohomology of the Orlik-Solomon algebra.

Let $\mathcal{K}$ be a commutative ring. Let E_1 be the linear space over $\mathcal{K}$ on n generators, $e_1, \ldots, e_n$. Let $E(\mathcal{A}) := \Lambda(E_1)$ be the exterior algebra on E_1. We have $E(\mathcal{A}) = \bigoplus_{p\geq 0} E_p$ is a graded algebra over $\mathcal{K}$. The standard $\mathcal{K}$-basis for E_p is given by

$$\{e_{i_1} \cdots e_{i_p} : \ 1 \leq i_1 < \ldots < i_p \leq p\}.$$

Any ordered subset $S = \{H_{i_1}, ..., H_{i_p}\}$ of $\mathcal{A}$ corresponds to an element $e_S := e_{i_1} \cdots e_{i_p}$ in $E(\mathcal{A})$.

We define the map $\partial : \ E(\mathcal{A}) \to E(\mathcal{A})$ via the usual differential. That is,

$$\begin{aligned} \partial(1) :=& 0, \\ \partial(e_i) :=& 1, \\ \text{and for } p \geq 2, \ \partial(e_{i_1} \cdots e_{i_p}) :=& \sum_{k=1}^{p} (-1)^{k-1} e_{i_1} \cdots \hat{e}_{i_k} \cdots e_{i_p}. \end{aligned}$$

Let $S = \{H_{i_1}, ..., H_{i_p}\}$ be a subset of $\mathcal{A}$. We say S is dependent if $\cap S \neq \emptyset$ and $\mathrm{rank}(\cap S) < |S|$. Equivalently, S is dependent if polynomials $\alpha_{i_k} \in F[x_1, \ldots, x_\ell]$ defining the hyperplanes H_{i_k} are linearly dependent.

We define $I(\mathcal{A})$ to be the ideal of $E(\mathcal{A})$ which is generated by the elements

$$\{\partial(e_S) : \ S \text{ is dependent } \} \cup \{e_S : \ \cap_{H\in S} H = \emptyset\}.$$

The Orlik-Solomon algebra, $A(\mathcal{A})$, is defined as

$$A(\mathcal{A}) := E(\mathcal{A})/I(\mathcal{A}).$$

Let $\pi : \ E(\mathcal{A}) \to A(\mathcal{A})$ be the canonical projection. We write a_S to represent the image of e_S under π.

The grading on $E(\mathcal{A})$ induces a grading on $A(\mathcal{A})$; hence, $A(\mathcal{A}) = \oplus A_p(\mathcal{A})$. The standard basis used for $A(\mathcal{A})$ is referred to as the broken circuit basis.

We construct a cochain complex on the homogeneous components of $A(\mathcal{A})$ as follows. Let $a \in A_1(\mathcal{A})$ with $a = \sum_{i=1}^{n} \lambda_i a_i$ for $\lambda_i \in \mathcal{K}$. Multiplication by a giving the differential $d_k : A_k(\mathcal{A}) \xrightarrow{a\cdot} A_{k+1}(\mathcal{A})$ forms a complex $(A(\mathcal{A}), a)$. The cohomology of this complex is said to be the cohomology of the Orlik-Solomon algebra and is denoted $H^*(A(\mathcal{A}), a)$.

In this section, we show that in the case $\mathcal{K} = \mathbb{R}$ or $\mathbb{C}$, the function $A_1(\mathcal{A}) \to \mathbb{Z}$ given by $t \mapsto \dim H^p(A(\mathcal{A}), t)$ is upper semicontinuous in t for any p. We show this in the more general setting of finite dimensional vector spaces and hence begin by establishing some standard definitions and notational conventions, which can be found in [4].

Let V be an n-dimensional vector space over $\mathcal{K}$. Relative to a basis $\{b_1, \dots, b_n\}$ for V over $\mathcal{K}$, for $v \in V$ we express $v = (v_1, \dots, v_n)$ as $v = \sum_{i=1}^{n} v_i b_i$.

Since $\mathcal{K} = \mathbb{C}$ or $\mathbb{R}$, we define the standard Euclidean norm, $\|\cdot\|_V$, on V as

$$\|v\|_V = \|(v_1, \dots, v_n)\|_V := \sqrt{v_1\bar{v}_1 + \cdots + v_n\bar{v}_n},$$

relative to the standard orthonormal basis $\{e_1, \dots, e_n\}$ for V.

With respect to the norm $\|\cdot\|_V$, we define the unit sphere in V by

$$S(V) := \{v \in V : \ \|v\|_V = 1\}.$$

We also have the corresponding standard inner product, $(\cdot, \cdot)$, defined on V as

$$(x, y) := x_1\bar{y}_1 + \cdots + x_n\bar{y}_n \text{ for } x, y \in V.$$

We define the orthogonal complement relative to the inner product. Let $X \subseteq V$ be a linear subspace. Then

$$X^\perp := \{v \in V : \ (v, x) = 0 \text{ for all } x \in X\}.$$

Let $X \subseteq V$ be a linear subspace and $v \in V$. We define the distance from v to X to be

$$\text{dist } (v, X) := \inf\{\|v - x\|_V : \ x \in X\}.$$

Note there exists an unique $x_0 \in X$ for which dist $(v, X) =$ dist (v, x_0).

For the remainder of this section, we fix two finite dimensional vector spaces V and W over $\mathcal{K}$. Fix $0 \neq A \in \text{Hom}_{\mathcal{K}}(V, W)$.

Definition 2.1. Define

$$\gamma(A) := \inf\{\|Ah\|_W : \ h \in S(V) \cap (\text{Ker}\, A)^\perp\}.$$

The following technical lemmas are well known and can be found in [4]; we include their proofs for clarity.

Lemma 2.2. *If* $0 \neq A \in \text{Hom}_{\mathcal{K}}(V, W)$*, then* $\gamma(A) > 0$.

