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On the Existence and Stability of Unfoldings of Equivariant Two-Parameter Bifurcation Problems

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Abstract

Under the action of Γ-equivariant **U***-equivalence group, we study the unfoldings of equivariant two-parameter bifurcation problems of finite codimension, where the bifurcating parameters are of vector value. The main results : the existence theorem , stability theorem and the versal unfolding theorem (a lemma is proved and by means of it we give simple proofs of the theorems), the uniqueness theorem.*

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Golubitsky and Schaeffer[1][2] introduced the idea of applying the methods and ideas in singularity theory and Lie group theory to study equivariant bifurcation problems. In many bifurcation problems, two or more bifurcation parameters appear. [3] introduced **E**-equivalence and **U**-equivalence for such problems which has one real state variable with two real parameters and considered the classification under **E**-equivalence.

When n bifurcation parameters are considered as a vector, a more general "**E**"-equivalence is employed (cf. [4][5]). But when bifurcation parameters are two vectors and one may be dependent on the other, a more general "**U**"-equivalence (which is still called **U**-equivalence in this paper) should be used. In addition, the **U**-equivalence is more general than **E**-equivalence . In fact, if we let one parameter be zero vector at all time in the **U**-equivalence, we get **E**-equivalence. Furthermore, suppose a compact Lie group Γ acts on the state space. Therefore we will consider Γ-equivariant **U**-equivalence.

The main results in this paper include: (1) the existence theorem , the stability theorem and the versal unfolding theorem, we prove a lemma by which we can give simple proofs of the theorems. (2) the orbits of two-parameter bifurcation problems are one-to-one corresponding to the equivalent classes of versal k-parameter unfoldings. It is a analogue of Mather's fundamental theorem (cf. [6] or [7]) and is fundamental in the classification.

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All mapping germs in this paper are of C^∞.

1 Preliminaries

Let Γ be a compact Lie group, it acts linearly on the state space $\mathbb{R}^n$ and $x=(x_1,\ldots,x_n)$ is the coordinate in it. The bifurcation parameters are $\lambda=(\lambda_1,\ldots,\lambda_s)\in\mathbb{R}^s$ and $\mu=(\mu_1,\ldots,\mu_t)\in\mathbb{R}^t$. Γ also acts on $\mathbb{R}^n\times\mathbb{R}^s\times\mathbb{R}^t$ naturally, the action of it on $\mathbb{R}^s\times\mathbb{R}^t$ is trivial.

Let $\mathbf{E}_{x,\lambda,\mu}(\Gamma)=\{g:(\mathbb{R}^n\times\mathbb{R}^s\times\mathbb{R}^t,0)\to\mathbb{R}^n|$ germ g satisfies the following condition (1)$\}$, and $\mathbf{E}_{\lambda,\mu}$ is the set of mapping germs $f:(\mathbb{R}^s\times\mathbb{R}^t,0)\to\mathbb{R}^s$, similarly, $\mathbf{E}_\mu$ is the set of germs $f:(\mathbb{R}^t,0)\to\mathbb{R}^t$. Let $\vec{\mathbf{E}}_{x,\lambda,\mu}(\Gamma)=\{S:(\mathbb{R}^n\times\mathbb{R}^s\times\mathbb{R}^t,0)\to gl(n)|$ germ S satisfies the following condition (2)$\}$.

$$g(\gamma x,\lambda,\mu)=\gamma g(x,\lambda,\mu),\forall\gamma\in\Gamma, \tag{1}$$

$$S(\gamma x,\lambda,\mu)=\gamma S(x,\lambda,\mu)\gamma^{-1},\forall\gamma\in\Gamma. \tag{2}$$

