Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISBN 1083-0464
Issue 1 June 2001, pp. 66 – 74
Submitted: March 22, 2001. Published: June 30, 2001

Submanifolds of F-Structure Manifold Satisfying $F^K + (-)^{K+1}F = 0$

LoveJoy S. Das

Abstract

Invariant submanifolds have been studied by Blair, Ludden, Yano [1], Kubo [2], Yano and Okumura [3], [4] and among others. Yano and Ishihara [5] have studied and shown that any invariant submanifold of codimension 2 in a contact Riemannian manifold is also a contact Riemannian manifold. The purpose of this paper is to study invariant submanifolds of an n-dimensional manifold M endowed with a F-structure satisfying $F^K+(-)^{K+1}F = 0$ and $F^W+(-)^{W+1}F \neq 0$ for $1 < W < K$, where K is a fixed positive integer greater than 2. The case when K is odd (≥ 3) has been considered in this paper. In the first part of the paper we have shown that an invariant submanifold $\widetilde{M}$ embedded in an F-structure manifold M in such a way that the complementary distribution D_m is never tangential to the invariant submanifold $\Psi(\tilde{M})$, is an almost complex manifold with the induced $\widetilde{F}$-structure. In the later part of the paper some theorems regarding the integrability conditions of induced $\widetilde{F}$-structure have been proved.

A.M.S. Subject Classification Codes: 53C15, 53C60, 53B40
Key Words and Phrases: Invariant submanifold, integrability, F-structure

1. F-STRUCTURE

Let F be a nonzero tensor field of the type (1,1) and of class C^∞ on an n-dimensional manifold M such that [6]

$$F^K+(-)^{K+1}F = 0 \text{ and } F^W+(-)^{W+1}F \neq 0 \text{ for } 1 < W < K \quad (1.1)$$

Department of Mathematics and Computer Science, Kent State University, USA

where K is a fixed positive integer greater than 2. Such a structure on M is called an F-structure of rank r and of degree K. If the rank of F is constant and $r = r(F)$, then M is called an F-structure manifold of degree $K(\geq 3)$.

Let the operator on M be defined as follows [6]

$$\ell = (-)^K F^{K-1} \quad \text{and} \quad m = I + (-)^{K+1} F^{K-1} \tag{1.2}$$

where I denotes the identity operator on M. For the operators defined by (1.2) we have

$$\ell + m = I, \quad \ell^2 = \ell; \quad \text{and} \quad m^2 = m. \tag{1.3}$$

For F-satisfying (1.1), there exist complementary distribution D_ℓ and D_m corresponding to the projection operators ℓ and m respectively. If rank (F) = constant on M then dim $D_\ell = r$ and dim $D_m = (n - r)$. We have the following results [6].

$$F\ell = \ell F = F \quad \text{and} \quad Fm = mF = 0 \tag{1.4a}$$

$$F^{K-1} = (-)^K \ell, \quad F^{K-1}\ell = -\ell \quad\quad F^{K-1}m = 0. \tag{1.4b}$$

Thus F^{K-1} acts on D_ℓ as an almost complex structure and on D_m as a null operator.

2. INVARIANT SUBMANIFOLDS OF F-STRUCTURE MANIFOLD

Let $\widetilde{M}$ be a differentiable manifold embedded differentially as a submanifold in an n-dimensional C^∞ Riemannian manifold M with an F-structure and let us denote its embedding by $\Psi : \widetilde{M} \to M$. Denote by $B : T(\widetilde{M}) \longrightarrow T(M)$ the differential mapping of Ψ, where $d\Psi = B$ is the Jacobson map of Ψ. $T(\widetilde{M})$ and $T(M)$ are tangent bundles of $\widetilde{M}$ and M respectively. Let us call $T(\widetilde{M}, M)$ as the set of all vectors tangent to the submanifold $\Psi(\widetilde{M})$. It is known that $B : T(\widetilde{M}) \longrightarrow T(\widetilde{M}, M)$ is an isomorphism [7].

