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INTERPOLATION AND THE LAGUERRE–PÓLYA CLASS

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Abstract A long-standing open problem in the theory of distribution of zeros of real entire functions requires the characterization of all real sequences $T = \{\gamma_k\}_{k=0}^{\infty}$ such that for any real polynomial $p(x) := \sum_0^n a_k x^k$, the polynomial $\sum_0^n \gamma_k a_k x^k$ has no more nonreal zeros than $p(x)$ has. The main result of this paper shows that under a mild growth restriction, an entire function of exponential type $f(z)$ for which the sequence $T = \{f(k)\}_{k=0}^{\infty}$ satisfies the above condition must have only real zeros. The paper concludes with some applications to the Riemann Hypothesis.

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1. Introduction.

In the vast literature dealing with the distribution of zeros of real polynomials and real entire functions, an important role is played by linear transformations T which possess the following property:

$$Z_c(T[p(x)]) \leq Z_c(p(x)), \tag{1.1}$$

where $p(x)$ is a polynomial and $Z_c(p(x))$ denotes the number of nonreal zeros of $p(x)$, counting multiplicities. If $T = D = \frac{d}{dx}$, then (1.1) is a consequence of Rolle's

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theorem. If $q(x)$ is a real polynomial with only real zeros and $T = q(D)$, then (1.1) follows from the classical Hermite-Poulain theorem [14, p. 4]. There are many other linear transformations T which satisfy inequality (1.1). Indeed, set $T = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k \in \mathbb{R}$, and for an arbitrary real polynomial $p(x) = \sum_0^n a_k x^k$, define

$$T[p(x)] := \sum_{k=0}^{n} \gamma_k a_k x^k. \tag{1.2}$$

If $Q(x)$ is a real polynomial with only real negative zeros and if $T = \{Q(k)\}_{k=0}^{\infty}$, then by Laguerre's theorem (cf. Theorem 2.4 below)

$$Z_c\left(\sum_{k=0}^{n} Q(k) a_k x^k\right) \leq Z_c(p(x)),$$

where $p(x) = \sum_0^n a_k x^k$ is an arbitrary real polynomial.

A long-standing open problem, suggested by the work of Laguerre in the 19th century [11, p. 116] (see also [10, p. 379]) is to characterize all real sequences $T = \{\gamma_k\}_{k=0}^{\infty}$ such that (1.1) holds, where $T[p(x)]$ is defined in (1.2).

On account of the work of Laguerre and Karlin cited above, it may be appropriate to refer to this problem as the *Karlin-Laguerre problem*. While the Karlin-Laguerre problem has remained open, recently there has been significant progress made in a series of investigations (see, for example [2], [1], [6] and [8]). In particular, in [1] (see Theorem 3.2 below) it was proved that *if* this sequence does not decrease "too rapidly", in the sense that $\overline{\lim}_{k\to\infty} \gamma_k^{1/k} > 0$, then the γ_k's can be interpolated by a function $\phi(x)$ in the Laguerre-Pólya class (Definition 2.1), that is $\gamma_k = \phi(k)$, $k = 0, 1, 2, \ldots$.

In Section 2 we begin with a brief review of some requisite terminology and we present some examples in order to motivate our main result which answers the following question raised in [7, Problem 9]. Suppose that $f(z)$ is a real entire function of exponential type such that the sequence $T = \{f(k)\}_{k=0}^{\infty}$ satisfies (1.1). Under what supplementary conditions does it follow that $f(z)$ has only real zeros? In Section 3, we make use of the result from [1, Theorem 2] as well as some classical uniqueness theorems and show that if the growth of $f(z)$ is suitably restricted along the imaginary axis, then $f(z)$ must possess only real zeros (Theorem 3.6). In fact, all the zeros of $f(z)$ are shown to be of one sign and thus Theorem 3.6 provides a converse, under additional assumptions, to Laguerre's theorem (Theorem 2.4). We also prove a slight extension of Laguerre's theorem to finite intervals (Theorem 3.7). By way of applications of the aforementioned results, we conclude the paper with two new necessary and sufficient conditions for the validity of the Riemann Hypothesis (Corollaries 3.10 and 3.11).

2. Definitions and background information.

We commence here with some pertinent definitions involving the Laguerre–Pólya class, multiplier sequences and complex zero decreasing sequences. Also we shall state Laguerre's theorem, mentioned in the introduction, and provide some examples to elucidate and motivate the main result of this paper.

