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Universal Factorization Equalities over Generalized Quaternion Algebras

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Abstract

A set of universal similarity factorization equalities are established over generalized quaternion algebras, their properties and some applications are also considered.

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1 Introduction

A generalized quaternion algebra over an arbitrary field $\mathbb{F}$ *of characteristic not two is well known as follows(see e.g., [4], [6] and [9]).*

Definition. *Let* $\mathbb{F}$ *be an arbitrary field of characteristic not two,* u *and* v *be two nonzero elements of* $\mathbb{F}$*. A generalized quaternion algebra* $\left(\frac{u,\,v}{\mathbb{F}}\right)$ *is a four dimensional vector space over* $\mathbb{F}$ *with the four basis* $1, i, j$ *and* k *satisfying the following multiplication laws*

$$i^2 = u, \qquad j^2 = v, \qquad k = ij = -ji, \tag{1.1}$$

and 1 acting as unity element.

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According to the above definition, the generalized quaternion algebra $\left(\frac{u,v}{\mathbb{F}}\right)$ *is a four-dimensional associative algebra with its center field* $\mathbb{F}$*. Three ordinary quaternion algebras over the real number field* $\mathbb{R}$ *and the complex number field* $\mathbb{C}$ *are well-known as follows:*

$$\mathbb{Q} := \left(\frac{-1,\,-1}{\mathbb{C}}\right), \qquad \text{the complex quaternion algebra,}$$
$$\mathbb{H} := \left(\frac{-1,\,-1}{\mathbb{R}}\right), \qquad \text{the real division quaternion algebra,}$$
$$\mathbb{S} := \left(\frac{-1,\,1}{\mathbb{R}}\right), \qquad \text{the split quaternion algebra.}$$

A well-known fact on generalized quaternions algebras is that $\left(\frac{u,v}{\mathbb{F}}\right)$ *is algebraically isomorphic to a subalgebra of the total matrix algebra* $M_4(\mathbb{F})$ *and each element* $a \in \left(\frac{u,v}{\mathbb{F}}\right)$ *has a matrix representation* $\phi(a)$ *over its central field* $\mathbb{F}$*.*

In this paper, we present some fundamental equalities for an element $a \in \left(\frac{u,v}{\mathbb{F}}\right)$ *and its matrix representation* $\phi(a)$*, and then derive from them some valuable consequences.*

Throughout, we use the following known definitions and notations. For any $a = a_0 + a_1 i + a_2 j + a_3 k \in \left(\frac{u,v}{\mathbb{F}}\right)$, $\overline{a} = a_0 - a_1 i - a_2 j - a_3 k$ *stands for the* conjugate *of* a*;* $n(a) = a\overline{a} = a_0^2 - ua_1^2 - va_2^2 + uva_3^2$ *stands for the* norm *of* a*;* $\overrightarrow{a} = [\,a_0,\ a_1,\ a_2,\ a_3\,]^T$ *stands for the* adjoint vector *of* a*. For a quaternion matrix* $A = (a_{st}) \in \left(\frac{u,v}{\mathbb{F}}\right)^{m\times n}$, $\overline{A} = (\overline{a_{st}})$ *denotes the* conjugate *of* A*; and* $A^* = (\overline{A})^T = (\overline{a_{ts}})$ *denotes the* conjugate transpose *of* A*. A square generalized quaternion matrix* A *of size* n *is said to be* invertible *if there is a matrix* B *over* $\left(\frac{u,v}{\mathbb{F}}\right)$ *such that* $AB = BA = I_n$*, the identity matrix.*

Some basic operation properties on generalized quaternions used in the sequel are listed below (see, e.g., [6] and [9]).

Lemma 1.1. Let $a = a_0 + a_1 i + a_2 j + a_3 k \in \left(\frac{u,v}{\mathbb{F}}\right)$ and $b \in \left(\frac{u,v}{\mathbb{F}}\right)$ be given. Then

(a) $\overline{\overline{a}} = a, \qquad \overline{a+b} = \overline{a} + \overline{b}, \qquad \overline{ab} = \overline{b}\,\overline{a}.$
(b) $a\overline{a} = \overline{a}a$, *i.e.*, $n(a) = n(\overline{a})$.
(c) $n(ab) = n(a)n(b)$.
(d) $a^2 - 2a_0 a + n(a) = 0$.
(e) if $n(a) \neq 0$, then a is invertible and $a^{-1} = n^{-1}(a)\,\overline{a}$.

Lemma 1.2. Suppose that $\left(\frac{u,v}{\mathbb{F}}\right)$ is a generalized quaternion algebra over an arbitrary field $\mathbb{F}$ of characteristic not two. Then

(a) $\left(\frac{u,v}{\mathbb{F}}\right)$ is a division algebra $\Longleftrightarrow n(a) \neq 0$, for any $0 \neq a \in \mathbb{F}(u,\,v)$.
(b) $\left(\frac{u,v}{\mathbb{F}}\right)$ is not a division algebra $\Longleftrightarrow$ $\mathbb{F}(u,\,v) \simeq M_2(\mathbb{F})$. In particular,

$$\left(\frac{u^2,\,v}{\mathbb{F}}\right) \simeq M_2(\mathbb{F}), \quad \textit{and} \quad \left(\frac{u,\,v^2}{\mathbb{F}}\right) \simeq M_2(\mathbb{F}). \tag{1.2}$$

2 A universal similarity factorrization equality over $\left(\frac{u,v}{\mathbb{F}}\right)$

In this section we first establish a universal similarity factorization equality over $\left(\frac{u,v}{\mathbb{F}}\right)$ and then present some of its consequences.

