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GEOMETRY OF STATIONARY SETS FOR THE WAVE EQUATION IN $\mathbb{R}^n$. THE CASE OF FINITELY SUPPORTED INITIAL DATA: AN ANNOUNCEMENT

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Abstract We consider the Cauchy problem for the wave equation in the whole space $\mathbb{R}^n$, with initial data which are distributions supported on finite sets. The main result is a precise description of the geometry of the sets of stationary points of the solutions to the wave equation. Complete proofs are given in [6].

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§1. Introduction.

Our goal is to understand the structure of stationary (nodal) sets of solutions to the wave equation $\mathbb{R}^n$, $\quad n \geq 2$:

$$\begin{aligned} &u_{tt} = \Delta u, \; u = u(x,t), \; x \in \mathbb{R}^n, \; t > 0, \\ &u\big|_{t=+0} = 0, \quad u_t\big|_{t=+0} = f\,. \end{aligned} \tag{1.1}$$

We will consider the case $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$, the set of distributions supported on a finite set of points. Stationary sets are sets of points $x \in \mathbb{R}^n$ for which the solution to the wave equation is always zero. This article is an announcement and proof sketches for the results in [6]; complete proofs will be given there.

We use standard notation for function spaces; for example, $\mathcal{D}(\mathbb{R}^n)$ is the space of all C^∞-functions with compact support; $\mathcal{E}'(\mathbb{R}^n)$ is the space of compactly supported distributions; $C^\infty_{\text{rad}}(\mathbb{R}^n)$ and $\mathcal{D}_{\text{rad}}(\mathbb{R}^n)$ are the subspaces of corresponding spaces, consisting of radial functions $f, i.e., f(x) = f(|x|)$. Finally, let $R_+ = (0, \infty)$.

For classical solutions to (1.1), we define stationary sets as follows.

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Definition 1. Let $f \in C^\infty(\mathbb{R}^n)$ and u be the (classical) solution for (1.1). Define the stationary set $S(f)$ as the set of time-invariant zeros of the solution u :

$$S(f) = \{x \in \mathbb{R}^n : u(x,t) = 0,\ t > 0\}. \tag{1.2}$$

We use regularization to extend Definition 1 to distributional solutions. Namely, if $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)$, the convolution $f * \varphi$ is smooth and $u * \varphi$ (convolution with respect to x) is in C^∞ and solves (1.1) for the data $f * \varphi$.

Definition 2. For $f \in \mathcal{D}'(\mathbb{R}^n)$ define

$$S(f) = \bigcap_{\varphi \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)} S(f * \varphi), \tag{1.3}$$

where $S(f * \varphi)$ is defined by (1.2).

For continuous f, one can use Theorem 2 below to show Definition 2 coincides with Definition 1. In fact, the set of all $\varphi \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)$ in (1.3) can be replaced by an δ-sequence φ_n, $\varphi_n \in \mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)$.

The main question under consideration is the following.

Problem. *Which sets $S \subset \mathbb{R}^n$ are stationary sets, $S = S(f)$, for some $f \in \mathcal{D}'(\mathbb{R}^n)$ $(f \in \mathcal{E}'(\mathbb{R}^n))$?*

The problem has been solved in the plane [3], [4] if f is an arbitrary distribution of compact support, but not much is known in general. For distributions in the plane, the stationary sets have very restrictive structure; they must be the union of a finite set and a Coxeter system of lines (lines through one point generated by a finite Coxeter group of reflections). This Coxeter set is contained in a translate of the zero set of a homogeneous harmonic polynomial, and it is conical about the point of intersection. Loosely speaking, we will prove that a similar pattern occurs for finitely supported distributions in $\mathbb{R}^n$.

Characterizing stationary sets in $\mathbb{R}^n$ for $n > 2$ is more difficult and only partial results are known. It is known for compactly supported initial data in $\mathbb{R}^n$ that stationary sets are contained in zero sets of harmonic polynomials union algebraic varieties of lower dimension, and it is conjectured that the harmonic polynomial can be assumed to be a translate of a homogeneous polynomial [3], [4]. It is shown in [2] for f sufficiently integrable at infinity that stationary sets cannot have bounded closed components. In [1] and [7] more precise analyses are given in $\mathbb{R}^n$ for stationary sets of lower dimension and conical stationary sets for f with arbitrary growth.