Definition 2.1. A real entire function $\phi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ is said to be in the *Laguerre-Pólya class*, written $\phi(x) \in \mathcal{L}\text{-}\mathcal{P}$, if $\phi(x)$ can be expressed in the form

$$\phi(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right) e^{-\frac{x}{x_k}}, \tag{2.1}$$

where $c, \beta, x_k \in \mathbb{R}$, $c \neq 0$, $\alpha \geq 0$, n is a nonnegative integer and $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$. We allow, by the usual convention, the possibility that the product is finite or empty. If $-\infty \leq a < b \leq \infty$ and if $\phi(x) \in \mathcal{L}\text{-}\mathcal{P}$ has all its zeros in (a, b) (or $[a, b]$), then we will use the notation $\phi \in \mathcal{L}\text{-}\mathcal{P}(a, b)$ (or $\phi \in \mathcal{L}\text{-}\mathcal{P}[a, b]$). If $\gamma_k \geq 0$ (or $(-1)^k \gamma_k \geq 0$ or $-\gamma_k \geq 0$) for all $k = 0, 1, 2 \ldots$, then $\phi \in \mathcal{L}\text{-}\mathcal{P}$ is said to be of *type I in the Laguerre-Pólya class*, and we will write $\phi \in \mathcal{L}\text{-}\mathcal{P}I$. We will also write $\phi \in \mathcal{L}\text{-}\mathcal{P}I^+$, if $\phi \in \mathcal{L}\text{-}\mathcal{P}I$ and if $\gamma_k \geq 0$ for all $k = 0, 1, 2 \ldots$.

In order to clarify the terminology above, we remark that if $\phi \in \mathcal{L}\text{-}\mathcal{P}I$, then $\phi \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0]$ or $\phi \in \mathcal{L}\text{-}\mathcal{P}[0, \infty)$, but that an entire function in $\mathcal{L}\text{-}\mathcal{P}(-\infty, 0]$ need not belong to $\mathcal{L}\text{-}\mathcal{P}I$. Indeed, if $\phi(x) = \frac{1}{\Gamma(x)}$, where $\Gamma(x)$ denotes the gamma function, then $\phi(x) \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0]$, but $\phi(x) \notin \mathcal{L}\text{-}\mathcal{P}I$. This can be seen, for example, by noting that functions in $\mathcal{L}\text{-}\mathcal{P}I$ are of exponential type (see, for example, [1, Proof of Corollary 2]), whereas $\frac{1}{\Gamma(x)}$ is an entire function of order one of maximal type (see [3, p. 8] or [12, Chapter 2] for the definition of the "type" of an entire function). Also, it is easy to see that $\mathcal{L}\text{-}\mathcal{P}I^+ = \mathcal{L}\text{-}\mathcal{P}I(-\infty, 0]$.

Definition 2.2. A sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ of real numbers is called a *multiplier sequence* if, whenever the real polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ has *only* real zeros, the polynomial $T[p(x)] = \sum_{k=0}^{n} \gamma_k a_k x^k$ also has *only* real zeros.

The following are well-known characterizations of multiplier sequences (cf. [18], [16, pp. 100–124] or [14, pp. 29–47]). A sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence if and only if

$$\phi(x) = T[e^x] := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}I. \tag{2.2}$$

Moreover, the algebraic characterization of multiplier sequences asserts that a sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence if and only if

$$g_n(x) := \sum_{j=0}^{n} \binom{n}{j} \gamma_j x^j \in \mathcal{L}\text{-}\mathcal{P}I \quad \text{for all } n = 1, 2, 3 \dots. \tag{2.3}$$

A particularly important collection of multiplier sequences were named in [6] as *complex zero decreasing sequences.* These sequences are defined as follows.

Definition 2.3. We say that a sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a *complex zero decreasing sequence (CZDS)*, if for any real polynomial $\sum_{k=0}^{n} a_k x^k$,

$$Z_c\left(\sum_{k=0}^{n} \gamma_k a_k x^k\right) \le Z_c\left(\sum_{k=0}^{n} a_k x^k\right), \tag{2.4}$$

where $Z_c(p(x))$ denotes the number of nonreal zeros of the polynomial $p(x)$, counting multiplicities. (The acronym CZDS will also be used in the plural.)

Now it follows from (2.4) that any complex zero decreasing sequence is also a multiplier sequence. If $T = \{\gamma_k\}_{k=0}^{\infty}$ is a CZDS of *nonzero* real numbers, then inequality (2.4) is equivalent to the statement that for any polynomial p, $T[p]$ has at least as many real zeros as p has. There are, however, CZDS which have zero terms (cf. [6, Section 3]) and consequently it may happen that $\deg T[p] < \deg p$. When counting the real zeros of p, the number generally increases with the application of T, but may in fact decrease due to a decrease in the degree of the polynomial. For this reason, we count nonreal zeros rather than real ones.

For the sake of clarification of the terminology used in the literature, we remark that if $T = \{\gamma_k\}_{k=0}^{\infty}$ is a CZDS of nonzero real numbers, then the sequence of reciprocals, $\Lambda = \{\frac{1}{\gamma_k}\}_{k=0}^{\infty}$ is sometimes called a *zero decreasing sequence* (cf. [1]). In the present setting it is more convenient for us to work with CZDS rather than with the sequences of reciprocals.