Theorem 2.1. Let $a = a_0 + a_1 i + a_2 j + a_3 k \in \left(\frac{u,v}{\mathbb{F}}\right)$ be given, where a_0—$a_3 \in \mathbb{F}$. Then the diagonal matrix diag(a, a, a, a) satisfies the following universal similarity factorization equality

$$P \begin{bmatrix} a & & & \\ & a & & \\ & & a & \\ & & & a \end{bmatrix} P^{-1} = \begin{bmatrix} a_0 & ua_1 & va_2 & -uva_3 \\ a_1 & a_0 & va_3 & -va_2 \\ a_2 & -ua_3 & a_0 & ua_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} := \phi(a) \tag{2.1}$$

where P has the following form

$$P = P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & i & j & k \\ u^{-1}i & 1 & -u^{-1}k & -j \\ v^{-1}j & v^{-1}k & 1 & i \\ -(uv)^{-1}k & -v^{-1}j & u^{-1}i & 1 \end{bmatrix}. \tag{2.2}$$

Proof. *It is easy to verify that*

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_0 & x_1 \\ x_1 & x_0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} x_0 + x_1 & 0 \\ 0 & x_0 - x_1 \end{bmatrix} \tag{2.3}$$

hold for any $x_0,\ x_1 \in \left(\frac{u,v}{\mathbb{F}}\right)$. On the basis of (2.3), we further obtain

$$\begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_0 & x_3 & x_2 \\ x_2 & x_3 & x_0 & x_1 \\ x_3 & x_2 & x_1 & x_0 \end{bmatrix} = V \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix} V, \tag{2.4}$$

where $x_0, x_1, x_2, x_3 \in \left(\frac{u,v}{\mathbb{F}}\right)$ and

$$d_1 = x_0 + x_1 + x_2 + x_3, \qquad d_2 = x_0 + x_1 - x_2 - x_3, \tag{2.5}$$

$$d_3 = x_0 - x_1 + x_2 - x_3, \qquad d_4 = x_0 - x_1 - x_2 + x_3, \tag{2.6}$$

$$V = V^T = V^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \tag{2.7}$$

Replacing x_0—x_3 in (2.4) by a_0, $a_1 i$, $a_2 j$ and $a_3 k$, respectively, we get

$$A := \begin{bmatrix} a_0 & a_1 i & a_2 j & a_3 k \\ a_1 i & a_0 & a_3 k & a_2 j \\ a_2 j & a_3 k & a_0 & a_1 i \\ a_3 k & a_2 j & a_1 i & a_0 \end{bmatrix} = V \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix} V, \tag{2.8}$$

where

$$d_1 = a_0 + a_1 i + a_2 j + a_3 k = a, \qquad d_2 = a_0 + a_1 i - a_2 j - a_3 k,$$

$$d_3 = a_0 - a_1 i + a_2 j - a_3 k, \qquad d_4 = a_0 - a_1 i - a_2 j + a_3 k.$$

It is easy to verify that $d_2 = iai^{-1}$, $d_3 = jaj^{-1}$ *and* $d_4 = kak^{-1}$. *Thus*

$$\mathrm{diag}(\, d_1,\ d_2,\ d_3,\ d_4\,) = J\mathrm{diag}(\, a,\ a,\ a,\ a\,)J^{-1}, \tag{2.9}$$

where $J = \mathrm{diag}(\, 1,\ i,\ j,\ k\,)$. *On the other hand, it is easy to verify that*

$$J^{-1}AJ = \begin{bmatrix} a_0 & a_1 ii & a_2 jj & a_3 kk \\ a_1 i^{-1} i & a_0 i^{-1} i & a_3 i^{-1} kj & a_2 i^{-1} jk \\ a_2 j^{-1} j & a_3 j^{-1} ki & a_0 j^{-1} j & a_1 j^{-1} ik \\ a_3 k^{-1} k & a_2 k^{-1} ji & a_1 k^{-1} ij & a_0 k^{-1} k \end{bmatrix} = \begin{bmatrix} a_0 & ua_1 & va_2 & -uva_3 \\ a_1 & a_0 & va_3 & -va_2 \\ a_2 & -ua_3 & a_0 & ua_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} = \phi(a). \tag{2.10}$$

Putting (2.8) and (2.9) in (2.10) yield

$$\phi(a) = J^{-1}AJ = J^{-1}V\mathrm{diag}(\, d_1,\ d_2,\ d_3,\ d_4\,)VJ = J^{-1}VJ\mathrm{diag}(\, a,\ a,\ a,\ a\,)J^{-1}VJ.$$

Let $P = J^{-1}VJ$. *Then we have (2.1) and (2.2).* □

Observe that the matrix P *in (2.1) is composed just by* u, v *and the basis of* $\left(\frac{u,\, v}{\mathbb{F}}\right)$, *which has no relation with expression of* a. *We can easily derive from (2.1) various operation properties on generalized quaternions and their matrix representations* $\phi(a)$ *over* $\mathbb{F}$.