Let $\widetilde{X}$ and $\widetilde{Y}$ be two C^∞ vector fields defined along $\Psi(\widetilde{M})$ and tangent to $\Psi(\widetilde{M})$. Let X and Y be the local extensions of $\widetilde{X}$ and $\widetilde{Y}$. The restriction of $[X, Y]_{\widetilde{M}}$ is determined independently of the choice of these local extensions X and Y. Therefore we can define

$$[\widetilde{X}, \widetilde{Y}] = [X, Y]_{\widetilde{M}}.$$

Since B is an isomorphism it is easy to see that $[B\widetilde{X}, B\widetilde{Y}] = B[\widetilde{X}, \widetilde{Y}]$ for all $\widetilde{X}, \widetilde{Y} \in T(\widetilde{M})$.

We denote by G, the Riemannian metric tensor of M and put

$$\tilde{g}(\widetilde{X}, \widetilde{Y}) = g(B\widetilde{X}, B\widetilde{Y}) \tag{2.1}$$

for all $\widetilde{X}, \widetilde{Y}$ in $T(\widetilde{M})$, where g is the Riemannian metric in M and $\tilde{g}$ is the induced metric of $\widetilde{M}$.

Definition(2.1): We say that $\widetilde{M}$ is an invariant submanifold of M if

(i) the tangent space $T_p(\Psi(\widetilde{M}))$ of the submanifold $\Psi(\widetilde{M})$ is invariant by the linear mapping F at each point p of $\Psi(\widetilde{M})$.

(ii) For each $\widetilde{X} \in T(\widetilde{M})$, we have

$$F^{\frac{K-1}{2}}(B\widetilde{X}) = B\widetilde{X}'.$$

Definition (2.2): Let $\widetilde{F}$ be a (1,1) tensor field defined in $\widetilde{M}$ such that $\widetilde{F}(\widetilde{X}) = \widetilde{X}'$ and M is an invariant submanifold then we have

$$F(B\widetilde{X}) = B(\widetilde{F}\widetilde{X}) \tag{2.2a}$$

and

$$F^{\frac{K-1}{2}}(B\widetilde{X}) = B(\widetilde{F}^{\frac{K-1}{2}}\widetilde{X}). \tag{2.2b}$$

We see that there are two cases for any invariant submanifold $\widetilde{M}$. We assume the following: Case I: The distribution D_m is never tangential to $\Psi(\widetilde{M})$.
Case II: The distribution D_m is always tangential to $\Psi(\widetilde{M})$.

We will consider Case I and assume that no vector field of the type mX where $X \in T(\Psi(\widetilde{M}))$ is tangential to $\Psi(\widetilde{M})$.

Theorem (2.1): *An invariant submanifold $\widetilde{M}$ is an almost complex manifold if the following two conditions are satisfied:*

(i) *the distribution D_m is never tangential to $\Psi(\widetilde{M})$ and*

(ii) $\widetilde{F}$ in $\widetilde{M}$ *defines an induced almost complex structure satisfying* $\widetilde{F}^{K-1} = (-)^K I$.

Proof: Applying $F^{\frac{K-1}{2}}$ in (2.2) we obtain

$$F^{\frac{K-1}{2}}, \quad \left(F^{\frac{K-1}{2}}(B\widetilde{X})\right) = F^{\frac{K-1}{2}}, \left(B(\tilde{F}^{\frac{K-1}{2}}, \widetilde{X})\right) \tag{2.3}$$

Making use of equation (2.2)a in (2.3) we get

$$F^{K-1}(B\widetilde{X}) = B(\tilde{F}^{K-1}\widetilde{X}) \tag{2.4}$$

In order to show that vector fields of the type $B\widetilde{X}$ belong to the distribution D_ℓ we suppose that $m(B\widetilde{X}) \neq 0$, then using (1.2) we have

$$\begin{aligned} m(B\widetilde{X}) &= (I + (-)^{K+1}F^{K-1})B\widetilde{X} \\ &= B\widetilde{X} + (-)^{K+1}F^{K-1}(B\widetilde{X}), \end{aligned}$$

which in view of (2.4) becomes

$$\begin{aligned} &= B\widetilde{X} + (-)^{K+1}B(\tilde{F}^{K-1}\widetilde{X}) \\ &= B[\widetilde{X} + (-)^{K+1}\tilde{F}^{K-1}\widetilde{X}] \end{aligned}$$

which contrary to our assumption shows that in $m(B\widetilde{X})$ is tangential to $\Psi(\widetilde{M})$. Thus $m(B\widetilde{X}) = 0$.