In this article we describe stationary sets for the case of the initial data with finite support for arbitrary dimension n. We prove that, up to a low-dimensional component, the stationary sets are affine algebraic cones with a special geometry.

A *cone* is understood to be a union of straight lines with a common point which is the vertex of the cone. We will call a cone $K \subset \mathbb{R}^n$ k*−flat* with edge L, where L is a k-dimensional plane in $\mathbb{R}^n$, if K is a union of $(k+1)$-planes containing L.

A union $\Sigma = H_1 \cup \cdots \cup H_q$ of hyperplanes $H_i \subset \mathbb{R}^n$ is called a *Coxeter system* of hyperplanes if Σ is invariant with respect to any reflection σ_i around the hyperplane H_i, $i = 1, \ldots, q$. The *Coxeter group* generated by the reflection $\sigma_1, \ldots, \sigma_q$ will be denoted by $W(\Sigma)$.

We will call a polynomial P in $\mathbb{R}^n$, with real coefficients *a harmonic divisor* if P divides a nonzero harmonic polynomial. Zero sets of homogeneous harmonic divisors will be called *harmonic cones.*

For any set $F \subset \mathbb{R}^n$, the affine subspace spanned by F will be denoted by span F.

Our main result is the following.

Theorem 1. *Let f be a distributions supported at a finite number of points, $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$. Assume $f \neq 0$. If $S(f) \neq \emptyset$, then*

(a) *$S(f)$ is an algebraic variety in $\mathbb{R}^n$, contained in the zero set of a nonzero harmonic polynomial.*

(b) *After a suitable translation, the set $S(f)$ can be represented in the form*

$$S(f) = S_0 \cup V,$$

where V is an algebraic variety of $\operatorname{codim} V > 1$ *and S_0, assuming it is nonempty, is a harmonic cone, which is a $(n-1)-$dimensional real algebraic variety.*

In addition, the following is true:

(c) *The conical component S_0, in general, has the two components*

$$S_0 = \Sigma \cup K,$$

where K contains $\operatorname{supp} f$ *but Σ does not, $\Sigma \cap K \neq \emptyset$ provided both Σ and K are nonempty, Σ is a Coxeter system and K is a k-flat harmonic cone with the edge $L =$ span $(\operatorname{supp} f)$, $k = \dim L < n$. If* $\operatorname{supp} f$ *is a generic set, i.e., $k = n$, then $K = \emptyset$. If $k = n-1$, then K is a hyperplane and $\Sigma \cup K$ is a Coxeter system.*

(d) *If $\tilde{\Sigma}$ is the union of all hyperplanes contained in S_0, then $\tilde{\Sigma}$ is again a Coxeter system; the distribution f is odd with respect to any reflection $\sigma \in W(\tilde{\Sigma})$, i.e. $f \circ \sigma = -f$; the sets S_0, V, $S(f)$ and* $\operatorname{supp} f$ *are $W(\tilde{\Sigma})$-invariant.*

Theorem 1 says that finite sets of point sources generate stationary sets which are necessarily algebraic varieties and which are either small (empty or low-dimensional) or up to a low-dimensional component, are $(n-1)$-dimensional cones which suitably translated are determined by zeros of spatial harmonics.

The geometry of the essential, conical part is as follows. If the set of points in $\operatorname{supp} f$ is generic then the cone is a Coxeter system of hyperplanes. These stationary sets may appear only as a result of a Coxeter skew-symmetry of the initial data. If $\operatorname{supp} f$ lies in a proper affine subspace in $\mathbb{R}^n$, then another component may appear which is a cone containing $\operatorname{supp} f$. In the plane, for any compactly supported f, $S(f)$ is, up to a finite set, a Coxeter system of lines [3]. However, in the plane the set K in Theorem 1 would be a collection of lines and therefore a Coxeter system by (d).

An important problem in studying of the wave equation is characterizing *nodal* sets (see [12], I, Ch.5, S.5), that is zero sets of eigenfunctions of the Laplace operator, or, equivalently, zero sets of time-harmonic solutions of the wave equation. This problem has been studied by many authors. Results on this subject mainly say that nodal sets are hypermanifolds with singularities and the eigenfunctions cannot vanish to high order on the nodal sets (see, *e.g.,* [13], [14], [11], [9] and others).