Proof. Clearly, $\gamma(A) \geq 0$. Suppose that $\gamma(A) = 0$. By definition of the infimum, there exists a sequence $\{h_j\} \subset S(V) \cap (\text{Ker}\, A)^\perp$ so that $\|Ah_j\|_W \to 0$. This implies $\lim_{j\to\infty} h_j \in \text{Ker}\, A \cap (\text{Ker}\, A)^\perp = \{0\}$. But $\|\cdot\|$ is continuous in the metric; hence, $\lim_{j\to\infty} \|h_j\|_V = 1$. ◻

Lemma 2.3. *If* $A \in \text{Hom}_{\mathcal{K}}(V, W)$ *and if* $h \in V$*, then we have*

$$\gamma(A) \cdot \text{dist } (h, \text{Ker}\, A) \leq \|Ah\|_W.$$

Proof. Let $\rho : V \to (\text{Ker}\, A)^\perp$ be orthogonal projection of V onto $(\text{Ker}\, A)^\perp$. We relate the norm to the distance by noticing $\|\rho h\|_V =$ dist $(h, \text{Ker}\, A)$. Hence,

$$\begin{aligned} \|Ah\|_W &= \|A\rho h\|_W \\ &\geq \gamma(A) \cdot \|\rho h\|_V \\ &= \gamma(A) \cdot \text{dist } (h, \text{Ker}\, A). \end{aligned}$$

The lemma now follows. ◻

Lemma 2.4. *If* $V_1, V_2 \subseteq V$ *are linear subspaces with* $\dim V_1 > \dim V_2$*, then there exists* $0 \neq v_1 \in V_1$ *so that* $\|v_1\|_V = \text{dist }(v_1, V_2)$.

Proof. Let ρ_1 be the orthogonal projection of V onto V_1. We have the inequality $\dim \rho_1(V_2) \leq \dim V_2 < \dim V_1$, so $\rho_1(V_2)$ is a proper linear subspace of V_1. Take $0 \neq v_1 \in V_1 \cap (\rho_1(V_2))^{\perp}$. Then for any $v_2 \in V_2$, we have

$$\begin{aligned} 0 =& (\rho_1(v_2), v_1) \\ =& (v_2, \rho_1(v_1)) \\ =& (v_2, v_1). \end{aligned}$$

Thus, $v_1 \in V_2^{\perp}$. Consequently, $\|v_1\|_V = \text{dist }(v_1, V_2)$. ▫

Definition 2.5. Let $B \in \text{Hom}_{\mathcal{K}}(V, W)$. The operator norm of B is defined to be

$$\|B\|_{\text{op}} := \sup\{\|Bh\|_W : \ h \in S(V)\}.$$

We note that for any $h \in V$, the inequality holds:

$$\|Bh\|_W \leq \|B\|_{\text{op}} \|h\|_V.$$

Proposition 2.6. *If* $B \in \text{Hom}_{\mathcal{K}}(V, W)$ *with* $\|B\|_{\text{op}} < \gamma(A)$ *then*

$$\dim \text{Ker}(A + B) \leq \dim \text{Ker}\, A.$$

Proof. If $0 \neq h \in \text{Ker}(A + B)$, then $Ah = -Bh$. By Lemma 2.3, we have

$$\begin{aligned} \gamma(A) \cdot \text{dist }(h, \text{Ker}\, A) \leq& \|Ah\|_W \\ =& \|Bh\|_W \\ \leq& \|B\|_{op} \cdot \|h\|_V \\ <& \gamma(A) \cdot \|h\|_V. \end{aligned}$$

Thus, $\text{dist }(h, \text{Ker}\, A) < \|h\|_V$, for all $0 \neq h \in \text{Ker}(A + B)$. By Lemma 2.4, we have $\dim \text{Ker}(A + B) \leq \dim \text{Ker}\, A$. ▫

Definition 2.7. Let $A \in \text{Hom}_{\mathcal{K}}(V, W)$. The adjoint of A, denoted by $A^* \in \text{Hom}_{\mathcal{K}}(W, V)$, is given by $(x, A^*y) := (Ax, y)$ for all $x \in V$ and for all $y \in W$.

Lemma 2.8. *If* $A \in \text{Hom}_{\mathcal{K}}(V, W)$*, then* $\text{Ker}\, A^* = (\text{range } \ A)^{\perp}$.

Proof. Let $y \in \text{Ker}\, A^*$. Then $y \in (Ax)^{\perp}$ for any $x \in V$. Thus $y \in (\text{range } \ A)^{\perp}$.

Let $y \in (\text{range } \ A)^{\perp}$. Then for any $x \in V$, we have $0 = (Ax, y) = (x, A^*y)$. This implies $A^*y = 0$; hence, $y \in \text{Ker}\, A^*$. ▫

Proposition 2.9. *Let* $A, B \in \text{Hom}_{\mathcal{K}}(V, W)$*. If* $\|B^*\|_{\text{op}} < \gamma(A^*)$*, then we have* $\text{rank}(A + B) \geq \text{rank}\, A$.

Proof. From Proposition 2.6, we have

$$\dim \text{Ker}(A + B)^* = \dim \text{Ker}(A^* + B^*) \leq \dim \text{Ker}\, A^*.$$

Since $\dim \text{Ker}\, A^* = \dim(\text{range } \ A)^{\perp} = \dim W - \text{rank}\, A$, it follows that

$$\dim W - \text{rank}(A + B) \leq \dim W - \text{rank}(A). \ ▫$$

Definition 2.10. Let X be a topological space. Let $f : X \to \mathbb{R}$ be a real-valued function; f is said to be upper semicontinuous if for any real number α the set $\{x \in X :\ f(x) < \alpha\}$ is open. Alternatively, for X a metric space we may define f to be upper semicontinuous at $x_0 \in X$ if

$$\limsup_{x \to x_0} f(x) \leq f(x_0).$$

We now apply these results to the graded algebra $A(\mathcal{A}) = \oplus A_p(\mathcal{A})$. Since $\mathcal{K} = \mathbb{R}$ or $\mathbb{C}$, we have $A_p(\mathcal{A})$ is a finite dimensional vector space. For any $t \in A_1(\mathcal{A})$, let the map $t\cdot :\ A_p(\mathcal{A}) \to A_{p+1}(\mathcal{A})$ be given by multiplication by t. Let $Z_p(A(\mathcal{A}), t)$ denote the kernel of the map $t\cdot :\ A_p(\mathcal{A}) \to A_{p+1}(\mathcal{A})$; let $B_p(A(\mathcal{A}), t)$ denote the image of the map $t\cdot :\ A_p(\mathcal{A}) \to A_{p+1}(\mathcal{A})$.