Also, in view of (1.4)b, (1.3) and (2.4) we obtain

$$\begin{aligned} B(\tilde{F}^{K-1}\widetilde{X}) &= F^{K-1}(B\widetilde{X}) \\ &= (-)^K\ell(B\widetilde{X}) \\ &= (-)^K(I - m)B\widetilde{X} \\ &= (-)^K B\widetilde{X} - (-)^K mB\widetilde{X} \\ B(\tilde{F}^{K-1}\widetilde{X}) &= (-)^K B\widetilde{X}. \end{aligned}$$

Since B is an isomorphism, we get

$$\tilde{F}^{K-1} = (-)^K I$$

Let $\mathcal{F}(M)$ be the ring of real valued differentiable functions on M and $\mathcal{X}(M)$ be the module of derivatives of $\mathcal{F}(M)$. Then $\mathcal{X}(M)$ is Lie algebra over the real numbers and the elements of $\mathcal{X}(M)$ are called vector fields. Then M is equipped with (1,1) tensor field F which is a linear map such that

$$F : \mathcal{X}(M) \to \mathcal{X}(M).$$

Let M be of degree K and let K be a positive odd integer greater than 2. Then we consider a positive definite Riemannian metric w, r, t which D_ℓ and D_m are orthogonal so that

$$g(X,Y) = g(HX, HY) + g(mX, Y), \tag{2.5}$$

where $H = F^{\frac{K-1}{2}}$ for all $X, Y \in \mathcal{X}(M)$.

Definition (2.3): The induced metric $\widetilde{g}$ defined by (2.1) is Hermitian if the following is satisfied

$$\widetilde{g}(H\widetilde{X}, H\widetilde{Y}) = \widetilde{g}(\widetilde{X}, \widetilde{Y}), \quad \text{where} \quad H = F^{\frac{K-1}{2}} \tag{2.6}$$

Theorem (2.2): *If F-structure manifold has the following two properties i.e.*

(a) $\widetilde{M}$ is an invariant submanifold of F-structure manifold M such that distribution D_m is never tangential to $\Psi(\widetilde{M})$.

(b) the Riemannian metric g on M is defined by (2.5), then the induced metric $\widetilde{g}$ of $\widetilde{M}$ defined by (2.1) is Hermitian.

Proof: In view of equations (2.1) and (2.6) we obtain

$$\widetilde{g}(\widetilde{F}^{\frac{K-1}{2}}\widetilde{X}, \widetilde{F}^{\frac{K-1}{2}}\widetilde{Y}) = g(B\widetilde{F}^{\frac{K-1}{2}}\widetilde{X}, B\widetilde{F}^{\frac{K-1}{2}}\widetilde{Y}).$$

Applying equation (2.2) and (2.5) in the above equation we get

$$\widetilde{g}(\widetilde{F}^{\frac{K-1}{2}}\widetilde{X}, \widetilde{F}^{\frac{K-1}{2}}\widetilde{Y}) \;=\; g(F^{\frac{K-1}{2}}B\widetilde{X}, F^{\frac{K-1}{2}}B\widetilde{Y}) \;=\; g(B\widetilde{X}, B\widetilde{Y}) - g(mB\widetilde{X}, B\widetilde{Y}) \tag{2.7}$$

Since the distribution D_m is never tangential to $\Psi(\widetilde{M})$ on using (2.1) we get:

$$\begin{aligned} &= g(B\widetilde{X}, B\widetilde{Y}) \\ &= \widetilde{g}(\widetilde{X}, \widetilde{Y}). \end{aligned}$$

Now we consider the second case and assume that the distribution D_m is always tangential to $\Psi(\widetilde{M})$. In view of the Case II, we have $m(B\widetilde{X}) = B\widetilde{X}^*$, where $\widetilde{X}^* \in T(\widetilde{M})$ for some $\widetilde{X}^* \in T(\widetilde{M})$.