The problem under consideration is directly related to describing nodal sets. Indeed, extending the solution $u(x,t)$ of (1.1) for $t<0$ by $u(x,t)=-u(x,-t)$ and applying Fourier transform in t to the both sides of (1.1) yields

$$-\lambda^2 v(x,\lambda)=\Delta v(x,\lambda),$$

where $v(x,\lambda)$ is the Fourier transform evaluated at arbitrary $\lambda\in\mathbb{R}$.

Thus the stationary set $S(f)$ (1.2) is just the intersection of nodal sets of all the eigenfunctions $v(\cdot,\lambda)$ which are, since the initial data f has compact support, nonzero for an infinite number of λ. Thus, while for a single eigenfunction or a finite linear combination of eigenfunctions, the available information is the general analytic structure of nodal sets, our result shows that joint nodal hypermanifolds of one-parameter families of eigenfunctions are strongly determined geometrically. It is worth noting in this connection the result of [10] which describes geometry of nodal lines of bounded membrane under the assumption that these lines contain a open piece of a straight line.

This announcement continues a series of works [3], [4], [5], [2], [1], [7], [8], started by [3] and devoted to the description of injectivity sets for the spherical transform, stationary sets for the wave and heat equations, and related problems. Our initial interest in the problem was motivated by a problem in approximation theory posed in [18] (cf. [3], [4]). Complete proofs of our results are given in [6].

§2. Proof outline for Theorem 1.

We will relate our main theorem to a spherical Radon transform and prove a support theorem for this transform.

Let $f\in C(\mathbb{R}^n)$. Define the spherical transform $C(\mathbb{R}^n)\ni f\to\hat f\in C(\mathbb{R}^n\times\mathbb{R}_+)$ by

$$\hat f(x,r)=Rf(x,r)=\int_{|\theta|=1} f(x+r\theta)dA(\theta), \tag{2.1}$$

where dA is the normalized surface measure on the unit sphere. Because this spherical transform is a Fourier integral operator, it can be defined on distributions.

Theorem 2. *Let $f\in C(\mathbb{R}^n)$ and let $S(f)$ be defined by* (1.2). *Then*

$$S(f)=\{x\in\mathbb{R}^n : \hat f(x,r)=0 \quad \textit{for all} \quad r>0\} \tag{2.2}$$

and also

$$S(f)=\{x\in\mathbb{R}^n : (f*u)(x)=0 \quad \textit{for all} \quad u\in\mathcal{D}_{\mathrm{rad}}(\mathbb{R}^n)\}. \tag{2.3}$$

The second equality is a key to proving the consistency of Definitions 1 and 2 for smooth functions. The proof of Theorem 2 follows from the Poisson-Kirchoff formula and integral equations techniques (e.g., [19]).

For $f\in\mathcal{E}'(\mathbb{R}^n)$, one can show that $S(f)$ satisfies equality (2.2) distributionally. One needs to observe that, for each $x\in\mathbb{R}^n$ one can define $\hat f(x,\cdot)$ as a distribution on even functions in $\mathcal{D}(\mathbb{R})$ using the natural map between this set of functions and the set of smooth, compactly supported radial functions centered at x. This is related to (2.3).

If Q is a polynomial, we let $N(Q)$ denote the zero set of Q in $\mathbb{R}^n$. Using ideas related to those in [3], [4], we prove

Theorem 3. *Let $f \in \mathcal{E}'(\mathbb{R}^n)$ and $f \neq 0$. Then*

a) *The set $S(f)$ is an algebraic variety in $\mathbb{R}^n$ contained in the zero sets of a nonzero harmonic polynomial.*
b) *$S(f) = S_0 \cup V$, where V is an algebraic variety of* $\operatorname{codim} V > 1$*, $S_0 = N(Q)$ and Q is a harmonic divisor.*

This gives us the key to understand the algebraic structure of $S(f)$.