Lemma 2.11. *If $\|t - t_0\|_{\mathcal{K}^n} \to 0$, then $\|t \cdot - t_0 \cdot\|_{\mathrm{op}} \to 0$.*

Proof. We have $A_p(\mathcal{A})$ and $E_p(\mathcal{A})$ are finite dimensional vector spaces over $\mathcal{K}$. We use the standard basis for $E_p(\mathcal{A})$ given by $\{e_{i_1} \cdots e_{i_p} :\ 1 \leq i_1 < \ldots < i_p \leq p\}$. We use the broken circuit basis for $A_p(\mathcal{A})$. Then $\|\cdot\|_{E_p(\mathcal{A})}$ and $\|\cdot\|_{A_p(\mathcal{A})}$ are defined as previously.

It will suffice to show $\|(t - t_0)(v)\|_{A_p(\mathcal{A})} \to 0$ for any $v \in A_{p-1}(\mathcal{A})$. But the maps given by $(t - t_0)\cdot :\ E_{p-1}(\mathcal{A}) \to E_p(\mathcal{A})$ commute with the projection map $\pi :\ E_p(\mathcal{A}) \to A_p(\mathcal{A})$; that is,

$$\begin{array}{ccc} E_{p-1}(\mathcal{A}) & \xrightarrow{(t-t_0)\cdot} & E_p(\mathcal{A}) \\ \downarrow \pi & \circ & \downarrow \pi \\ A_{p-1}(\mathcal{A}) & \xrightarrow{(t-t_0)\cdot} & A_p(\mathcal{A}). \end{array}$$

Hence, it will suffice to show $\|(t - t_0) \cdot v\|_{E_p(\mathcal{A})} \to 0$ for any $v \in E_{p-1}(\mathcal{A})$.

We need only show that $\|(t - t_0) \cdot b\|_{E_p(\mathcal{A})} \to 0$ for any standard basis element $b \in E_{p-1}(\mathcal{A})$. We write $b = e_{i_1} \cdots e_{i_{p-1}}$, where $1 \leq i_1 < \ldots < i_{p-1} \leq n$. Then

$$\begin{aligned} \|(t - t_0) \cdot b\|^2_{E_p(\mathcal{A})} &= \|(t - t_0) \cdot e_{i_1} \cdots e_{i_p}\|^2_{E_p(\mathcal{A})} \\ &= \|\left(\sum_{i=1}^{n} (t^i - t_0^i) e_i\right) e_{i_1} \cdots e_{i_{p-1}}\|^2_{E_p} \\ &= \sum_{i \neq i_1, \ldots, i_{p-1}} (t^i - t_0^i)^2 \end{aligned}$$

As $\|t - t_0\|_{\mathcal{K}^n} \to 0$, we have $\|(t - t_0) \cdot b\|^2_{E_p(\mathcal{A})} \to 0$ as required. ▫

Theorem 2.12. *Let $\mathcal{A}$ be a hyperplane arrangement with n hyperplanes. Let $A(\mathcal{A})$ be the Orlik-Solomon algebra on $\mathcal{A}$ over the field $\mathcal{K}$, where $\mathcal{K}$ is either $\mathbb{C}$ or $\mathbb{R}$. The function $t \mapsto \dim H^p(A(\mathcal{A}), t)$ from $A_1(\mathcal{A})$ to $\mathbb{Z}$ is upper semicontinuous.*

Proof. We first identify $A_1(\mathcal{A})$ with $\mathcal{K}^n$.

The result clearly holds for $t_0 = 0 \in A_1(\mathcal{A})$. That is,

$$\limsup_{t \to 0} \dim H^p(A(\mathcal{A}), t) \leq \dim H^p(A(\mathcal{A}), 0) = \dim A_p(\mathcal{A}).$$

Fix $0 \neq t_0 \in \mathcal{K}^n$. Let $\epsilon = \min\{\gamma(t_0\cdot), \gamma(t_0\cdot^*)\}$. By Lemma 2.2, $\epsilon > 0$. As $t \to t_0$ in $\mathcal{K}^n$, by Lemma 2.11, we have $\|t \cdot - t_0 \cdot\|_{\text{op}} \to 0$. Hence, there exists $\delta > 0$ so that $\|t \cdot - t_0 \cdot\|_{\text{op}} < \epsilon$ whenever $\|t - t_0\|_{\mathcal{K}^n} < \delta$.

Consequently, we use Proposition 2.6, Proposition 2.9, and Lemma 2.11 to see $\dim \operatorname{Ker}(A(\mathcal{A}), t) \leq \dim \operatorname{Ker}(A(\mathcal{A}), t_0)$ and $\operatorname{rank}(A(\mathcal{A}), t) \geq \operatorname{rank}(A, t_0)$. Thus,

$$\begin{aligned} \dim H^p(A(\mathcal{A}), t) &= \dim Z_p(A(\mathcal{A}), t) - \dim B_{p-1}(A(\mathcal{A}), t) \\ &\leq \dim Z_p(A(\mathcal{A}), t_0) - \dim B_{p-1}(A(\mathcal{A}), t_0) \\ &= \dim H^p(A(\mathcal{A}), t_0). \end{aligned}$$

The assertion now follows. □

3 Tensor Products in the Category of Graded Commutative Algebras

Let $\mathcal{K}$ be a commutative ring. We introduce the following definitions and notational conventions, as can be found in [7].

Definition 3.1. (Tensor Product of Modules) Let M and N be $\mathcal{K}$-modules. The tensor product $M \otimes N$ is the abelian group with generators being all symbols $m \otimes n$ for $m \in M$ and $n \in N$ subject to the relations ($k \in \mathcal{K}$)
(i) $(m + m', n) - (m, n) - (m', n)$
(ii) $(m, n + n') - (m, n) - (m, n')$
(iii) $(km, n) - (m, kn)$.
There exists a bilinear map $\phi : M \times N \to M \otimes N$ so that $\phi(m, n) = m \otimes n$. We have the following universal property. Let A be a $\mathcal{K}$-module. For any bilinear homomorphism $f : M \times N \to A$, there exists an unique $\hat{f} : M \otimes N \to A$ so that $f(m, n) = \hat{f}(m \otimes n)$.

Definition 3.2. (Graded Module) We say M is a graded $\mathcal{K}$-module if there is a family of $\mathcal{K}$-modules $\{M_n\}_{n \geq 0}$ so that $M = \bigoplus_{n \geq 0} M_n$. For $m \in M_n$, we write $\deg(m) = n$.