The existence of a *nontrivial* CZDS is a consequence of the following theorem proved by Laguerre and extended by Pólya (see Pólya [15] or [16, pp. 314-321]).

Theorem 2.4. (Laguerre [14, Satz 3.2])

(1) *Let* $f(x) = \sum_{k=0}^{n} a_k x^k$ *be an arbitrary real polynomial of degree* n *and let* $h(x)$ *be a polynomial with only real zeros, none of which lie in the interval* $(0, n)$*. Then* $Z_c(\sum_{k=0}^{n} h(k) a_k x^k) \leq Z_c(f(x))$.

(2) *Let* $f(x) = \sum_{k=0}^{n} a_k x^k$ *be an arbitrary real polynomial of degree* n*, let* $\phi \in \mathcal{L}\text{-}\mathcal{P}$ *and suppose that none of the zeros of* ϕ *lie in the interval* $(0, n)$*. Then the inequality* $Z_c(\sum_{k=0}^{n} \phi(k) a_k x^k) \leq Z_c(f(x))$ *holds.*

(3) *Let* $\phi \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0]$*, then the sequence* $\{\phi(k)\}_{k=0}^{\infty}$ *is a complex zero decreasing sequence.*

Partial converses of Laguerre's theorem may be found in [6, 8]. While the theory of multiplier sequences commenced with the work of Laguerre [11] in the 19th century, and while in the ensuing decades this theory gained prominence at the hands of such renowned mathematicians as Pólya, Schur, Schoenberg and Obreschkov among others (see [5] and the references therein), today the *only* known CZDS are those sequences $T = \{\gamma_k\}_{k=0}^{\infty}$ which can be interpolated by functions ϕ in $\mathcal{L}\text{-}\mathcal{P}$ (i.e. $\gamma_k = \phi(k)$). The vexing problem of the existence of other types of CZDS has eluded all previous investigations and so we highlight this problem as

Open Problem 2.5. *Do there exist complex zero decreasing sequences which cannot be interpolated by functions in the Laguerre-Pólya class? In particular, are the sequences* $T = \{\gamma_k\}_{k=0}^{\infty}$*, where* $\gamma_k = e^{-k^p}$*,* $p \geq 3$*, CZDS?*

Next, suppose that $\phi(x) \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0)$ and set $f(z) := \phi(z) + \sin(\pi z)$ for $z \in \mathbb{C}$. Then, since $f(k) = \phi(k)$ for $k = 0, 1, 2, \ldots$, the sequence $\{f(k)\}_{k=0}^{\infty}$ is a CZDS, but, in general, $f \notin \mathcal{L}\text{-}\mathcal{P}$ (cf. Example 2.8 below). Thus, in the absence of additional assumptions, the converse of Laguerre's theorem need not hold true. These considerations have led to the following problem.

Problem 2.6. ([7, Problem 9]) *Let $f(z)$ be a real entire function of exponential type. Suppose that all the zeros of $f(z)$ lie in the strip $|\Im z| < M$ for some $M > 0$. If $\{f(k)\}_{k=0}^{\infty}$ is a CZDS, when does it follow that $f(z) \in \mathcal{L}\text{-}\mathcal{P}$?*

Remark 2.7. We recall (cf. [3, p. 8] or [12, p. 84]) here that entire functions $f(z)$ of at most order one and of normal type are called *entire functions of exponential type* and the *exponential type* of $f(z)$ is

$$\sigma = \overline{\lim_{r\to\infty}} \frac{\log M(r,f)}{r} < \infty,$$

where $M(r,f) = \max_{|z|=r} |f(z)|$. Thus, entire functions of order less than one, or of order one and of minimal type, are said to be of exponential type zero. Equivalently, an entire function $f(z)$ is of exponential type if $f(z) = O(e^{c|z|})$ for some $c \in \mathbb{R}$ as $|z| \to \infty$. Moreover, if $f(z)$ is analytic in $\Re z \geq 0$ and $f(z) = O(e^{c|z|})$ (as $|z| \to \infty$), then we will say that $f(z)$ is of exponential type in $\Re z \geq 0$.

In order to motivate and elucidate the supplementary hypothesis which we will use to prove our main theorem (Theorem 3.6), we conclude this section with the following elementary example.