We now need to understand the geometric structure of $S(f)$. We introduce some new notation. Let $r \in \mathbb{R}_+$, $S \subset \mathbb{R}^n$, and $x \in S$. The point x is called a *regular point* of S if and only if there is a connected real-analytic hypersurface, A, (an $(n-1)-$dimensional real-analytic submanifold of $\mathbb{R}^n$) such that $x \in A \subset S$. Let x be a regular point of S, and let A be such an associated hypersurface ($x \in A \subset S$). Then, we let T_x denote the hyperplane tangent to A at x. The points y and y' in $\mathbb{R}^n$ are said to be T_x-*mirror* if and only if they are reflections about T_x. If $y \in T_x$, then y is its own mirror point, and we say y is *self-mirror*. Our next theorem is a microlocal version of a reflection principle of Courant and Hilbert.

Theorem 4. *Let $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$, $f \neq 0$. We let $x_0 \in S = S(f)$ be a regular point. Let $A \subset S$ be a connected real-analytic hypersurface containing x_0, and let T_{x_0} be the hyperplane tangent to A at x_0. Let $y_0 \in \operatorname{supp} f \setminus x_0$. Then, the $T_{x_0}-$mirror point to y_0 must also be in* $\operatorname{supp} f$.

Courant and Hilbert [12 II, pp. 699 *ff.*] proved the *reflection principle* that if $A = T_{x_0}$ is a hyperplane and f has zero integrals over all spheres centered on T_{x_0}, then f has to be an odd function about T_{x_0}. Therefore, if f has zero integrals over such spheres, and f is zero at one mirror point, it must be zero at the other. Our theorem is a microlocal version of that fact where the tangent plane provides the microlocal symmetry. It is proven by calculating the microlocal properties of the spherical transform and using a theorem of Kawai, Kashiwara and Hörmander about analytic wavefront at boundary points of supp f [17].

Proof sketch for Theorem 1. Let $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$. Let $x_0 \in A \subset S$ be a regular point, let A be a real-analytic hypersurface such that $x_0 \in A \subset S$. Let T_{x_0} be the plane tangent to A at x_0. Using the result of Theorem 3, we have $S(f) = N(Q) \cup V$ where Q is a harmonic divisor; $S_0 = N(Q)$.

The geometric part of the proof has two cases.

First, assume $\operatorname{supp} f \not\subset T_{x_0}$. Then, there is an $r_0 > 0$ and a point $a_0 \in S(x_0, r_0)$ that is in $\operatorname{supp} f \setminus T_{x_0}$. By the support theorem, the mirror point, a_m, to a_0 must also be in $\operatorname{supp} f$. By definition, T_{x_0} is the perpendicular bisector of the segment $\overline{a_0 a_m}$. For $x \in A$ near x_0, the sphere $S(x, |x - a_0|)$ must meet $\operatorname{supp} f$ at a_0 and at its T_x mirror point by Theorem 4. Because there are only a finite number of points in $\operatorname{supp} f$ and x is close to x_0, this mirror point must be a_m. So, T_x is the perpendicular bisector of $\overline{a_0 a_m}$ for all $x \in A$ sufficiently close to x_0. Therefore, $T_x = T_{x_0}$. This shows locally near x_0, A is flat. Because $S(f)$ is an algebraic variety, this shows that $T_{x_0} \subset S(f)$. Now, we use the reflection principle [12 II, pp. 699 *ff.*] to conclude f is odd about T_{x_0}.

We do this construction for each regular point in $x \in S$ for which $\operatorname{supp} f \notin T_x$. This gives us a collection of hyperplanes contained in $S(f)$ such that f is odd about each one. We know this collection is finite since $S(f)$ is an algebraic variety. It also gives us a Coxeter group of reflections about these hyperplanes, and the set of all such hyperplanes generated by the Coxeter group is Σ. Using properties of Coxeter groups [15, Ch. 8, S.10, Th. 8], [16, Prop. 4.1.3], we show the Coxeter group is finite and the intersection $\cap\Sigma$ is nonempty. We use Theorem 3 and the irreducible factors of the harmonic divisor Q corresponding to Σ to show these factors are all homogeneous about the points in $\cap\Sigma$.