Definition 3.3. (Tensor Product of Graded Modules) Let M and N be graded $\mathcal{K}$-modules. The tensor product $M \otimes N$ is the graded module given by

$$(M \otimes N)_n = \bigoplus_{p+q=n} M_p \otimes N_q. \tag{3.3.a}$$

Let A be a graded $\mathcal{K}$-module. Let $f : M \times N \to A$ be any bilinear graded homomorphism, there exists an unique graded homomorphism $\hat{f} : M \otimes N \to A$ so that $f(m, n) = \hat{f}(m \otimes n)$.

Definition 3.4. (Graded Commutative Algebra) M is said to be a graded commutative $\mathcal{K}$-algebra if the following are satisfied:
1. M is a graded $\mathcal{K}$-module.
2. There is an associative multiplication in M so that $M_p M_q \subseteq M_{p+q}$.
3. (Commutative) For homogeneous elements $a, b \in M$ we have

$$ab = (-1)^{\deg(a) \cdot \deg(b)}\, ba.$$

Definition 3.5. (Tensor Product of Graded Commutative Algebras) Let M and N be graded commutative $\mathcal{K}$-algebras. The tensor product $M \otimes N$ is the graded commutative $\mathcal{K}$-algebra given by
1. $M \otimes N$ is a graded $\mathcal{K}$-module defined in (3.3.a).
2. Multiplication is defined by $(m \otimes n)(m' \otimes n') := (-1)^{\deg(n)\deg(m')} mm' \otimes nn'$.
   Note: One can check that this multiplication is commutative.

We have the following universality description of $M \otimes N$. Let A be graded commutative $\mathcal{K}$-algebra. Let $f: M \times N \to A$ be a bilinear graded homomorphism with

$$f((m,n)(m',n')) = (-1)^{\deg(n)\deg(m')} f(m,n)f(m',n').$$

There exists an unique $\hat{f}: M \otimes N \to A$ so that $f(m,n) = \hat{f}(m \otimes n)$.

Example 3.6. Let $\mathcal{A}_i$ be arrangements. Let $A(\mathcal{A}_i)$ denote the Orlik-Solomon algebra on the arrangement $\mathcal{A}_i$ over the commutative ring $\mathcal{K}$. Then $A(\mathcal{A}_i)$ is a graded commutative algebra over $\mathcal{K}$. Hence, we have defined $\bigotimes_i A(\mathcal{A}_i)$.

We recall the product arrangement. Let $\mathcal{A}_1$ be an arrangement in V_1, and let $\mathcal{A}_2$ be an arrangement in V_2. If $V = V_1 \oplus V_2$, then we put

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 = \{H \oplus V_2 : H \in \mathcal{A}_1\} \cup \{V_1 \oplus H : H \in \mathcal{A}_2\}.$$

We recall the broken circuit basis for the Orlik-Solomon algebra $A(\mathcal{A})$. Let $S = \{H_{i_1}, ..., H_{i_p}\}$ be an ordered subset of $\mathcal{A}$ with $i_1 < \cdots < i_p$. We say a_S is basic in $A_p(\mathcal{A})$ if
1. S is independent, and
2. For any $1 \le k \le p$, there does not exist a hyperplane $H \in \mathcal{A}$ so that $H < H_{i_k}$ with $\{H, H_{i_k}, H_{i_{k+1}}, ..., H_{i_p}\}$ dependent.

Lemma 3.7. *If $a_\gamma \in A_p(\mathcal{A}_1)$ and $a_\sigma \in A_q(\mathcal{A}_2)$ are basic, then $a_\gamma a_\sigma$ is basic in $A(\mathcal{A}_1 \times \mathcal{A}_2)$.*

Proof. Order the hyperplanes in $\mathcal{A}_1 \times \mathcal{A}_2$ via
1. $H \oplus V_2 < K \oplus V_2$ if $H < K$ in $\mathcal{A}_1$
2. $V_1 \oplus H < V_1 \oplus K$ if $H < K$ in $\mathcal{A}_2$
3. $H \oplus V_2 < V_1 \oplus K$ if $H \in \mathcal{A}_1$ and $K \in \mathcal{A}_2$

Let $a_\gamma \in A_p(\mathcal{A}_1)$ and $a_\sigma \in A_q(\mathcal{A}_2)$ be basic (i.e. in the broken circuit basis). Suppose $\mathcal{A}_1$ is an arrangement in V_1 and $\mathcal{A}_2$ is an arrangement in V_2. Suppose $a_\gamma a_\sigma$ is not basic in $A(\mathcal{A}_1 \times \mathcal{A}_2)$. By definition of the broken circuit basis, there are only two possibilities. Suppose $(\cap_{H\in\gamma}(H \oplus V_2)) \cap (\cap_{H\in\sigma}(V_1 \oplus H)) = \emptyset$. This happens only if $\cap_{H\in\gamma} H = \emptyset$ or $\cap_{H\in\sigma} H = \emptyset$. This is not possible since a_σ and a_γ are basic.

Suppose there exists a hyperplane $H \in \mathcal{A}_1 \times \mathcal{A}_2$ and a subset ρ of $\gamma \cup \sigma$ with $H < \rho$ so that $\{H, \rho\}$ is dependent. But this implies the linear functionals defining the hyperplanes are linearly dependent. Since $Q(\mathcal{A}_1)$ and $Q(\mathcal{A}_2)$ have no common variables, this implies H is dependent upon $\rho \cap \gamma$ or $\rho \cap \sigma$. This contradicts the fact a_γ and a_σ are basic. Our assertion now follows. ◻

Lemma 3.8. *For the product arrangement, we have $A(\mathcal{A}_1 \times \mathcal{A}_2) \cong A(\mathcal{A}_1) \otimes A(\mathcal{A}_2)$.*

Proof. We define the map $\phi: A(\mathcal{A}_1) \times A(\mathcal{A}_2) \to A(\mathcal{A}_1 \times \mathcal{A}_2)$ on the generators by $\phi(a_\gamma, a_\sigma) := a_\gamma a_\sigma$, and we extend the map ϕ bilinearly.