Example 2.8. Let $f(z) := \phi(z) - \sin(\pi z)$ for $z \in \mathbb{C}$, where $\phi(z) = 2$ for all $z \in \mathbb{C}$. Then $\phi \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0)$, the sequence $\{\phi(k)\}_{k=0}^{\infty} = \{2, 2, 2, \dots\}$ is a CZDS and $f(z)$ is an entire function of exponential type. Moreover, a calculation shows (with $z = x + iy$, $x, y \in \mathbb{R}$) that

$$|f(z)| = |2 - \sin(\pi z)| \geq |\sinh(\pi y) - 2| > 0$$

for $y > A := \frac{1}{\pi}\log(2+\sqrt{5})$. Thus the zeros of $f(z)$ lie in the strip $|\Im z| \leq A$. But $f \notin \mathcal{L}\text{-}\mathcal{P}$ as it has the nonreal zeros $(\frac{1}{2} + 2n) + \frac{i}{\pi}\log(2+\sqrt{3})$, $n = 0, 1, 2, \dots$.

These observations suggest that the assumption in Problem 2.6 that the zeros of f lie in a strip of the form $|\Im z| < M$ is not necessary. On the other hand, if we calculate the exponential type of $f(z)$ (see Remark 2.7), we find that

$$\sigma = \overline{\lim_{r\to\infty}} \frac{\log M(r,f)}{r} = \overline{\lim_{y\to\infty}} \frac{\log |f(\pm iy)|}{y} = \pi. \tag{2.5}$$

This, together with the familiar uniqueness theorems (see, for example, [3, Chapter 9] or [12, Chapter IV]) gives us the clue that a converse of Laguerre's theorem is valid for entire functions $f(z)$ of exponential type, provided we restrict the growth of $f(z)$ along the imaginary axis.

3. The main theorem and applications.

Let $f(z)$ be analytic in a domain D which contains a ray making an angle θ with the positive real axis. In order to measure the growth of $|f(z)|$ along this ray, we will make use of the *(Phragmén–Lindelöf) indicator function*, which is defined as

$$h_f(\theta) := h(\theta) := \overline{\lim_{r\to\infty}} \frac{\log |f(re^{i\theta})|}{r}. \tag{3.1}$$

The proof of our main theorem requires several results which we include here for the reader's convenience.

Theorem 3.1. ([3, Theorem 9.8.1, p. 166]) *Let $f(z)$ be analytic and of exponential type for $|\arg z| \le \alpha$, $\alpha > \pi/2$. Suppose that $h_f(\pm\pi/2) < \pi$. Let $\{\lambda_k\}_{k=0}^{\infty}$ be a sequence such that $\lim_{n\to\infty} \frac{n}{\lambda_n} = 1$ and $|\lambda_m - \lambda_n| \ge \delta|n-m|$ for some $\delta > 0$. If*

$$\lim_{n\to\infty} \frac{\log |f(\lambda_n)|}{\lambda_n} = -\infty,$$

then $f(z) \equiv 0$.

Using the terminology and the notation adopted in Section 2, we can formulate a special case of a recent result proved in [1] as follows.

Theorem 3.2. ([1, Theorem 2]) *Let $\{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k > 0$, be a CZDS. If*

$$\overline{\lim_{k\to\infty}} \gamma_k^{1/k} > 0, \tag{3.2}$$

then there is a function $\phi(z) \in \mathcal{L}\text{-}\mathcal{P}$ of the form

$$\phi(z) := be^{az}\psi(z) := be^{az} \prod_{n=1}^{\infty} \left(1 + \frac{z}{x_n}\right), \tag{3.3}$$

where $a, b \in \mathbb{R}$, $b \neq 0$, $x_n > 0$ and $\sum_{n=1}^{\infty} 1/x_n < \infty$, such that $\phi(z)$ interpolates the sequence $\{\gamma_k\}_{k=0}^{\infty}$; that is, $\gamma_k = \phi(k)$ for $k = 0, 1, 2, \ldots$.

Remark 3.3. If $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence (cf. Definition 2.2) with $\gamma_k \neq 0$, then it is known [18] that either $\gamma_k > 0$, $-\gamma_k > 0$ or $(-1)^k \gamma_k > 0$ for all k. Thus, Theorem 3.2 could be stated in a slightly more general form with the concomitant conclusion that $\gamma_k = \phi(k)$, $-\gamma_k = \phi(k)$ or $(-1)^k \gamma_k = \phi(k)$. We also note that if $a \geq 0$, then $\phi \in \mathcal{L}\text{-}\mathcal{P}I$.

Theorem 3.4. (Carlson's theorem [3, p. 153] or [12, p. 168]) *Let $f(z)$ be analytic and of exponential type in the half plane $\Re z \geq 0$. Suppose that $h_f(-\pi/2) + h_f(\pi/2) < 2\pi$. If $f(k) = 0$ for $k = 0, 1, 2, \ldots$, then $f(z) \equiv 0$.*

Remark 3.5. In Carlson's theorem, the assumption $f(k) = 0$ for $k = 0, 1, 2, \ldots$ can be weakened to $f(k) = 0$ for $k = m, m+1, m+2, \ldots$ for some positive integer m. In this case, let $p(z) = z(z-1)\cdots(z-m+1)$. We can apply the theorem to $g(z) = p(z)f(z)$ since $h_g(-\pi/2) + h_g(\pi/2) < 2\pi$ still holds.