Theorem 2.2. Let $a,\, b \in \left(\frac{u,\, v}{\mathbb{F}}\right)$ be given and $\lambda \in \mathbb{F}$. Then
(a) $a = b \Longleftrightarrow \phi(a) = \phi(b)$.
(b) $\phi(a + b) = \phi(a) + \phi(b)$, $\phi(ab) = \phi(a)\phi(b)$, $\phi(\lambda a) = \phi(a\lambda) = \lambda\phi(a)$, $\phi(1) = I_4$.

The results in Theorem 2.2(a) and (b) together with (2.1) manifest that through the bijective map

$$\phi : a \in \left(\frac{u,\, v}{\mathbb{F}}\right) \longrightarrow \phi(a) \in M_4(\mathbb{F}),$$

the generalized quaternion algebra $\left(\frac{u,\, v}{\mathbb{F}}\right)$ *is algebraically isomorphic to the matrix algebra constructed by all matrices with the form as* $\phi(a)$. *Another remarkable fact implied by (2.1) is that any* $a \in \left(\frac{u,\, v}{\mathbb{F}}\right)$ *is in fact an eigenvalue of its representation matrix* $\phi(a)$.

Theorem 2.3. Let $a \in \left(\frac{u,\, v}{\mathbb{F}}\right)$ be given. Then $\phi(\overline{a})$ can be expressed as $\phi(\overline{a}) = K^{-1}\phi^T(a)K$, where $K = \mathrm{diag}(\, 1,\ -u,\ -v,\ uv\,)$.

Proof. *Follows from a direct verification.* □

Theorem 2.4. Let $a = a_0 + a_1 i + a_2 j + a_3 k \in \left(\frac{u,\, v}{\mathbb{F}}\right)$ be given. Then the determinant of matrix representation $\phi(a)$ of a is $|\phi(a)| = n^2(a) = (\, a_0^2 - ua_1^2 - va_2^2 + uva_3^2\,)^2$.

Proof. *From* $\phi(a\overline{a}) = \phi(a)\phi(\overline{a})$ *and* $n(a) = a\overline{a}$, *we obtain* $|\phi(a)|\,|\phi(\overline{a})| = |\phi(a\overline{a})| = |n(a)I_4| = n^4(a)$. *On the other hand, by Theorem 2.3, it follows that* $|\phi(\overline{a})| = |K^{-1}|\,|\phi^T(a)|\,|K| = |\phi(a)|$. *Hence* $|\phi(a)|^2 = n^4(a)$. *Note that the term* a_0^4 *occurs in* $|\phi(a)|$. *Thus* $|\phi(a)| = n^2(a)$. $\square$

Theorem 2.5. Let $a \in \left(\frac{u,\,v}{\mathbb{F}}\right)$ be given. Then

$$a = \frac{1}{4}[\,1,\ i,\ j,\ k\,]\phi(a)[\,1,\ i^{-1},\ j^{-1},\ k^{-1}\,]^T = \frac{1}{4}E_4\phi(a)F_4^T, \tag{2.11}$$

where $E_4 = [\,1,\ i,\ j,\ k\,]$ and $F_4 = [\,1,\ i^{-1},\ j^{-1},\ k^{-1}\,]$.

Proof. *The equality (2.1) can equivalently be written as* $\mathrm{diag}(\,a,\ a,\ a,\ a\,) = P\phi(a)P^{-1}$. *Hence a can be expressed as* $a = [\,1,\ 0,\ 0,\ 0\,]P\phi(a)P^{-1}[\,1,\ 0,\ 0,\ 0\,]^T$. *Written in an explicit form, it is (2.11).* $\square$

Theorem 2.6. Let $a \in \left(\frac{u,\,v}{\mathbb{F}}\right)$ be given. Then a is invertible if and only if $\phi(a)$ is invertible, in which case,

$$\phi(a^{-1}) = \phi^{-1}(a) \quad \textit{and} \quad a^{-1} = \frac{1}{4}E_4\phi^{-1}(a)F_4^T, \tag{2.12}$$

where E_4 and F_4 are as in *(2.11)*.

Proof. *Follows from (2.1) and (2.11).* $\square$

Another matrix representation of $a \in \left(\frac{u,\,v}{\mathbb{F}}\right)$ *can be introduced as follows*

$$\tau(a) := G^{-1}\phi^T(a)G = \begin{bmatrix} a_0 & ua_1 & va_2 & -uva_3 \\ a_1 & a_0 & -va_3 & va_2 \\ a_2 & ua_3 & a_0 & -ua_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}, \tag{2.13}$$

where $G = \mathrm{diag}(\,1,\ u,\ v,\ -uv\,)$. *This representation together with (2.1) will play key roles for solving various linear equations over* $\left(\frac{u,\,v}{\mathbb{F}}\right)$.