Moreover, we have

$$\begin{aligned}\phi((a_\gamma, a_\sigma)\cdot(a_{\gamma'}, a_{\sigma'})) &= \phi(a_\gamma a_{\gamma'}, a_\sigma a_{\sigma'})\\ &= a_\gamma a_{\gamma'} a_\sigma a_{\sigma'}\\ &= (-1)^{\deg(\sigma)\cdot\deg(\gamma')} a_\gamma a_\sigma a_{\gamma'} a_{\sigma'}\\ &= (-1)^{\deg(\sigma)\cdot\deg(\gamma')}\phi(a_\gamma, a_\sigma)\cdot\phi(a_{\gamma'}, a_{\sigma'}).\end{aligned}$$

By the universal mapping property, there exists

$$\hat{\phi}:\ A(\mathcal{A}_1)\otimes A(\mathcal{A}_2)\to A(\mathcal{A}_1\times A_2)$$

so that $\phi=\hat{\phi}\pi$, where $\pi: A(\mathcal{A}_1)\times A(\mathcal{A}_2)\to A(\mathcal{A}_1)\otimes A(\mathcal{A}_2)$ is the canonical projection. Now, $\hat{\phi}$ is clearly surjective. All that remains is to verify injectivity.

Let $a_\gamma\in A_p(\mathcal{A}_1)$ and $a_\sigma\in A_q(\mathcal{A}_2)$ be basic. Suppose $\hat{\phi}(\sum\alpha_i a_{\gamma_i}\otimes a_{\sigma_i})=0$. Then by the linearity of $\hat{\phi}$, we have $\sum\alpha_i a_{\gamma_i}a_{\sigma_i}=0$. Since a_{γ_i} and a_{σ_i} are basic in $A(\mathcal{A}_1)$ and $A(\mathcal{A}_2)$, we have $a_{\gamma_i}a_{\sigma_i}$ is basic in $A(\mathcal{A}_1\times\mathcal{A}_2)$. Hence, we must have $\alpha_i=0$ for each i. ◻

Suppose $\mathcal{A}=\mathcal{A}_1\times\mathcal{A}_2$. By Lemma 3.7, $A_1(\mathcal{A})$ can be identified with the linear space $A_1(\mathcal{A}_1)\oplus A_1(\mathcal{A}_2)$. Let $a\in A_1(\mathcal{A})$. We may express $a=a_1+a_2$ uniquely for $a_1\in A_1(\mathcal{A})$ and $a_2\in A_1(\mathcal{A}_2)$. For the chain complexes $(A(\mathcal{A}_1),a_1)$ and $(A(\mathcal{A}_2),a_2)$, we recall tensor products of chain complexes; see [8].

Let the differential (multiplication by a_i) for the complex $(A(\mathcal{A}_i),a_i)$ be denoted d_i for $i=1,2$. The differential for the chain complex $(A(\mathcal{A}_1),a_1)\otimes(A(\mathcal{A}_2),a_2)$, written $d_1\otimes d_2$, is defined on generators as

$$(d_1\otimes d_2)(a_\gamma\otimes a_\sigma):=a_1a_\gamma\otimes a_\sigma+(-1)^{\deg(a_\gamma)}a_\gamma\otimes a_2a_\sigma.$$

Lemma 3.9. *Let $\mathcal{A}=\mathcal{A}_1\times\mathcal{A}_2$. Let $a\in A_1(\mathcal{A})$ with $a=a_1+a_2$ for $a_1\in A_1(\mathcal{A}_1)$ and for $a_2\in A_1(\mathcal{A}_2)$. As chain complexes, $(A(\mathcal{A}_1\times\mathcal{A}_2),a)=(A(\mathcal{A}_1)\otimes A(\mathcal{A}_2),d_1\otimes d_2)$*

Proof. From Lemma 3.8, we have $A(\mathcal{A}_1\times\mathcal{A}_2)=A(\mathcal{A}_1)\otimes A(\mathcal{A}_2)$. For a basic element $a_\rho\in A_p(\mathcal{A}_1\times\mathcal{A}_2)$, we may write $a_\rho=a_\gamma\cdot a_\sigma$ with $a_\gamma\in A_m(\mathcal{A}_1)$ and $a_\sigma\in A_n(\mathcal{A}_2)$ and $m+n=p$. Hence, multiplying by a in the chain complex $A(\mathcal{A}_1\times\mathcal{A}_2)$ gives the differential defined on generators as

$$\begin{aligned}a\cdot a_\rho=&(a_1+a_2)(a_\gamma\cdot a_\sigma)\\ =&a_1a_\gamma\cdot a_\sigma+(-1)^{\deg(a_\gamma)}a_\gamma\cdot a_2a_\sigma.\end{aligned}$$

The result follows immediately. ◻

Theorem 3.10. *Let $\mathcal{A}=\mathcal{A}_1\times\mathcal{A}_2$ be a product arrangement. Let $a\in A_1(\mathcal{A}_1\times\mathcal{A}_2)$. Write $a=a_1+a_2$ for $a_1\in A_1(\mathcal{A}_1)$ and $a_2\in A_1(\mathcal{A}_2)$. Let $\mathcal{K}$ be a field. We have:*

$$H^m(A(\mathcal{A}_1\times\mathcal{A}_2),a)=\bigoplus_{p+q=m}H^p(A(\mathcal{A}_1),a_1)\otimes H^q(A(\mathcal{A}_2),a_2).$$

Proof. By Lemma 3.9, this is a direct application of the Künneth Formula (see [7]) to the cochain complex $(A(\mathcal{A}_1)\otimes A(\mathcal{A}_2),d_1\otimes d_2)$. ◻

4 $H^*(A(\mathcal{A}), a)$

In this section, we use the results of §2 and §3 to establish necessary and sufficient conditions for $H^*(A(\mathcal{A}), a) = 0$.

We recall an arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ is central if $\cap_{i=1}^n H_i \neq \emptyset$. An affine arrangement $\mathcal{A}$ is an arrangement that may or may not be central.

Let $\mathcal{A}$ be an affine arrangement. We may write

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k \times \mathcal{B},$$

where $\mathcal{A}_i$ are each central and $\mathcal{B}$ is not central. Moreover, we may assume each $\mathcal{A}_i$ contains no proper central factors and $\mathcal{B}$ contains no central factors; otherwise, we would decompose the arrangement further. For $a \in A_1(\mathcal{A})$, write $a = a_1 + \cdots + a_k + a_{\mathcal{B}}$ for $a_j \in A_1(A_j)$ and $a_{\mathcal{B}} \in A_1(\mathcal{B})$.

Example 4.1. Let $Q(\mathcal{A}) = x(x-1)y(y-1)$. Although $\mathcal{A}$ is a product of affine arrangements, $\mathcal{A}$ contains no central factors. Hence, $\mathcal{A} = \mathcal{B}$ in this case.