Preliminaries aside, we can now prove the following theorem.

Theorem 3.6. *Let $f(z)$ be an entire function of exponential type. Suppose that $\{f(k)\}_{k=0}^{\infty}$ is a CZDS, where $f(0) = 1$. Let $h_f(\theta)$ denote the indicator function of $f(z)$ (cf. (3.1)), where $\theta \in [-\pi, \pi]$. If $h_f(\pm\pi/2) < \pi$, then $f(z)$ is in $\mathcal{L}\text{-}\mathcal{P}$ and $f(z)$ can be expressed in the form*

$$f(z) = e^{az} \prod_{n=1}^{\infty} \left(1 + \frac{z}{x_n}\right), \tag{3.4}$$

where $a \in \mathbb{R}$, $x_n > 0$ and $\sum_{n=1}^{\infty} 1/x_n < \infty$.

Proof. We first show that the hypotheses imply that $f(k) > 0$ for $k = 1, 2, \ldots$. If $f(m) = 0$ for any positive integer m, then $f(k) = 0$ for $k \geq m$ since any multiplier sequence (see Definition 2.2) has this property [4, Theorem 3.4(a)]. By Theorem 3.4 and Remark 3.5, this cannot happen for the function $f(z)$. Furthermore, by Remark 3.3, the values $f(k)$ are either all positive or alternate in sign. But it is known [3, Theorem 9.9.4, p. 169] that if $f(z)$ is an entire function of exponential type with $h_f(\pm\pi/2) < \pi$, then it is impossible for the numbers $f(k)$ to alternate in sign.

For convenience of notation, we set $\gamma_k := f(k)$, $k = 0, 1, 2, \ldots$ and we begin by showing that the limit

$$L = \lim_{k\to\infty} (f(k))^{1/k} = \lim_{k\to\infty} \gamma_k^{1/k} \tag{3.5}$$

exists. Since $T = \{\gamma_k\}_{k=0}^{\infty}$ is a CZDS, it is known (see, for example, [5]) that T satisfies the *Turán inequalities*

$$\gamma_{k+1}^2 \geq \gamma_k\gamma_{k+2}, \qquad \text{or equivalently,} \qquad \frac{\gamma_{k+2}}{\gamma_{k+1}} \leq \frac{\gamma_{k+1}}{\gamma_k},$$

for $k = 0, 1, 2, \ldots$, where we have used the assumption that $\gamma_k > 0$. Also, from the calculus we know that [19, p.68]

$$\varliminf_{k\to\infty} \frac{\gamma_{k+2}}{\gamma_{k+1}} \leq \varliminf_{k\to\infty} \gamma_k^{1/k} \leq \varlimsup_{k\to\infty} \gamma_k^{1/k} \leq \varlimsup_{k\to\infty} \frac{\gamma_{k+1}}{\gamma_k}. \tag{3.6}$$

But the sequence $\left\{\frac{\gamma_{k+1}}{\gamma_k}\right\}$ is monotone decreasing and so it follows from (3.6) that the limit in (3.5) exists.

Now, if in (3.5) the limit $L = 0$, then $\lim_{k\to\infty} \frac{\log f(k)}{k} = -\infty$ and by Theorem 3.1 (with $\lambda_k = k$) $f(z) \equiv 0$, which is impossible. Thus $L > 0$ and consequently, by Theorem 3.2, there is a function $\phi(z) \in \mathcal{L}\text{-}\mathcal{P}$ of the form (3.3) (with $b = 1$) such that

$$f(k) = \phi(k) \qquad (k = 0, 1, 2, \ldots). \tag{3.7}$$

In order to estimate the indicator function of $f - \phi$ at $\theta = \pm\pi/2$, we recall that (see, for example [1, proof of Corollary 2]) the canonical product $\psi(z)$ in (3.3) is an entire function of order at most one and of minimal type, and hence $h_\psi(\theta) = 0$ for $\theta \in [-\pi, \pi]$. Setting $E(z) := e^{az}$, $a \in \mathbb{R}$, it is easy to verify that $h_E(\pm\pi/2) = 0$ and thus

$$h_\phi(\pm\pi/2) = 0. \tag{3.8}$$

Let $F(z) := f(z) - \phi(z)$ for $z \in \mathbb{C}$. Then $F(z)$ is an entire function of exponential type and by the well-known properties of the indicator function (cf. [12, p. 52]) and (3.8) we have

$$h_F(\pm\pi/2) \leq \max(h_f(\pm\pi/2), h_\phi(\pm\pi/2)) = \max(h_f(\pm\pi/2), 0). \tag{3.9}$$