Combining them with Theorem 2.3, we have the following.

Theorem 2.7. Let $a \in \left(\frac{u,\,v}{\mathbb{F}}\right)$ be given. Then

$$\tau(a) = L\phi(\overline{a})L \quad \textit{and} \quad \phi(a) = L\tau(\overline{a})L,$$

where $L = \mathrm{diag}(\,1,\ -1,\ -1,\ -1\,)$.

Proof. *Follows from a direct verification.* $\square$

Theorem 2.8. Let $a,\ b \in \left(\frac{u,\,v}{\mathbb{F}}\right)$ be given and $\lambda \in \mathbb{F}$. Then

(a) $a = b \Longleftrightarrow \tau(a) = \tau(b)$.

(b) $\tau(a + b) = \tau(a) + \tau(b), \quad \tau(ab) = \tau(b)\tau(a), \quad \tau(\lambda a) = \tau(a\lambda) = \lambda\tau(a)$.

(c) If $a = a_0 + a_1 i + a_2 j + a_3 k$, then $|\tau(a)| = n^2(a) = (\,a_0^2 - ua_1^2 - va_2^2 + uva_3^2\,)^2$.

(d) a is invertible if and only if $\tau(a)$ is invertible, in which case, $\tau(a^{-1}) = \tau^{-1}(a)$.

The results in Theorem 2.8(a) and (b) clearly show that through the bijective map

$$\tau : a \in \left(\frac{u,\, v}{\mathbb{F}}\right) \;\longrightarrow\; \tau(a) \in M_4(\mathbb{F}),$$

the generalized quaternion algebra $\left(\frac{u,v}{\mathbb{F}}\right)$ *is algebraically anti-isomorphic to the matrix algebra constructed by all matrices with the same form as* $\tau(a)$ *in (2.13).*

Combining two matrix representations with adjoint vectors of generalized quaternions, we can derive the following basic results.

Theorem 2.9. Let $a,\, b,\, x \in \left(\frac{u,v}{\mathbb{F}}\right)$ be given. Then

$$\overrightarrow{ax} = \phi(a)\overrightarrow{x}, \qquad \overrightarrow{xb} = \tau(b)\overrightarrow{x}, \qquad \overrightarrow{axb} = \phi(a)\tau(b)\overrightarrow{x} = \tau(b)\phi(a)\overrightarrow{x}. \tag{2.14}$$

and

$$\phi(a)\tau(b) = \tau(b)\phi(a). \tag{2.15}$$

Proof. *It is easily seen from the structure of* $\phi(a)$ *and* $\tau(a)$ *that* $\overrightarrow{a} = \phi(a)\alpha_4^T$ *and* $\overrightarrow{a} = \tau(a)\alpha_4^T$*, where* $\alpha_4 = [\,1,\ 0,\ 0,\ 0\,]$*. From both of them and Theorems 2.2(b) and 2.8(b), we easily find*

$$\overrightarrow{ax} = \phi(ax)\alpha_4^T = \phi(a)\phi(x)\alpha_4^T = \phi(a)\overrightarrow{x}, \qquad \overrightarrow{xb} = \tau(xb)\alpha_4^T = \tau(b)\tau(x)\alpha_4^T = \tau(b)\overrightarrow{x},$$

$$\overrightarrow{axb} = \overrightarrow{a(xb)} = \phi(a)\overrightarrow{xb} = \phi(a)\tau(b)\overrightarrow{x}, \qquad \overrightarrow{axb} = \overrightarrow{(ax)b} = \tau(b)\overrightarrow{ax} = \tau(b)\phi(a)\overrightarrow{x},$$

as required for (2.14). Observe that $\overrightarrow{x}$ *is arbitrarily given. Thus we get (2.15) from the third equality in (2.14).* □

In term of the three formulas in (2.14), all two-sided linear equations over $\mathbb{F}(u,\, v)$ *can equivalently be transformed into ordinary linear equation systems over* $\mathbb{F}$*, such as*

$$axb = c \Longleftrightarrow \phi(a)\tau(b)\overrightarrow{x} = \overrightarrow{c},$$

$$ax - xb = c \Longleftrightarrow [\,\phi(a) - \tau(b)\,]\overrightarrow{x} = \overrightarrow{c},$$

$$axb + cxd = e \Longleftrightarrow [\,\phi(a)\tau(b) + \phi(c)\tau(d)\,]\overrightarrow{x} = \overrightarrow{e}.$$