Let us define (1,1) tensor fields $\widetilde{m}$ and $\widetilde{\ell}$ in $\widetilde{M}$ as follows

$$\widetilde{\ell} = (-)^K \widetilde{F}^{K-1} \text{ and } \widetilde{m} = \widetilde{I} + (-)^{K+1}\widetilde{F}^{K-1} \tag{2.7a}$$

$$\widetilde{m}\widetilde{X} = \widetilde{X}^* \text{ and } m(B\widetilde{X}) = B(\widetilde{m}\widetilde{X}) \tag{2.7b}$$

Theorem (2.4): *We have*

$$B(\widetilde{\ell}\,\widetilde{X}) = \ell(B\,\widetilde{X}) \tag{2.8}$$

Proof: In view of (2.7)a, (2.8) assumes the following form

$$\begin{aligned} B(\widetilde{\ell}\,\widetilde{X}) &= B((-)^K \widetilde{F}^{K-1}\widetilde{X}), \\ &= (-)^K B(\widetilde{F}^{K-1}\widetilde{X}), \end{aligned}$$

On making use of (2.4) and (2.7) in the above equation, we get

$$\begin{aligned} B(\widetilde{\ell}\,\widetilde{X}) &= (-)^K \widetilde{F}^{K-1}(B\widetilde{X}), \\ &= \widetilde{\ell}(B\widetilde{X})\ . \end{aligned}$$

Theorem (2.5): *For $\widetilde{\ell}$ and $\widetilde{m}$ satisfying (2.7)a, we have*

$$\widetilde{\ell} + \widetilde{m} = \widetilde{I}, \qquad \widetilde{\ell}^2 = \widetilde{\ell} \quad \text{and} \quad \widetilde{m}^2 = \widetilde{m} \tag{2.9}$$

Proof: From (1.3) we have $\ell + m = I$, which can be written as $(\ell + m)B\widetilde{X} = B\widetilde{X}$, thus we have

$$\ell B\widetilde{X} + mB\widetilde{X} = B\widetilde{X}\ .$$

which in view of (2.7)b and (2.8) becomes

$$\begin{aligned} B(\widetilde{\ell}\widetilde{X}) + B(\widetilde{m}\widetilde{X}) &= B(\widetilde{\ell} + \widetilde{m})\widetilde{X} \\ &= B\widetilde{X}. \end{aligned}$$

Therefore $\widetilde{\ell} + \widetilde{m} = I$ since B is an isomorphism. Proof of the other relations follows in a similar manner.

Theorem (2.5) shows that $\widetilde{\ell}$ and $\widetilde{m}$ defined by (2.7)a are complementary projection operators on $\widetilde{M}$.

Theorem (2.6): *If F-structure manifold has the following properties i.e., $\widetilde{M}$ is an invariant submanifold of F-structure manifold M such that distribution D_m is always tangential to $\Psi(\widetilde{M})$. Then there exists an induced $\widetilde{F}$-structure manifold which admits a similar Riemannian metric $\widetilde{g}$ satisfying*

$$\widetilde{g}(\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{H}\widetilde{X}, \widetilde{H}\widetilde{Y}) + \widetilde{g}(\widetilde{m}\widetilde{X}\widetilde{Y}) \tag{2.10}$$

Proof: From (2.2)b we get

$$B(\tilde{F}^{\frac{K-1}{2}}\widetilde{X}) = F^{\frac{K-1}{2}}(B\widetilde{X}).$$