Since by assumption $h_f(\pm\pi/2) < \pi$, (3.9) yields the estimate

$$h_F\left(-\frac{\pi}{2}\right) + h_F\left(\frac{\pi}{2}\right) < 2\pi. \tag{3.10}$$

Therefore, by (3.7) and (3.10), $F(z)$ satisfies the hypothesis of Carlson's theorem. Thus, $f(z) = \phi(z)$ for all $z \in \mathbb{C}$ and *a fortiori* $f(z)$ is in $\mathcal{L}$-$\mathcal{P}$ and $f(z)$ has a representation of the form (3.4). $\Box$

In order to state some applications of Theorem 3.6, we provide here a slight generalization of Laguerre's theorem. While this result may be found, *mutatis mutandis*, in [1], we will sketch a different proof which may be of interest because of its elementary character.

Theorem 3.7. (Laguerre's theorem on intervals) *Let*

$$\phi(x) := \prod_{n=1}^{\infty}\left(1 + \frac{x}{x_n}\right), \tag{3.11}$$

where $x_n > 0$ *and* $\sum_{n=1}^{\infty} 1/x_n < \infty$. *Then, for any real polynomial* $p(x) = \sum_{k=0}^{n} a_k x^k$,

$$Z_{[0,1]}\left(\sum_{k=0}^{n} \phi(k) a_k x^k\right) \geq Z_{[0,1]}\left(\sum_{k=0}^{n} a_k x^k\right), \tag{3.12}$$

where $Z_{[0,1]}(p(x))$ *denotes the number of real zeros of* $p(x)$ *in the interval* $[0,1]$, *counting multiplicities.*

Proof. Only a sketch of the proof will be given since the main ideas of the proof may be found in [14, p. 6]. We first outline the proof under the simplifying assumptions that (a) $p(0) > 0$ and (b) the real zeros of $p(x)$ in $[0,1]$ are simple.

Consider the special case when $\phi_1(x) := x + \alpha_1$, $\alpha_1 > 0$, so that for the arbitrary polynomial $p(x)$, we have

$$\sum_{k=0}^{n} \phi_1(k) a_k x^k = \alpha_1 p(x) + x p'(x) =: f(x).$$

If $p(x)$ has no real zeros in $[0,1]$, then (3.12) is clear. Otherwise, let $0 < x_1 < x_2 < \cdots < x_m \leq 1$ denote the zeros of $p(x)$ in $[0,1]$. Then $f(x_j) = x_j p'(x_j)$, $j = 1, \ldots, m$, and it follows from Rolle's theorem that $f(x)$ has an odd number of zeros in (x_j, x_{j+1}), whence $f(x)$ has at least $m-1$ zeros in $[0,1]$. Since $p(0) > 0$ and x_1 is the first zero of $p(x)$ to the right of the origin, $p'(x_1) < 0$. Hence $f(x_1) = \alpha_1 p(x_1) + x_1 p'(x_1) < 0$, so that $f(x)$ has a zero in $(0, x_1)$ by the intermediate value theorem. Therefore, $f(x)$ has at least m zeros in $(0,1)$ and so this case of (3.12) is clear.

The argument above remains valid, *mutatis mutandis*, if $p(0) = 0$, since (i) $p(x)$ and $f(x)$ will then vanish at $x = 0$ with the same multiplicity and (ii) $p(x)$ and $f(x)$ have the same sign for sufficiently small $x > 0$. The case when $p(x)$ has a multiple zero in $[0,1]$ can be handled by standard continuity arguments.

Next, if $\phi(x) := \phi_N(x) := \prod_{n=1}^{N} (1 + x/x_n)$, $N \geq 1$, then repeated application of the foregoing argument shows that

$$Z_{[0,1]} \left(\sum_{k=0}^{n} \phi_N(k) a_k x^k \right) \geq Z_{[0,1]} \left(\sum_{k=0}^{n} a_k x^k \right).$$

Finally, taking the limit as $N \to \infty$, we obtain the desired conclusion (3.12). $\square$

Remark 3.8. Theorem 3.7 remains valid for functions of the form $\psi(x) := e^{ax}\phi(x)$, where $a \geq 0$ and $\phi(x)$ is given by (3.11). Moreover, in place of the interval $[0,1]$, we could have stated the theorem for any interval of the form $[-R, 0]$ or $[0, R]$, $0 \leq R \leq \infty$. We hasten to add that Theorem 3.7 is *not* valid for arbitrary functions $\phi \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0)$. Indeed, consider the simple examples $\phi_1(x) := e^{-x}$ and $\phi_2(x) := 1/\Gamma(x+1)$ and apply the sequences $\{\phi_1(k)\}_{k=0}^{\infty}$ or $\{\phi_2(k)\}_{k=0}^{\infty}$ to the polynomial $p(x) := 1 - 2x + x^2$.