We recall the deconing process. Suppose $\mathcal{A}$ is central. We form the deconed arrangement, $d\mathcal{A}$, of $\mathcal{A}$ as follows. Let α_i be the functional corresponding to H_i. Without loss of generality, we may assume $\alpha_1 = x_1$. Decone at $\alpha_1 = x_1$ by setting $x_1 = 1$.

We recall the coning process. Let $\mathcal{A}$ be an affine arrangement given by a polynomial $Q = \prod_{i=1}^n \alpha_i$ in the affine space $V = \{(x_1, \ldots, x_\ell)\}$. Put $cV = \{(x_0, x_1, \ldots, x_\ell)\}$ and regard V as the hyperplane of cV defined by $x_0 = 1$. Let $S = F[x_1, \ldots, x_\ell]$ and $S^c = F[x_0, x_1, \ldots, x_\ell]$ be the coordinate rings of V and cV, respectively. For each $g \in S$ define the homogenization of g as

$$g^h = x_0^{\deg g} g(x_1/x_0, \ldots, x_\ell/x_0)$$

where $\deg g$ is the degree of g. Then the coned arrangement $\mathcal{A}^c$ is the central arrangement in cV defined by the polynomial $Q^c = x_0 \prod_{i=1}^n \alpha_i^h$.

Lemma 4.2. *If $\mathcal{A}$ is a central hyperplane arrangement and $\mathcal{A}$ contains no proper central factor, then $d\mathcal{A}$ contains no central factor.*

Proof. Suppose $d\mathcal{A}$ contains a central factor. There exist subarrangements $\mathcal{C}_1$ and $\mathcal{C}_2$ of $d\mathcal{A}$ so that $d\mathcal{A} = \mathcal{C}_1 \times \mathcal{C}_2$. Moreover, wemay assume $\mathcal{C}_1$ is central. Since $\mathcal{C}_1$ is central, by taking a linear change of coordinates if necessary, we may assume the hyperplanes of $\mathcal{C}_1$ pass through the origin; i.e. we are assuming the defining equation $Q(\mathcal{C}_1)$ consists of linear functionals. Then by coning, we obtain $\mathcal{A}$. Since the defining equation of $\mathcal{C}_1$ is unaffected by coning the arrangement $d\mathcal{A}$, we have constructed a central factor of $\mathcal{A}$. This contradicts the assumption that $\mathcal{A}$ contains no central factors. ◻

Example 4.3. To demonstrate the proof of Lemma 4.2, we consider an arrangement $\mathcal{A}$ where both $\mathcal{A}$ and $d\mathcal{A}$ contain a central factor. Let $Q(\mathcal{A}) = z(x-z)(y-z)(w-z)w$. When deconing at $z = 1$, we have $Q(d\mathcal{A}) = (x-1)(y-1)(w-1)w$. Take $Q(\mathcal{C}_1) = (x-1)(y-1)$ and $Q(\mathcal{C}_2) = (w-1)w$. Let $\hat{x} = x - 1$ and $\hat{y} = y - 1$. We have $Q(d\mathcal{A}) = \hat{x}\hat{y}(w-1)w$. When coning, we have $Q((d\mathcal{A})^c) = z\hat{x}\hat{y}(w-z)w$. By taking the linear change of coordinate $\hat{x} = x - z$ and $\hat{y} = y - z$, we see that $\mathcal{A}$ and $(d\mathcal{A})^c$ are linearly isomorphic.

The rank of $\mathcal{A}$ is defined by

$$r(\mathcal{A}) = \max_{X \in L(\mathcal{A})} \operatorname{rank}(X).$$

Let $\operatorname{rank}(\mathcal{A}) = \ell$. The Euler characteristic is given by

$$\chi(\mathcal{A}) = \sum_{i=1}^{\ell} (-1)^i \dim A_i(\mathcal{A}).$$

We also note that $\chi(d\mathcal{A})$ depends only on $L(\mathcal{A})$, see [11].

Let $\mathcal{A}$ be an arrangement. Let $H_0 \in \mathcal{A}$. We recall the arrangements given by deletion and restriction

$$\mathcal{A}' = \{H : \ H \in \mathcal{A} \setminus H_0\}, \text{and}$$

$$\mathcal{A}'' = \{H_0 \cap H : \ H \in \mathcal{A} \text{ and } H \cap H_0 \neq \emptyset\}.$$

We need the following lemmas and proposition, established in [5].

Lemma 4.4. *Let $\mathcal{A}$ be an affine arrangement with $r(\mathcal{A}) > 1$. If $\mathcal{A}$ does not contain a central factor, then for any distinguished hyperplane $H_0 \in \mathcal{A}$ either $\mathcal{A}'$ or A'' does not contain a central factor.*

Proof. We refer to the proof given in Lemma to Theorem II in [5]. ▫

We define

$$\beta(\mathcal{A}) := (-1)^{r(\mathcal{A})} \chi(\mathcal{A}).$$

Lemma 4.5. *Let $\mathcal{A}$ be an arrangement with $|\mathcal{A}| > 1$. If $\mathcal{A}$ is not central, then there exists $H_0 \in \mathcal{A}$ so that $\operatorname{rank}(\mathcal{A}') = \operatorname{rank}(\mathcal{A})$. With respect to H_0, we have the equality $\beta(\mathcal{A}) = \beta(\mathcal{A}') + \beta(\mathcal{A}'')$. If $\mathcal{A}$ is central, then this inequality holds for any $H \in \mathcal{A}$.*

Proof. Suppose $\mathcal{A}$ is not central. Then there exists a maximal element $T \in L(\mathcal{A})$ and a hyperplane $H_0 \not< T$. Hence, T is a maximal element in $L(\mathcal{A}')$. Since $\chi(A) = \chi(\mathcal{A}') - \chi(\mathcal{A}'')$ by Theorem 2.56 in [10], we have

$$(-1)^{r(\mathcal{A})} \chi(\mathcal{A}) = (-1)^{r(\mathcal{A})} \chi(\mathcal{A}') - (-1)^{r(\mathcal{A})} \chi(\mathcal{A}'').$$

We have $r(\mathcal{A}'') = r(\mathcal{A}) - 1$ and $\operatorname{rank}(\mathcal{A}') = \operatorname{rank}(\mathcal{A})$, so

$$\beta(\mathcal{A}) = \beta(\mathcal{A}') + \beta(\mathcal{A}'').$$

If $\mathcal{A}$ is central, then $\mathcal{A}'$ and $\mathcal{A}''$ are both central, so $\chi(A) = \chi(\mathcal{A}') = \chi(\mathcal{A}'') = 0$. ▫

Lemma 4.6. *If $\mathcal{A}$ is an arrangement, then $\beta(\mathcal{A}) \geq 0$.*

Proof. We induct on $|\mathcal{A}|$.