Now, for the second case, assume $\operatorname{supp} f \subset T_{x_0}$. This implies $\dim L < n$. Let $a_0 \in \operatorname{supp} f$. In this case, a_0 is a $T_{x_0}-$self-mirror point. If there were an $x \in A$ near x_0 for which a_0 was not T_x-self-mirror, then this would contradict Theorem 4 because by finiteness of $\operatorname{supp} f$, there would be no T_x-mirror point to a_0 in $\operatorname{supp} f$. This shows that locally A generates a subset of S_0 that is conical about a_0. By a linearity argument, A generates a subset of S_0 conical about any point in $L = \text{span } \operatorname{supp} f$. We use Theorem 3 and the irreducible factors of the harmonic divisor Q corresponding to K to show these factors are all homogeneous about points in $L = \text{span } \operatorname{supp} f$.

This finishes the geometric part of the proof. If $K \neq \emptyset$ and $\Sigma \neq \emptyset$, we show, using Kakutani' Theorem [15] and invariance of L under $\Sigma-$ reflections, that L and Σ have common points and therefore $S_0 = K \cup \Sigma$ is a cone about any such a point. This finishes the proof sketch. $\square$

§3. Sufficient Conditions for Stationary Sets.

Theorem 1 says that the stationary sets $S(f)$ for $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$ may consist of three parts: a low-dimensional variety V, a Coxeter system Σ, and a cone K having all points in span ($\operatorname{supp} f$) as vertices. In addition, the union $S_0 = \Sigma \cup K$ must be a cone containing the zero set of some shifted harmonic homogeneous polynomial and the entire stationary set $S(f) = \Sigma \cup K \cup V$ must belong to the zero set of some nonzero harmonic (not necessarily homogeneous, if V is not a cone with the common vertex with S_0).

Now the question is whether all the possibilities are realizable. Namely, whether each of the sets Σ, K, V and any unions of the sets of these three types are the stationary set $S(f)$ for some $f \in \mathcal{E}'_{\text{fin}}(\mathbb{R}^n)$ or, more generally, are contained in a stationary set?

Below we give positive answers for the sets Σ, K, V and $\Sigma \cup V$. The case of $\Sigma \cup K \cup V$, where each of the three sets are nonempty, remains unsolved.

Given a polynomial $G \in \mathbb{R}[x_1, \dots, x_n]$, denote by T_G the distribution $\langle T_G, \varphi\rangle = G(\partial)\varphi(0)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Here $G(\partial) = G\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$.

The following theorem shows that the stationary set generated by a homogeneous distribution (of finite order) supported at a single point coincides with common zeros of iterated Laplacians of the symbol of the corresponding differential operator:

Theorem 5. *For any homogeneous polynomial $G \in \mathbb{R}[x_1 \dots x_n]$, we have*

$$S(T_G) = \bigcap_{j \geq 0} N(\Delta^j(G)).$$

The following two corollaries prove that zero sets of harmonics are stationary sets of a homogeneous distribution supported at a single point and describe all such distributions:

Corollary 6. *If Ψ is a homogeneous harmonic polynomial, then $N(\Psi) = S(T_\Psi)$.*

Corollary 7. *Let Ψ be a homogeneous harmonic polynomial and G a polynomial in $\mathbb{R}^n$. Then $N(\Psi) \subset S(T_G)$ if and only if Ψ divides all the polynomials G, ΔG, $\Delta^2 G, \dots$.*

Proof. Since Ψ is homogeneous, then it is easy to check that $N(\Psi) \subset S(T_G)$ is equivalent to $N(\Psi) \subset S(T_{G_m})$, where G_m is any homogeneous term of G. In turn, by Theorem 5, this is equivalent to $G_m, \Delta G_m, \Delta^2 G_m, \dots$ vanishing on $N(\Psi)$. It can be proven that vanishing on zeros of a real harmonic polynomial is equivalent to divisibility, therefore all the homogeneous terms G_m, along with their iterated Laplacians, are divisible by Ψ. This proves the corollary. □

Finally, any low-dimensional real algebraic variety can be stationary for some solution of the wave equation with point supported initial data and, moreover, any Coxeter system can be added:

Theorem 8. [1] *Let V be an algebraic variety in $\mathbb{R}^n$, $\operatorname{codim} V > 1$. Let Σ be either empty or a Coxeter system of hyperplanes. Then there exists a nontrivial polynomial $G \in \mathbb{R}[x_1, \dots, x_n]$ such that $\Sigma \cup V \subset S(T_G)$.*