The converse of Theorem 3.7, proved in [1], is a deeper theorem. The more precise result may be expressed as follows.

Theorem 3.9. ([1, Theorem 2]) *Let $\{\gamma_k\}_{k=0}^{\infty}$, $\gamma_0 = 1$, $\gamma_k > 0$, be a sequence of positive real numbers. Then, for any real polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$,*

$$Z_{[0,1]}\left(\sum_{k=0}^{n} \gamma_k a_k x^k\right) \geq Z_{[0,1]}\left(\sum_{k=0}^{n} a_k x^k\right)$$

holds if and only if the sequence $\{\gamma_k\}_{k=0}^{\infty}$ can be interpolated by a function of the form (3.11).

For the reader's convenience, we remark that the sequences $\{\gamma_k\}_{k=0}^{\infty}$ which satisfy the inequality above are the reciprocals of sequences belonging to the class τ_1 as defined in [1, equation (7)].

In order to state our applications of the foregoing results, we need to recall some of the nomenclature pertaining to the Riemann Hypothesis. To begin with, Riemann's ξ-function can be defined (cf. [17] or [16]) by

$$\xi(iz) := \frac{1}{2}\left(z^2 - \frac{1}{4}\right)\pi^{-\frac{z}{2}-\frac{1}{4}}\Gamma\left(\frac{1}{4}+\frac{z}{2}\right)\zeta\left(\frac{1}{2}+z\right), \tag{3.13}$$

where ζ is the Riemann ζ-function. It is known that $\xi(z)$ is an entire function of order one (see [20, p. 29] and note that Titchmarsh uses the symbol Ξ for ξ) and of maximal type [9, p. 493]. Moreover, the Riemann ξ-function admits the Fourier integral representation ([17] or [20, p. 255])

$$H(x) := \frac{1}{8}\xi\left(\frac{x}{2}\right) = \int_0^{\infty} \Phi(t)\cos(xt)\,dt = \frac{1}{2}\int_{-\infty}^{\infty} \Phi(t)e^{ixt}\,dt, \tag{3.14}$$

where

$$\Phi(t) := \sum_{n=1}^{\infty} \left(2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t}\right)\exp\left(-\pi n^2 e^{4t}\right).$$

The Riemann Hypothesis is the statement that all the zeros of $H(x)$ are real and so $H \in \mathcal{L}\text{-}\mathcal{P}$. Since

$$H(x) = \sum_{m=0}^{\infty} \frac{b_m}{(2m)!}(-x^2)^m, \tag{3.15}$$

where the b_m's are the moments defined by

$$b_m = \int_0^{\infty} t^{2m}\Phi(t)\,dt, \quad m = 0, 1, 2, \ldots,$$

the change of variables $u = -x^2$ in (3.15) yields

$$F(u) := \sum_{m=0}^{\infty} \frac{b_m}{(2m)!} u^m = \int_0^{\infty} \Phi(t) \cosh(t\sqrt{u})\, dt. \tag{3.16}$$

Thus F is an entire function of order $\frac{1}{2}$ and the Riemann Hypothesis is equivalent to the statement that F has only real negative zeros (so that $F \in \mathcal{L}\text{-}\mathcal{P}I$).

We can now state and prove the following consequence of Theorem 3.6 and Theorem 3.7.

Corollary 3.10. *Let $F(z)$ be the entire function defined by (3.16). Then the following statements are equivalent.*

(1) *The Riemann Hypothesis is true.*
(2) *The sequence $\{F(k)\}_{k=0}^{\infty}$ is a complex zero decreasing sequence.*
(3) *For every polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$, $a_k \in \mathbb{R}$,*

$$Z_{[0,1]}\left(\sum_{k=0}^{n} F(k) a_k x^k\right) \geq Z_{[0,1]}(p(x)). \tag{3.17}$$

Proof. (1) $\Longleftrightarrow$ (2) Suppose that the Riemann Hypothesis is true. Then $F \in \mathcal{L}\text{-}\mathcal{P}I$, whence by Laguerre's theorem (Theorem 2.4) the sequence $\{F(k)\}_{k=0}^{\infty}$ is a CZDS. Conversely, suppose that $\{F(k)\}_{k=0}^{\infty}$ is a CZDS. Then, from the discussion preceding this corollary, we know that the order of the entire function F is $\frac{1}{2}$ and consequently F is of exponential type zero. Also, the indicator function of F (cf. (3.1)) satisfies $h_F(\pm\frac{\pi}{2}) = 0$. Thus by Theorem 3.6, we have $F \in \mathcal{L}\text{-}\mathcal{P}I$.