It can be proved that $\mathbf{E}_{x,\lambda,\mu}(\Gamma)$ and $\vec{\mathbf{E}}_{x,\lambda,\mu}(\Gamma)$ are finitely generated modules over the ring $\varepsilon_{x,\lambda,\mu}(\Gamma)$ of Γ-invariant function germs (cf. [8]). If $g\in\mathbf{E}_{x,\lambda,\mu}(\Gamma)$ satisfies $g(0)=0$ and $(D_xg)_0$=0, we call it a $\Gamma-$ *equivariant bifurcation problem* with two-parameters (or **bifurcation problem** for short), where $(D_xg)_0$ is the derivative $\frac{\partial g}{\partial x}$ at 0. Let $\mathbf{M}^l_{x,\lambda,\mu}(\Gamma)$ be the subspace of $\mathbf{E}_{x,\lambda,\mu}(\Gamma)$, a germ $g\in\mathbf{M}^l_{x,\lambda,\mu}(\Gamma)$ iff all of the derivatives of g at $0\in\mathbb{R}^n\times\mathbb{R}^s\times\mathbb{R}^t$ with orders less than l are equal to zero. When l=1, $\mathbf{M}^1_{x,\lambda,\mu}(\Gamma)$ is denoted by $\mathbf{M}_{x,\lambda,\mu}(\Gamma)$. We define the subspace $\mathbf{M}^l_{\lambda,\mu}$ of $\mathbf{E}_{\lambda,\mu}$ and $\mathbf{M}^l_\mu$ of $\mathbf{E}_\mu$ similarly. Let

$$\mathfrak{U}=\{(S,X,\Lambda,M)\in\vec{\mathbf{E}}_{x,\lambda,\mu}(\Gamma)\times\mathbf{M}_{x,\lambda,\mu}(\Gamma)\times\mathbf{M}_{\lambda,\mu}\times\mathbf{M}_\mu;$$

$$S(0)\in\mathfrak{L}_\Gamma(\mathbb{R}^n)^o,(D_xX)_0\in\mathfrak{L}_\Gamma(\mathbb{R}^n)^o,(D_\lambda\Lambda)_0\in\mathfrak{L}(\mathbb{R}^s)^o,(D_\mu M)_0\in\mathfrak{L}(\mathbb{R}^t)^o\}.$$

Where $\mathfrak{L}_\Gamma(\mathbb{R}^n)$ is the linear space of Γ-equivariant linear maps on $\mathbb{R}^n$ and $\mathfrak{L}_\Gamma(\mathbb{R}^n)^o$ is the connected component of $\mathfrak{L}_\Gamma(\mathbb{R}^n)\cap GL(\mathbb{R}^n)$ which contains the unity.

We say bifurcation problems g and h are $\Gamma-$*equivariant* $\mathbf{U}-$*equivalent* (or **equivalent**) if $\exists(S,X,\Lambda,M)\in\mathfrak{U}$ such that

$$h(x,\lambda,\mu)=S(x,\lambda,\mu)g(X(x,\lambda,\mu),\Lambda(\lambda,\mu),M(\mu)).$$

It is clearly a equivalent relation. When g and h are equivalent, the equality above is also denoted by $h=u\cdot g$, $u=(S,X,\Lambda,M)\in\mathfrak{U}$. Let

$$T(g,\mathfrak{U})=\vec{\mathbf{E}}_{x,\lambda,\mu}(\Gamma)\cdot g+(D_xg)\mathbf{E}_{x,\lambda,\mu}(\Gamma)+(D_\lambda g)\mathbf{E}_{\lambda,\mu}+(D_\mu g)\mathbf{E}_\mu, \tag{3}$$

$$\tilde{T}(g)=\vec{\mathbf{E}}_{x,\lambda,\mu}(\Gamma)\cdot g+(D_xg)\mathbf{E}_{x,\lambda,\mu}(\Gamma), \tag{4}$$

then $\tilde{T}(g)$ is a submodule of $\mathbf{E}_{x,\lambda,\mu}(\Gamma)$ on the ring $\varepsilon_{x,\lambda,\mu}(\Gamma)$. g is of *finite codimension* if $dim_{\mathbb{R}}\mathbf{E}_{x,\lambda,\mu}(\Gamma)/\tilde{T}(g)<+\infty$. A Γ-equivariant mapping germ $G:(\mathbb{R}^n\times\mathbb{R}^s\times\mathbb{R}^t\times\mathbb{R}^k,0)\to$