3 Some particular universal similarity factorization equalities over $\left(\frac{u,v}{\mathbb{F}}\right)$

As pointed out in Lemma 1.2(b), if $\left(\frac{u,v}{\mathbb{F}}\right)$ *is not a division algebra, then it is algebraically isomorphic to the matrix algebra* $M_2(\mathbb{F})$*. Motivated by this result and the equality in (2.1), a question can naturally be asked: Does there exist an universal similarity factorization equality between elements in* $\left(\frac{u,v}{\mathbb{F}}\right)$ *and matrices over* $M_2(\mathbb{F})$*? For some particular choices of u and v in* $\left(\frac{u,v}{\mathbb{F}}\right)$*, or some* *specified fields $\mathbb{F}$, the answer to this question is positive. To do so, we first present a simple but vital result.*

Lemma 3.1. Let $\mathbb{F}$ be an arbitrary field, and the $\mathcal{A}$ is a four dimensional algebra over $\mathbb{F}$ with the four bases e_{11}, e_{12}, e_{21} and e_{22} in it satisfying the following multiplication rules

$$e_{st}e_{pq} = \begin{cases} e_{sq}, & t = p, \\ 0, & t \neq p, \end{cases} \qquad s,\, t,\, p,\, q = 1,\, 2. \tag{3.1}$$

Then for any $a = a_{11}e_{11} + a_{12}e_{12} + a_{21}e_{21} + a_{22}e_{22} \in \mathcal{A}$, where $a_{st} \in \mathbb{F}$, the corresponding diagonal matrix $\mathrm{diag}(\, a,\ a\,)$ satisfies the following universal similarity factorization equality

$$Q \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} Q^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{3.2}$$

where Q has the independent form

$$Q = Q^{-1} = \begin{bmatrix} e_{11} & e_{21} \\ e_{12} & e_{22} \end{bmatrix}. \tag{3.3}$$

Proof. *According to (3.1), it is easy to verify that the unity element in $\mathcal{A}$ is $e = e_{11} + e_{22}$. In that case, the matrix Q in (3.3) satisfies*

$$Q^2 = \begin{bmatrix} e_{11} & e_{21} \\ e_{12} & e_{22} \end{bmatrix} = \begin{bmatrix} e_{11}^2 + e_{21}e_{12} & e_{11}e_{21} + e_{21}e_{22} \\ e_{12}e_{11} + e_{22}e_{12} & e_{12}e_{21} + e_{22}^2 \end{bmatrix} = \begin{bmatrix} e_{11} + e_{22} & 0 \\ 0 & e_{11} + e_{22} \end{bmatrix} = eI_2,$$

which implies that Q is invertible over $\mathcal{A}$ and $Q = Q^{-1}$. As to (3.2), it comes from a direct calculation of the matrix product in the left-hand side of (3.2). $\square$

We can easily establish from (3.2) some universal similarity factorization equalities between elements in $\left(\frac{u,\,v}{\mathbb{F}}\right)$ and matrices in $M_2(\mathbb{F})$, when u and v in $\left(\frac{u,\,v}{\mathbb{F}}\right)$ are chosen some particular values, or the field $\mathbb{F}$ has some particular structure.

Theorem 3.2. Let $a = a_0 + a_1 i + a_2 j + a_3 k \in \left(\frac{u^2,\,v}{\mathbb{F}}\right)$ be given, where u and v are two nonzero elements of $\mathbb{F}$, a_0—$a_3 \in \mathbb{F}$, and

$$i^2 = u^2, \qquad j^2 = v, \qquad k = ij = -ji. \tag{3.4}$$

Then a satisfies the following universal similarity factorization equality

$$Q_1 \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} Q_1^{-1} = \begin{bmatrix} a_0 + ua_1 & v(\, a_2 + ua_3\,) \\ a_2 - ua_3 & a_0 - ua_1 \end{bmatrix} := \psi_1(a), \tag{3.5}$$

where Q_1 has the universal form

$$Q_1 = Q_1^{-1} = \frac{1}{2}\begin{bmatrix} 1 + u^{-1}i & j - u^{-1}k \\ v^{-1}j + (uv)^{-1}k & 1 - u^{-1}i \end{bmatrix}. \tag{3.6}$$

Proof. *According to (3.4), we choose a group of new basis for* $\left(\frac{u^2,\, v}{\mathbb{F}}\right)$ *as follows*

$$e_{11} = \frac{1}{2}\left(1 + \frac{1}{u}i\right), \quad e_{12} = \frac{1}{2}\left(\frac{1}{v}j + \frac{1}{uv}k\right), \quad e_{21} = \frac{1}{2}\left(j - \frac{1}{u}k\right), \quad e_{22} = \frac{1}{2}\left(1 - \frac{1}{u}i\right). \tag{3.7}$$

Then it is not difficult to verify that the basis (3.7) satisfies the multiplication rules in (3.1). Under this group of new basis, any element a *of* $\left(\frac{u^2,\, v}{\mathbb{F}}\right)$ *can be expressed as*

$$a = (\,a_0 + ua_1\,)e_{11} + (\,va_2 + uva_3\,)e_{12} + (\,a_2 - ua_3\,)e_{21} + (\,a_0 - ua_1\,)e_{22}. \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.2), we obtain (3.5). □

We easily find from (3.5) the following result.