If $\mathcal{A} = \emptyset$, then $\beta(\mathcal{A}) = 1$. If $|\mathcal{A}| = 1$, then $\beta(\mathcal{A}) = 0$.

Assume $\beta(\mathcal{B}) \geq 0$ for all arrangements $\mathcal{B}$ with $|\mathcal{B}| < k$. Suppose $|\mathcal{A}| = k > 1$. By Lemma 4.5, we have $\beta(\mathcal{A}) = \beta(\mathcal{A}') + \beta(\mathcal{A}'')$ for some hyperplane $H_0 \in \mathcal{A}$. By the induction hypothesis, we have $\beta(\mathcal{A}'), \beta(\mathcal{A}'') \geq 0$. We therefore have $\beta(\mathcal{A}) \geq 0$ as required. ▫

Proposition 4.7. *Let $\mathcal{A}$ be an affine arrangement. We have $\chi(\mathcal{A}) \neq 0$ if and only if $\mathcal{A}$ contains no central factors.*

Proof. Suppose $\mathcal{A}$ contains a central factor; that is, $\mathcal{A} = \mathcal{B} \times \mathcal{C}$, where $\mathcal{B}$ is central. Then $\chi(\mathcal{A}) = \chi(\mathcal{B})\chi(\mathcal{C})$; see Lemma 2.50 in [10]. Since $\mathcal{B}$ is central, we have $\chi(\mathcal{B}) = 0$; see Proposition 2.51 in [10]. Hence, $\chi(\mathcal{A}) = 0$.

Suppose $\mathcal{A}$ contains no central factors. We want to show $\chi(\mathcal{A}) \neq 0$. It will suffice to show $\beta(\mathcal{A}) \neq 0$. We proceed by induction. Suppose $|\mathcal{A}| = 2$, then $\mathcal{A}$ consists of two hyperplanes which don't intersect; hence, $\chi(\mathcal{A}) = -1$.

Suppose for any $\mathcal{B}$ with $|\mathcal{B}| < k$ $(k > 1)$ for which $\mathcal{B}$ contains no central factors, we have $\beta(\mathcal{B}) \neq 0$ (hence, $\chi(\mathcal{B}) \neq 0$). Fix $H_0 \in \mathcal{A}$ so that $\beta(\mathcal{A}) = \beta(\mathcal{A}') + \beta(\mathcal{A}'')$. We apply Lemma 4.6 to see that $\mathcal{A}'$ or $\mathcal{A}''$ contains no central factors. By the induction hypothesis, $\beta(\mathcal{A}') > 0$ or $\beta(\mathcal{A}'') > 0$. Since $\beta(\mathcal{A}) = \beta(\mathcal{A}') + \beta(\mathcal{A}'')$ and $\beta(\mathcal{A}'), \beta(\mathcal{A}'') \geq 0$ with at least one positive, we have $\beta(\mathcal{A}) > 0$. Hence, $\chi(\mathcal{A}) \neq 0$ as required. ▫

Theorem 4.8. *Let $\mathcal{A}$ be an affine arrangement. We may write*

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k \times \mathcal{B},$$

where $\mathcal{A}_j$ are each central and $\mathcal{B}$ not central and they contain no proper central factors. For $a \in A_1(\mathcal{A})$, write $a = a_1 + \cdots + a_k + a_{\mathcal{B}}$ for $a_j \in A_1(A_j)$ and $a_{\mathcal{B}} \in A_1(\mathcal{B})$. We have $H^*(A(\mathcal{A}), a) \neq 0$ if and only if $\sum_{H_i \in \mathcal{A}_j} \lambda_i = 0$ for all $1 \leq j \leq k$.

Proof. We use the Künneth Formula from Theorem 3.10; that is,

$$H^m(A(\mathcal{A}), a) = \bigoplus_{p_1 + \cdots p_k + p_{k+1} = m} H^{p_1}(A(\mathcal{A}_1), a_1) \otimes \cdots \otimes H^{p_k}(A(\mathcal{A}_k), a_k) \otimes H^{p_{k+1}}(A(\mathcal{B}), a_{\mathcal{B}}).$$

Suppose $\sum_{A_1(\mathcal{A}_j)} \lambda_i \neq 0$ for some j. We have $H^*(\mathcal{A}_j, a_j) = 0$ since $\mathcal{A}_j$ is central; we refer to [11]. By the Künneth Formula, it follows that $H^*(\mathcal{A}, a) = 0$.

Suppose $\sum_{A_1(\mathcal{A}_j)} \lambda_i = 0$ for all j. By the Künneth Formula, it will suffice to show $H^*(A(\mathcal{A}_i), a_i) \neq 0$ and $H^*(A(\mathcal{B}), a_{\mathcal{B}}) \neq 0$. Since $\mathcal{B}$ contains no central factors, we have $\chi(\mathcal{B}) \neq 0$. Hence, $H^*(A(\mathcal{B}), a_{\mathcal{B}}) \neq 0$.

Take $\tilde{a} = \displaystyle\sum_{i=2}^{n} \lambda_i a_i$. We consider the chain complex formed by multiplication by $\tilde{a}$, $(d\mathcal{A}, \tilde{a})$; here, $d\mathcal{A}$ is $\mathcal{A}$ deconed at H_1. Since we have the short exact sequences, see [11]

$$0 \to H^{p-1}(A(d\mathcal{A}_i), \tilde{a}) \to H^p(A(\mathcal{A}_i), a) \to H^p(A(d\mathcal{A}_i), \tilde{a}) \to 0,$$

it will suffice to show $H^*(A(d\mathcal{A}_i), \tilde{a}) \neq 0$. But by Lemma 4.2, $d\mathcal{A}_i$ contains no central factors, so by Proposition 4.7 $\chi(d\mathcal{A}_i) \neq 0$; hence, $H^*(A(d\mathcal{A}_i), \tilde{a}) \neq 0$ as required. ▫

We recall the following theorem from [11].