Furthermore,

$$B(\tilde{F}^{K}\widetilde{X}) = F^{K}(B\widetilde{X}),$$

which in view of (1.1) and (2.2)a yields

$$= B(-(-)^{K+1}\tilde{F}\widetilde{X}),$$

which shows that $\tilde{F}$ defines an $\tilde{F}$-structure manifold which satisfies

$$\tilde{F}^{K} + (-)^{K+1}\tilde{F} = 0. \tag{2.10}$$

In consequence of equations (2.1), (2.2)b and (2.5) we obtain:

$$\tilde{g}(\widetilde{H},\widetilde{X},\widetilde{H}\tilde{Y}) + \tilde{g}(\widetilde{m}\widetilde{X},\tilde{Y}), \quad \text{where} \quad \widetilde{H} = \tilde{F}^{\frac{K-1}{2}} \tag{2.11}$$

$$\begin{aligned} &= g(B\widetilde{H}\widetilde{X}, B\widetilde{H}\tilde{Y}) + g(B\widetilde{m}\widetilde{X}, B\tilde{Y}) \\ &= g(H\,B\widetilde{X}, H\,B\tilde{Y}) + g(m\,B\widetilde{X}, B\tilde{Y}) \\ &= g(B\widetilde{X}, B\tilde{Y}) \end{aligned}$$

which in view of the fact that B is an isomorphism gives

$$= \tilde{g}(\widetilde{X},\tilde{Y}).$$

3. INTEGRABILITY CONDITIONS

The Nijenhuis tensor N of the type (1.2) of F satisfying (1.1) in M is given by [14]

$$N(X,Y) = [FX,FY] - F[FX,Y] - F[X,F,Y] + F^2[X,Y] \tag{3.1}$$

and the Nijenhuis tensor $\widetilde{N}$ of $\tilde{F}$ satisfying (2.10) in $\widetilde{M}$ is given by

$$N(\widetilde{X},\tilde{Y}) = [\tilde{F}\widetilde{X},\tilde{F}\tilde{Y}] - \tilde{F}[\tilde{F}\widetilde{X},\tilde{Y}] - \tilde{F}[\widetilde{X}\tilde{F}\tilde{Y}] + \tilde{F}^2[\widetilde{X},\tilde{Y}] \tag{3.2}$$

Theorem (3.1): *The Nijenhuis tensors N and $\widetilde{N}$ of M and $\widetilde{M}$ given by (3.1) and (3.2) satisfy the following relation*

$$N(B\widetilde{X}, B\tilde{Y}) = B\widetilde{N}(\widetilde{X},\tilde{Y}) \tag{3.3}$$

Proof: We have

$$\begin{aligned} N(B\widetilde{X}, B\widetilde{Y}) &= [F(B\widetilde{X}), F(B\widetilde{Y})] - F[F(B\widetilde{X}), B\widetilde{Y}] \\ &\quad -F[B\widetilde{X}, F(B\widetilde{Y})] + F^2[B\widetilde{X}, B\widetilde{Y}], \end{aligned}$$

which in view of (2.2)a becomes

$$\begin{aligned} &= B[\widetilde{F}\widetilde{X}, \widetilde{F}\widetilde{Y}] - F[B(\widetilde{F}\widetilde{X}), B\widetilde{Y}] - F[B\widetilde{X}, B\widetilde{F}\widetilde{Y})] + F^2[B\widetilde{X}, B\widetilde{Y}] \\ &= B[\widetilde{F}\widetilde{X}, \widetilde{F}\widetilde{Y}] - FB[\widetilde{F}\widetilde{X}, \widetilde{Y}] - FB[\widetilde{X}, \widetilde{F}\widetilde{Y}] + BF^2[\widetilde{X}, \widetilde{Y}] \\ &= B[\widetilde{F}\widetilde{X}, \widetilde{F}\widetilde{Y}] - B\widetilde{F}[\widetilde{F}, \widetilde{X}, \widetilde{Y}] - B\widetilde{F}[\widetilde{X}, \widetilde{F}\widetilde{Y}] + B\widetilde{F}^2[\widetilde{X}, \widetilde{Y}] \\ &= B\widetilde{N}(\widetilde{X}, \widetilde{Y}) \end{aligned}$$