Theorem 3.3. Let $a,\ b \in \left(\frac{u^2,\, v}{\mathbb{F}}\right)$, and $\lambda \in \mathbb{F}$. Then

(a) $a = b \Longleftrightarrow \psi_1(a) = \psi_1(b)$.

(b) $\psi_1(a + b) = \psi_1(a) + \psi_1(b), \quad \psi_1(ab) = \psi_1(a)\psi_1(b), \quad \psi_1(\lambda a) = \psi_1(a\lambda) = \lambda\psi_1(a), \quad \psi_1(1) = I_2$.

(c) $|\psi_1(a)| = n(a)$.

(d) $\psi_1(\overline{a}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \psi_1^T(a) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(e) $a = \frac{1}{4}[\,1 + u^{-1}i,\ j - u^{-1}k\,]\psi_1(a) \begin{bmatrix} 1 + u^{-1}i \\ v^{-1}j + (uv)^{-1}k \end{bmatrix}$.

The equality (3.5) and the results in Theorem 3.3(a) and (b) manifest that through the bijective map

$$\psi_1 : a \in \left(\frac{u^2,\, v}{\mathbb{F}}\right) \longrightarrow \psi_1(a) \in M_2(\mathbb{F}),$$

the generalized quaternion algebra $\left(\frac{u^2,\, v}{\mathbb{F}}\right)$ *is algebraically isomorphic to the 2-by-2 total matrix algebra* $M_2(\mathbb{F})$. *Another important fact implied by (3.5) is that any 2-by-2 matrix over the field* $\mathbb{F}$ *can be diagonalized over a generalized quaternion algebra* $\left(\frac{u^2,\, v}{\mathbb{F}}\right)$, *where* u *and* v *are any two nonzero elements in* $\mathbb{F}$.

An analogous result is given below.

Theorem 3.4. Let $a = a_0 + a_1 i + a_2 j + a_3 k \in \left(\frac{u,\, v^2}{\mathbb{F}}\right)$ be given, where u and v are two nonzero elements of $\mathbb{F}$, a_0— $a_3 \in \mathbb{F}$, and

$$i^2 = u, \qquad j^2 = v^2, \qquad k = ij = -ji. \tag{3.9}$$

Then a satisfies the following universal similarity factorization equality

$$Q_2 \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} Q_2^{-1} = \begin{bmatrix} a_0 + va_2 & u(\,a_1 - va_3\,) \\ a_1 + va_3 & a_0 - va_2 \end{bmatrix} := \psi_2(a), \tag{3.10}$$

where Q_2 has the universal form

$$Q_2 = Q_2^{-1} = \frac{1}{2}\begin{bmatrix} 1 + v^{-1}j & i + v^{-1}k \\ u^{-1}i - (uv)^{-1}k & 1 - v^{-1}j \end{bmatrix}. \tag{3.11}$$

Proof. *Setting a group of new basis for* $\left(\frac{u,\, v^2}{\mathbb{F}}\right)$ *as follows*

$$e_{11} = \frac{1}{2}\left(1 + \frac{1}{v}j\right), \quad e_{12} = \frac{1}{2}\left(\frac{1}{u}i - \frac{1}{uv}k\right), \quad e_{21} = \frac{1}{2}\left(i + \frac{1}{v}k\right), \quad e_{22} = \frac{1}{2}\left(1 - \frac{1}{v}j\right). \tag{3.12}$$

Then it is not difficult to verify that the basis (3.12) satisfies the multiplication laws in (3.1). Under this group of new basis, any element a *of* $\left(\frac{u,\, v^2}{\mathbb{F}}\right)$ *can be expressed as*

$$a = (\, a_0 + va_2\,)e_{11} + (\, ua_1 - uva_3\,)e_{12} + (\, a_1 + va_3\,)e_{21} + (\, a_0 - va_2\,)e_{22}. \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.2), we obtain (3.10). □

The operation properties on $\psi_2(a)$ *in (3.10) are much like to those in Theorem 3.3, hence we omit them here.*

Some direct consequences of the above several theorems are given as follows.

Theorem 3.5. Let the field $\mathbb{F}$ be algebraically closed, and $\left(\frac{u,v}{\mathbb{F}}\right)$ be a generalized quaternion algebra over $\mathbb{F}$ with u and v nonzero. Then any $a = a_0 + a_1 i + a_2 j + a_3 k \in \left(\frac{u,v}{\mathbb{F}}\right)$ satisfies the following two universal similarity factorization equalities

$$Q_1 \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} Q_1^{-1} = \begin{bmatrix} a_0 + u_0 a_1 & v(\, a_2 + u_0 a_3\,) \\ a_2 - u_0 a_3 & a_0 - u_0 a_1 \end{bmatrix}, \tag{3.14}$$

$$Q_2 \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} Q_2^{-1} = \begin{bmatrix} a_0 + v_0 a_2 & u(\, a_1 - v_0 a_3\,) \\ a_1 + v_0 a_3 & a_0 - v_0 a_2 \end{bmatrix}, \tag{3.15}$$

where u_0, v_0 and w are the elements in $\mathbb{F}$ satisfying $u_0^2 = u$, $v_0^2 = v$ and $w^2 = -uv$, and Q_1 and Q_2 have the independent forms

$$Q_1 = Q_1^{-1} = \frac{1}{2}\begin{bmatrix} 1 + u_0^{-1}i & j - u_0^{-1}k \\ v^{-1}j + (u_0 v)^{-1}k & 1 - u_0^{-1}i \end{bmatrix},$$

$$Q_2 = Q_2^{-1} = \frac{1}{2}\begin{bmatrix} 1 + v_0^{-1}j & i + v_0^{-1}k \\ u^{-1}i - (uv_0)^{-1}k & 1 - v_0^{-1}j \end{bmatrix}.$$