Theorem 4.9. (*Yuzvinsky* [11]) *Let* $\mathcal{A} = \{H_1, \dots, H_n\}$ *be an affine arrangement with* $\operatorname{rank}(\mathcal{A}) = \ell$. *Let* $a = \sum_{i=1}^{n} \lambda_i a_i \in A_1(\mathcal{A})$ *and satisfy the condition* $\sum_{X \subset H_i} \lambda_i \neq 0$ *for all* $X(\mathcal{A})$ *such that* $\chi(dA(\mathcal{A}_X)) \neq 0$. *Then* $H^p(A(\mathcal{A}), a) = 0$ *for every* $p < \ell$.

We use Theorem 4.9 in conjunction with the upper semicontinuity of the map $t \mapsto \dim H^p(A(\mathcal{A}), t)$ discussed in §2 in order to establish conditions under which $\dim H^\ell(A(\mathcal{A}), a) \neq 0$ for an affine ℓ-arrangement $\mathcal{A}$.

Theorem 4.10. *Let* $\mathcal{A}$ *be an arbitrary* ℓ*-arrangement with* $|\mathcal{A}| = n$. *Suppose* $\mathcal{A}$ *contains no central factors. Let* $a = \sum_{i=1}^{n} \lambda_i a_i \in A_1(\mathcal{A})$. *We have:*

$$H^\ell(A(\mathcal{A}), a) \neq 0.$$

Proof. If $a = 0$, then $H^\ell(A(\mathcal{A}), a) = Z_\ell(A(\mathcal{A}), a) = A_\ell(\mathcal{A}) \neq 0$. Hence, we may assume $a \neq 0$ for the remainder of the proof.

Let

$$S = \{X \in L(\mathcal{A}) : \sum_{X \subset H_j} \lambda_j \neq 0\}.$$

Since $a \neq 0$, we have $S \neq \emptyset$. We define

$$\kappa := \min \{|\sum_{X \subset H_j} \lambda_j| : \ X \in S\}.$$

Since $S \neq \emptyset$, $\kappa > 0$.

We now construct a sequence $\alpha_i \in F^n$ so that $(\alpha_i)_j \to \lambda_j$ as $i \to \infty$. For $i \in \mathbb{N}$, define

$$(\alpha_i)_j := \lambda_j + \frac{\kappa}{n \cdot 2^i}.$$

It is clear that $(\alpha_i)_j \to \lambda_j$ as $i \to \infty$. Moreover, we now show $\sum_{X \subset H_j} (\alpha_i)_j \neq 0$ for any $X \in L(\mathcal{A})$ and any $i \in \mathbb{N}$.

Fix $X \in L(\mathcal{A})$. If $\sum_{X \subset H_j} \lambda_j = 0$, then $\sum_{X \subset H_j} (\alpha_i)_j \neq 0$ since $\kappa > 0$.

Suppose $\sum_{X \subset H_j} \lambda_j \neq 0$. If $\sum_{X \subset H_j} \lambda_j > 0$, then $\sum_{X \subset H_j} (\alpha_i)_j > 0$ since $\dfrac{\kappa}{n \cdot 2^i} > 0$.

Suppose $\sum_{X \subset H_j} \lambda_j < 0$. Then

$$\begin{aligned}
\sum_{X \subset H_j} (\alpha_i)_j &= \sum_{X \subset H_j} (\lambda_j + \frac{\kappa}{n \cdot 2^i}) \\
&= (\sum_{X \subset H_j} \lambda_j) + \frac{|X| \cdot \kappa}{n \cdot 2^i} \\
&\leq (\sum_{X \subset H_j} \lambda_j) + \frac{\kappa}{2^i} \\
&< 0,
\end{aligned}$$

where the last inequality is true because of the definition of κ.

Therefore, for any $i \in \mathbb{N}$, we have α_i satisfies the condition of Theorem 4.9 ensuring $\dim H^p(A(\mathcal{A}), \alpha_i) = 0$ for $p < \ell$. Since $\mathcal{A}$ contains no central factors, we have $H^*(A(\mathcal{A}), \alpha_i) \neq 0$; hence, $\dim H^\ell(A(\mathcal{A}), \alpha_i) \neq 0$. By Theorem 2.12, the function $t \mapsto \dim H^\ell(A(\mathcal{A}), t)$ is upper semicontinuous in t; therefore, $\dim H^\ell(A(\mathcal{A}), a) > 0$ as required. □

Theorem 4.11. *Let $\mathcal{A}$ be an affine ℓ-arrangement. We may write*

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k \times \mathcal{B},$$

where $\mathcal{A}_j$ are each central and $\mathcal{B}$ not central and they contain no proper central factors. Let $a \in A_1(\mathcal{A})$. We have $\dim H^\ell(A(\mathcal{A}), a) \neq 0$ *if and only if* $\sum_{H_i \in \mathcal{A}_j} \lambda_i = 0$ *for all j.*

Proof. Let $\mathcal{A}_j$ be an ℓ_j-arrangement, and let $\mathcal{B}$ be an $\ell_{\mathcal{B}}$-arrangement. Then $\mathcal{A}$ is an $(\ell_{\mathcal{B}} + \sum_{i=1}^{k} \ell_j)$-arrangement. Since we have the short exact sequences

$$0 \to H^{p-1}(A(d\mathcal{A}_i), \tilde{a}) \to H^p(A(\mathcal{A}_i), a) \to H^p(A(d\mathcal{A}_i), \tilde{a}) \to 0$$

and the Künneth Formula

$$H^m(A(\mathcal{A}), a) = \bigoplus_{p_1 + \cdots p_k + p_{k+1} = m} H^{p_1}(A(\mathcal{A}_1), a_1) \otimes \cdots \otimes H^{p_k}(A(\mathcal{A}_k), a_k) \otimes H^{p_{k+1}}(A(\mathcal{B}), a_{\mathcal{B}}),$$

it will suffice to show $H^{\ell_j - 1}(A(d\mathcal{A}_j), \tilde{a}_j) \neq 0$ and $H^{\ell_{\mathcal{B}}}(A(\mathcal{B}), a_{\mathcal{B}}) \neq 0$. This result was established in Theorem 4.10. □

Theorem 4.12. *Let $\mathcal{A}$ be an arrangement with $\ell = \operatorname{rank}(\mathcal{A})$. Fix $a \in A_1(\mathcal{A})$. Then $H^*(A(\mathcal{A}), a) = 0$ if and only if $H^\ell(A(\mathcal{A}), a) = 0$.*

Proof. This follows immediately from Theorem 4.11 and Theorem 4.8. □

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