Theorem (3.2): *The following identities hold*

$$B\widetilde{N}(\widetilde{\ell}\widetilde{X}, \widetilde{\ell}\widetilde{Y}) = N(\widetilde{\ell}B\widetilde{X}, \widetilde{\ell}B\widetilde{Y}) \tag{3.4}$$

$$B\widetilde{N}(\widetilde{m}\widetilde{X}, \widetilde{m}\widetilde{Y}) = N(\widetilde{m}B\widetilde{X}, \widetilde{m}B\widetilde{Y}) \tag{3.5}$$

$$B\{\widetilde{m}\widetilde{n}(\widetilde{X}, \widetilde{Y})\} = m\, N(B\widetilde{X}, B\widetilde{Y}) \tag{3.6}$$

Proof: The proof of (3.4) to (3.6) follows by virtue of theorem (3.1), equation (1.4)a, (2.2)a, (2.7)a,b and (3.3).

For $\widetilde{F}$ satisfying (2.10), there exists complementary distribution $D_{\widetilde{\ell}}$ and $D_{\widetilde{m}}$ corresponding to the projection operators $\widetilde{\ell}$ and $\widetilde{m}$ in $\widetilde{M}$ given by (2.7)a. Then in view of the intergrability conditions of $\widetilde{F}$ structure we state the following theorem:

Theorem (3.3): *If D_ℓ is integrable in M, then $D_{\widetilde{\ell}}$ is also integrable in $\widetilde{M}$. If D_m is integrable in M then $D_{\widetilde{m}}$ is also integrable in $\widetilde{M}$.*

Theorem (3.4): *If D_ℓ and D_m are both integrable in M, then $D_{\widetilde{\ell}}$ and $D_{\widetilde{m}}$ are also integrable in $\widetilde{M}$.*

Theorem (3.5): *If F-structure is integrable in M then the induced structure $\widetilde{F}$ is also integrable in $\widetilde{M}$.*

References

1. Blair, D., Ludden, G. and Yano, K. *Semi-Invariant immersions*, Kodai Math., Sem,, Rep., Vol 25(1976), 313-319.

2. Kubo, Y., *Invariant submanifolds of codimension 2 of a manifold with (F, G, u, v, λ)-structure,* Kodai Math., Sem., Rep., Vol. 24(1972), 50-61.

3. Yano, K. and Okumura, M., *On (f, g, u, v, λ)-structures*, Kodai, Math., Sem., Rep., Vol 22(1970), 401-423.

4. Yano, K. and Okumura, M., *Invariant hypersurfaces of a manifold with (f, g, u, v, λ)-structure*, Kodai, Math., Sem., Rep., Vol 23(1971), 290-304.

5. Yano, K. and Isaihara, S., *Invariant submanifolds of almost contact manifolds*, Kodai, Math., Sem., Rep., Vol 21(1969), 350-364.

6. Kim, J.B., *Notes on f-manifold*, Tensor(N.S.), Vol. 29(1975), 299-302.

7. Nekogova, H., *f-structures induced on submanifolds in spaces*, almost Hermitian or Kählerian, Kodai, Math., Sem., Rep., Vol 18(1966), 161-183.

8. Das, Lovejoy S., *Prolongations of F-structure to the tangent bundle of order 2.* International J. Math. & Math. Sci. Vol 16 No.1(1993), 201-204.

9. Das, Lovejoy S. *Fiberings on almost r-contact manifolds.* Publications Mathematicae, Debrecen, Hongrie, Vol. 43(1993), 1-7.

10. Hicks, N.J., *Notes on Differential Geometry*, D. Van Nostrand Company, Inc., New York.

11. Das, Lovejoy S. and Upadhyay, M.D., *F-structure manifold*, Kyung Pook Mathematical Journal, Korea, 18(1978), 272-283.

12. Das, Lovejoy S., *Complete lift of F-structure manifold*, Kyung Pook Mathematical Journal, Korea, 20(1980), 231-237.

13. Helgason, S., *Differential Geometry, Lie groups and Symmetric Spaces*, Academic Press, New York 1998.

14. Yano, k. & Ishihara, S., *On integrability of a structure of satisfying $f^3 + f = 0$*, Quart. J. Math. Oxford, Vol. 25(1964), 217-222.