4 Universal factorization equalities for matrices over $\left(\frac{u,v}{\mathbb{F}}\right)$

Observe that the matrix P *in (2.1) has a universal form. Hence we can easily extend (2.1) to any* $m \times n$ *matrix over* $\left(\frac{u,v}{\mathbb{F}}\right)$.

Theorem 4.1. Let $A = A_0 + A_1 i + A_2 j + A_3 k$ be an $m \times n$ matrix over $\left(\frac{u,\,v}{\mathbb{F}}\right)$, where A_0—A_3 are four $m \times n$ matrices over $\mathbb{F}$. Then the diagonal block matrix diag$(A,\ A,\ A,\ A)$ satisfies the following universal factorization equality

$$J_{4m} \begin{bmatrix} A & & & \\ & A & & \\ & & A & \\ & & & A \end{bmatrix} J_{4n} = \begin{bmatrix} A_0 & uA_1 & vA_2 & -uvA_3 \\ A_1 & A_0 & vA_3 & -vA_2 \\ A_2 & -uA_3 & A_0 & uA_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} := \Phi(A), \quad (4.1)$$

where

$$J_{4t} = J_{4t}^{-1} = \frac{1}{2} \begin{bmatrix} I_t & iI_t & jI_t & kI_t \\ u^{-1}I_t & I_t & -u^{-1}kI_t & -jI_t \\ v^{-1}jI_t & v^{-1}kI_t & I_t & iI_t \\ -(uv)^{-1}kI_t & -v^{-1}jI_t & u^{-1}iI_t & I_t \end{bmatrix}, \qquad t = m,\ n.$$

In particular, when $m = n$, (4.1) becomes a universal similarity factorization equality over $\left(\frac{u,\,v}{\mathbb{F}}\right)$.

The proof of (4.1) is much like that of (2.1), hence omitted. We call the $4m \times 4n$ *matrix* $\Phi(A)$ *in (4.1) the* central representation matrix *of* A.

We can also derive from the equality (4.1) some operation properties on generalized quaternion matrices and their central representation matrices.

Theorem 4.2. Let $A,\ B \in \left(\frac{u,\,v}{\mathbb{F}}\right)^{m\times n}$, $C \in \left(\frac{u,\,v}{\mathbb{F}}\right)^{n\times p}$ be given, and $\lambda \in \mathbb{F}$. Then

(a) $A = B \Longleftrightarrow \Phi(A) = \Phi(B)$.

(b) $\Phi(A+B) = \Phi(A)+\Phi(B)$, $\Phi(AC) = \Phi(A)\Phi(C)$, $\Phi(\lambda A) = \lambda\Phi(A)$, $\Phi(I_m) = I_{4m}$.

(c) Cayley-Hamilton theorem: $p(A) = 0$, where A is square and $p(A) = |\,\lambda I - \Phi(A)\,|$, called the characteristic polynomial of A.

Theorem 4.3. Let $A = A_0 + A_1 i + A_2 j + A_3 k \in \left(\frac{u,\,v}{\mathbb{F}}\right)^{m\times n}$. Then $\Phi(A^*)$ can be expressed as $\Phi(A^*) = K_{4n}^{-1}\, \Phi^T(A)\, K_{4m}$, where $K_{4t} = \mathrm{diag}(I_{4t},\ -uI_{4t},\ -vI_{4t},\ uvI_{4t})$, $t = m,\ n$.

Proof. *According to (4.1),* $\Phi(A^*)$ *and* $\Phi^T(A)$ *has the following forms*

$$\Phi(A^*) = \begin{bmatrix} A_0^T & -uA_1^T & -vA_2^T & uvA_3^T \\ -A_1^T & A_0^T & -vA_3^T & vA_2^T \\ -A_2^T & uA_3^T & A_0^T & -uA_1^T \\ -A_3^T & A_2^T & -A_1^T & A_0^T \end{bmatrix},$$

and

$$\Phi^T(A) = \begin{bmatrix} A_0^T & A_1^T & A_2^T & A_3^T \\ uA_1^T & A_0^T & -uA_3^T & -A_2^T \\ vA_2^T & vA_3^T & A_0^T & A_1^T \\ -uvA_3^T & -vA_2^T & uA_1^T & A_0^T \end{bmatrix}.$$