$(\mathbb{R}^n, 0) \in \mathbf{E}_{x,\lambda,\mu\,\alpha}(\Gamma)$, is called a ($k$-parameter) *unfolding* of g if it satisfies $G(x, \lambda, \mu, 0) = g(x, \lambda, \mu)$. A k-parameter unfolding H of g is called *factors through* another l-parameter unfolding G of g if

$$H(x, \lambda, \mu, \beta) = S(x, \lambda, \mu, \beta)G(X(x, \lambda, \mu, \beta), \Lambda(\lambda, \mu, \beta), M(\mu, \beta), \psi(\beta)), \tag{5}$$

where $S : (\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^k, 0) \to gl(n)$ and $\psi : (\mathbb{R}^k, 0) \to \mathbb{R}^l$ and

$$\psi(0) = 0, M(\mu, 0) = \mu, \Lambda(\lambda, \mu, 0) = \lambda; \tag{6}$$

$$S(x, \lambda, \mu, 0) = I_n, X(x, \lambda, \mu, 0) = x; \tag{7}$$

$$S(\gamma x, \lambda, \mu, \beta) = \gamma S(x, \lambda, \mu, \beta)\gamma^{-1}, X(\gamma x, \lambda, \mu, \beta) = \gamma X(x, \lambda, \mu, \beta), \forall \gamma \in \Gamma. \tag{8}$$

Suppose H factors through G and (5)-(8) hold, then we say H and G are $\Gamma - isomorphic$ if ψ is a diffeomorphism. A unfolding G of g is *versal* if every unfolding H of g factors through G. Suppose G_1, G_2, G_3 are unfoldings of g. If G_1 factors through G_2 and G_2 factors through G_3, then G_1 factors through G_3 .

2 Existence and Uniqueness

Definition 1 *Let f,g be bifurcation problems, F and G be k-parameter unfoldings of f and g respectively. F and G are called* **equivalent** *if F can be written in the form:*

$$F(x, \lambda, \mu, \alpha) = S(x, \lambda, \mu, \alpha)G(X(x, \lambda, \mu, \alpha), \Lambda(\lambda, \mu, \alpha), M(\mu, \alpha), \phi(\alpha)), \tag{9}$$

where $S(x, \lambda, \mu, \alpha), X(x, \lambda, \mu, \alpha)$ satisfy (8) in §1 and

$$(S(\cdot, \cdot, \cdot, 0), X(\cdot, \cdot, \cdot, 0), \Lambda(\cdot, \cdot, 0), M(\cdot, , 0)) \in \mathfrak{U}, \tag{10}$$

$$\phi : (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)\ is\ a\ differential\ homeomorphism. \tag{11}$$

Corollary 1 *Let f,g be bifurcation problems, F and G be k-parameter unfoldings of f and g respectively. If F and G are equivalent, then f and g are equivalent.*

Theorem 1 *Let bifurcation problem g be of finite codimension,*

$$l = dim_{\mathbb{R}}\mathbf{E}_{x,\lambda,\mu}(\Gamma)/T(g, \mathfrak{U}).$$

Then g has a versal l-parameter unfolding.

Proof. It follows from Theorem 5 (in §3), we omit the details.

Theorem 2 *Suppose F, G are versal k-parameter unfoldings of $f(x, \lambda, \mu) = F(x, \lambda, \mu, 0)$ and $g(x, \lambda, \mu) = G(x, \lambda, \mu, 0)$ respectively, where f and g are bifurcation problems of finite codimension, $k \in \mathbb{N}$. Then F and G are equivalent if and only if f and g are equivalent.*

Proof. We only need to prove the "if" part. Because f and g are equivalent, there is an $u = (S, X, \Lambda, M) \in \mathfrak{U}$, such that $g = u \cdot f$. Let

$$H(x, \lambda, \mu, \alpha) = S(x, \lambda, \mu) F(X(x, \lambda, \mu), \Lambda(\lambda, \mu), M(\mu), \alpha), \tag{12}$$

where $\alpha \in (\mathbb{R}^k, 0)$. Then H is a k-parameter unfolding of g. We divide the following proof into three steps: 1.H is a versal unfolding of g; 2.H and G are equivalent ; 3.F and G are equivalent.

Fact 1. H is a versal unfolding of g.