Remark. In order to prove that $N(\Psi) \cup V$, Ψ is a homogeneous harmonic polynomial, $\operatorname{codim} V > 1$, can be realized as a stationary set, it would be sufficient to prove, according to Corollary 7, that the set of homogeneous polynomials G such that Ψ divides all $\Delta^s G$, $s = 0, 1, \dots$ is big enough to satisfy the additional condition on the low-dimensional part: $\Delta^s G\big|_V = 0$, $s = 0, 1, \dots$. Due to what has been proven in Theorem 5, this means that all the harmonic homogeneous polynomials h_{k-2j} in the decomposition

$$G(x) = h_k(x) + |x|^2 h_{k-2}(x) + \dots, \quad k = \deg G$$

are divisible by Ψ and vanish on V. However, the question about whether the space of harmonic homogeneous polynomials h divisible by a given harmonic Ψ is big enough turned out to be very nontrivial in $\mathbb{R}^n$ for $n > 2$ (*cf.* [1]). We even do not know whether this space is always infinite dimensional or not.

§4. The Case of Balls.

Similar arguments can be used to prove a theorem similar to Theorem 1 if $\operatorname{supp} f$ is (roughly) the union of disjoint balls. Let $\mathcal{E}'_D(\mathbb{R}^n)$ be the set of distributions whose support is contained in the union of a finite number of disjoint closed balls and whose support contains the boundaries of these balls. Distributions in $\mathcal{E}'_D(\mathbb{R}^n)$ can be arbitrary inside each closed ball, but their support must contain the entire boundary of each ball. Here is the geometric analogue of Theorem 1 for $\mathcal{E}'_D(\mathbb{R}^n)$.

Theorem 9 (Support Theorem). *Let $f \in \mathcal{E}'_D(\mathbb{R}^n)$, $f \neq 0$. Assume $S = S(f) \neq \emptyset$. Assume there are regular points in S, and let $x_0 \in S$ be a regular point and $x_0 \notin \operatorname{supp} f$. Let A be a connected real-analytic hypersurface in $\mathbb{R}^n$ such that $x_0 \in A \subset S$. Let T_{x_0} be the hyperplane tangent to A at x_0. Let C be the set of centers of the disks making up $\operatorname{supp} f$. There are two possibilities.*

(a) *For some $c_0 \in C$, $c_0 \notin T_{x_0}$. In this case $A \subset T_{x_0} \subset S$ and $\operatorname{supp} f$ is symmetric about T_{x_0}. Furthermore, f is odd about T_{x_0}.*
(b) *Or, $C \subset T_{x_0}$. In this case, near x_0, S is conical about $L = \operatorname{span} C$. Precisely, A generates a subset of S that is conical with edge L. In this case, $k = \dim L < n$.*

§5. Concluding Remarks.

Theorem 1 asserts that for an initial distributions with finite support, the essential $(n-1)-$dimensional part of the stationary set is a cone. From Section 3, we learn this cone appears as the set of common zeros of spatial harmonics in the Fourier decomposition of the initial distribution. Correspondingly, this happens only when these harmonics have a large set of common zeros (are coherent). More specifically, the cone may contain a system of Coxeter mirrors, if the initial data (sources) admit a corresponding symmetry. In this case vanishing of the solution of the wave equation on the mirrors is the result of cancelling of waves propagated by symmetric sources.

We expect that the stationary sets have a similar geometry for compactly supported initial data and, more generally, for distributions vanishing sufficiently fast at infinity. The main difficulty in proving that is obtaining the conical structure of the essential part of stationary sets. This was done in [3], [4] for $n = 2$. There, the simple structure of zero sets of harmonic polynomials of two variables and the support theorem, Theorem 4, play important roles. Lack of information about zero sets of harmonic polynomials of more than two variables was our main obstacle in extending our approach to $n > 2$. Nevertheless, we hope to succeed using a deeper analysis of the algebraic and geometric structure of stationary sets and by refining the microlocal results that go into the proof of Theorem 1 to be valid more generally, such as for rapidly decreasing functions.

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