(2) $\Longleftrightarrow$ (3) Suppose that $\{F(k)\}_{k=0}^{\infty}$ is a CZDS. Then, as above, we conclude that $F \in \mathcal{L}\text{-}\mathcal{P}I$, whence F can be expressed in the form (3.11). Therefore by Theorem 3.7, (3.17) holds. Conversely, suppose that (3.17) holds. Then by Theorem 3.9 the sequence $\{F(k)\}_{k=0}^{\infty}$ can be interpolated by a function of the form (3.11) and hence by Theorem 2.4, $\{F(k)\}_{k=0}^{\infty}$ is a CZDS. $\square$

We remark that the proof of the implication (3) $\Longrightarrow$ (2) of Corollary 3.10 can be based on an elementary argument which does not use Theorem 3.9. Indeed, suppose that (3.17) holds. Now if the sequence $\{F(k)\}_{k=0}^{\infty}$ is not a CZDS, then there exists a polynomial $q(x) := \sum_0^n b_k x^k$, $b_k \in \mathbb{R}$ such that

$$Z_c\left(\sum_{k=0}^{n} F(k) b_k x^k\right) > Z_c(q(x)). \tag{3.18}$$

Let $Q(x) := \sum_0^n F(k) b_k x^k$ and choose $R > 0$ so large that all of the zeros of $Q(x)$ and $q(x)$ are contained in the disk $|z| < R$. Since $Z_c(Q(Rx)) = Z_c(Q(x))$, (3.18) implies that

$$Z_c(Q(Rx)) > Z_c(q(Rx)). \tag{3.19}$$

Using the fact that $F(k) > 0$ for $k = 0, 1, 2, \ldots$, we can express (3.19) in the equivalent form

$$Z_{[-1,1]}(Q(Rx)) < Z_{[-1,1]}(q(Rx)). \tag{3.20}$$

But then either $Z_{[-1,0]}(Q(Rx)) < Z_{[-1,0]}(q(Rx))$ or $Z_{[0,1]}(Q(Rx)) < Z_{[0,1]}(q(Rx))$. By considering $q(-x)$ if necessary, we may conclude that the latter holds. But this contradicts (3.17), and thus $\{F(k)\}_{k=0}^{\infty}$ is a CZDS.

In light of the importance of the theory of the zeta function, it is desirable to express the foregoing result (say the equivalence of (1) and (3) of Corollary 3.10) in terms of the ζ-function. To this end, we use (3.13), (3.14) and (3.16) and the fact that (at least for positive x)

$$8F(4x) = \frac{1}{2}\left(x - \frac{1}{4}\right) \pi^{-\frac{\sqrt{x}}{2} - \frac{1}{4}} \Gamma\left(\frac{1}{4} + \frac{\sqrt{x}}{2}\right) \zeta\left(\frac{1}{2} + \sqrt{x}\right). \tag{3.21}$$

Therefore, as a consequence of (3.21) and Corollary 3.10 and the fact that $F(x) \in \mathcal{L}\text{-}\mathcal{P}I$ if and only if $F(Rx) \in \mathcal{L}\text{-}\mathcal{P}I$ for $R > 0$, we have

Corollary 3.11. *Let $R > 0$. The Riemann Hypothesis is true if and only if*

$$\begin{aligned} Z_{[0,1]}\left(\sum_{k=0}^{n} a_k \left(4R^2 k - \frac{1}{4}\right) \pi^{-R\sqrt{k}} \Gamma\left(\frac{1}{4} + R\sqrt{k}\right) \zeta\left(\frac{1}{2} + 2R\sqrt{k}\right) x^k\right) \\ \geq Z_{[0,1]}\left(\sum_{k=0}^{n} a_k x^k\right), \end{aligned}$$

for every positive integer n *and for arbitrary* $a_0, \dots, a_n \in \mathbb{R}$.

Remark 3.12. It may be of interest to note here that Corollary 3.11, in conjunction with some standard results from the theory of moments (see, for example, [13, p. 131 and p. 142]), yield several *necessary* conditions for the validity of the Riemann Hypothesis. We mention here two such necessary conditions. Let $R > 0$ and for nonnegative integers n and m set

$$\mu_{n,m}(R) := \frac{\pi^{R\sqrt{n+m}}}{(4R^2(n+m)-\frac{1}{4})\Gamma(\frac{1}{4}+R\sqrt{n+m})\zeta(\frac{1}{2}+2R\sqrt{n+m})}.$$

If the Riemann Hypothesis is true, then (a)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\mu_{k,m}(R) > 0 \qquad (m,n=0,1,2,\dots)$$

and (b) there exists a bounded, nondecreasing function $\mu_R(t)$ on $[0,1]$ such that

$$\mu_{n,0}(R) = \int_0^1 t^n \, d\mu_R(t) \qquad (n=0,1,2,\dots).$$

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