Hence we have the desired result. □

Theorem 4.4. Let $A \in \left(\frac{u,\,v}{\mathbb{F}}\right)^{m\times n}$ be given. Then A and its central representation matrix $\Phi(A)$ satisfies the following equality

$$A = \frac{1}{4}[\,I,\ iI_m,\ jI_m,\ kI_m\,]\Phi(A)[\,I_n,\ i^{-1}I_n,\ j^{-1}I_n,\ k^{-1}I_n\,]^T = \frac{1}{4}E_{4m}\Phi(A)F_{4n}^T, \tag{4.2}$$

where $E_{4m} = [\,I_m,\ iI_m,\ jI_m,\ kI_m\,]$ and $F_{4n} = [\,I_n,\ i^{-1}I_n,\ j^{-1}I_n,\ k^{-1}I_n\,]$.

Proof. *The equality (4.1) can also be written as* $\mathrm{diag}(\,A,\ A,\ A,\ A\,) = J_{4m}\phi(A)J_{4n}^{-1}$. *Hence A can be expressed as* $A = [\,I_m,\ 0,\ 0,\ 0\,]J_{4m}\Phi(A)J_{4n}^{-1}[\,I_n,\ 0,\ 0,\ 0\,]^T$. *Written in an explicit form, it is (4.2).* □

Theorem 4.5. Let $A \in \left(\frac{u,\,v}{\mathbb{F}}\right)^{m\times m}$ be given. Then A is invertible if and only if its central representation matrix $\Phi(A)$ is invertible, in which case,

$$\Phi(A^{-1}) = \Phi^{-1}(A), \qquad A^{-1} = \frac{1}{4}E_{4m}\Phi^{-1}(A)F_{4m}^T, \tag{4.3}$$

where E_{4m} and F_{4m} are as in (4.2).

Proof. *Follows directly from (4.1) and (4.2).* □

In addition to the universal factorization equality in (4.1), we can also extend all the universal similarity factorization equalities in Section 3 to $m\times n$ matrices over $\left(\frac{u,\,v}{\mathbb{F}}\right)$ and derive a variety of consequences analogous to those in Theorem 3.2—3.5. Next we only list two of them.

Theorem 4.6. Let $A = A_0 + A_1 i + A_2 j + A_3 k \in \left(\frac{u^2,\,v}{\mathbb{F}}\right)^{m\times n}$ be given, where u and v are two nonzero elements of $\mathbb{F}$, A_0—$A_3 \in \mathbb{F}^{m\times n}$, and $i^2 = u^2,\ j^2 = v,\ k = ij = -ji$. Then A satisfies the following universal factorization equality

$$G_{2m}\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}G_{2n} = \begin{bmatrix} A_0 + uA_1 & v(\,A_2 + uA_3\,) \\ A_2 - uA_3 & A_0 - uA_1 \end{bmatrix}, \tag{4.4}$$

where G_{2t} has the independent form

$$G_{2t} = G_{2t}^{-1} = \frac{1}{2}\begin{bmatrix} (\,1 + u^{-1}i\,)I_t & (\,j - u^{-1}k\,)I_t \\ [\,v^{-1}j + (uv)^{-1}k\,]\,I_t & (\,1 - u^{-1}i\,)I_t \end{bmatrix}, \qquad t = m,\ n. \tag{4.5}$$

In particular, when $m = n$, (4.4) becomes a universal similarity factorization equality over $\left(\frac{u^2,\,v}{\mathbb{F}}\right)$.

Theorem 4.7. Let $A = A_0 + A_1 i + A_2 j + A_3 k \in \left(\frac{u,\,v^2}{\mathbb{F}}\right)^{m\times n}$ be given, where u and v are two nonzero elements of $\mathbb{F}$, A_0—$A_3 \in \mathbb{F}^{m\times n}$, and $i^2 = u,\ j^2 = v^2,\ k = ij = -ji$. Then A satisfies the following universal factorization equality

$$L_{2m}\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}L_{2n} = \begin{bmatrix} A_0 + vA_2 & u(\,A_1 - vA_3\,) \\ A_1 + vA_3 & A_0 - vA_2 \end{bmatrix}, \tag{4.6}$$

where L_{2t} has the independent form

$$L_{2t} = L_{2t}^{-1} = \frac{1}{2} \left[\begin{array}{cc} (\,1 + v^{-1}j\,)I_t & (\,i + v^{-1}k\,)I_t \\ \left[\,u^{-1}i - (uv)^{-1}k\,\right] I_t & (\,1 - v^{-1}j\,)I_t \end{array} \right], \qquad t = m, \ n. \quad (4.7)$$

In particular, when $m = n$, Eq.(4.6) becomes a universal similarity factorization equality over $\left(\frac{u,\, v^2}{\mathbb{F}}\right)$.

The results given in the preceding sections are all applicable to matrices over the real quaternion algebra $\mathbb{H}$, the real split quaternion algebra $\mathbb{S}$, and the complex quaternion algebra $\mathbb{Q}$. Using those results, one can easily develop a perfect theory for matrices analysis over these algebras.

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