By $u = (S, X, \Lambda, M)$, we get a Γ-equivariant diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t, 0) \to (\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t, 0)$, $\Phi(x, \lambda, \mu) = (X(x, \lambda, \mu), \Lambda(\lambda, \mu), M(\mu)) \equiv (x', \lambda', \mu')$. Then the inverse of Φ , $\Phi^{-1} : (\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t, 0) \to (\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t, 0)$ is also a Γ-equivariant diffeomorphism germ of the form $(x', \lambda', \mu') \overset{\Phi^{-1}}{\longmapsto} (X'(x', \lambda', \mu'), \Lambda'(\lambda', \mu'), M'(\mu'))$. By (12) we have

$$F(x', \lambda', \mu', \alpha) = S^{-1}(\Phi^{-1}(x', \lambda', \mu')) H(X'(x', \lambda', \mu'), \Lambda'(\lambda', \mu'), M'(\mu'), \alpha), \tag{13}$$

where S^{-1} is the inverse matrix of S.

Let $K(x, \lambda, \mu, \beta)$ be a l-parameter unfolding of g. We prove that K factors through H. Because $u \cdot f = g = K(x, \lambda, \mu, 0)$, we have

$$S^{-1}(\Phi^{-1}(x', \lambda', \mu')) K(\Phi^{-1}(x', \lambda', \mu'), 0) = f(x', \lambda', \mu').$$

Then $S^{-1}(\Phi^{-1}(x', \lambda', \mu')) K(\Phi^{-1}(x', \lambda', \mu'), \beta)$ is an unfolding of f. Since $F(x', \lambda', \mu', \alpha)$ is a versal unfolding of f, so $S^{-1}(\Phi^{-1}(x', \lambda', \mu')) K(\Phi^{-1}(x', \lambda', \mu'), \beta)$ factors through F:

$$\begin{gathered} S^{-1}(\Phi^{-1}(x', \lambda', \mu')) K(\Phi^{-1}(x', \lambda', \mu'), \beta) = \\ \widetilde{S}(x', \lambda', \mu', \beta) F(\widetilde{X}(x', \lambda', \mu', \beta), \widetilde{\Lambda}(\lambda', \mu', \beta), \widetilde{M}(\mu', \beta), \widetilde{\phi}(\beta)), \end{gathered} \tag{14}$$

where $\widetilde{S}, \widetilde{X}, \widetilde{\Lambda}, \widetilde{M}, \widetilde{\phi}$ satisfy the conditions corresponding to (6),(7),(8). By (13) and (14) we have

$$\begin{gathered} S^{-1}(\Phi^{-1}(x', \lambda', \mu')) K(\Phi^{-1}(x', \lambda', \mu'), \beta) = \\ \widetilde{S}(x', \lambda', \mu', \beta) S^{-1}(\Phi^{-1}(\widetilde{X}, \widetilde{\Lambda}, \widetilde{M})) H(X'(\widetilde{X}, \widetilde{\Lambda}, \widetilde{M}), \Lambda'(\widetilde{\Lambda}, \widetilde{M}), M'(\widetilde{M}), \widetilde{\phi}(\beta)), \end{gathered} \tag{15}$$

where $\widetilde{X} = \widetilde{X}(x', \lambda', \mu', \beta), \widetilde{\Lambda} = \widetilde{\Lambda}(\lambda', \mu', \beta), \widetilde{M} = \widetilde{M}(\mu', \beta)$. Let $S_1(x, \lambda, \mu, \beta) = S(x, \lambda, \mu) \widetilde{S}(\Phi(x, \lambda, \mu), \beta) S^{-1}(\Phi^{-1}(\widetilde{X}(\Phi(x, \lambda, \mu), \beta), \widetilde{\Lambda}(\Lambda(\lambda, \mu), M(\mu), \beta), \widetilde{M}(M(\mu), \beta)))$, therefore we have

$$K(x, \lambda, \mu, \beta) = S_1(x, \lambda, \mu, \beta) H(\Psi(x, \lambda, \mu, \beta)), \tag{16}$$

where Ψ is determined by the following commutative diagram:

$$
\begin{tikzcd}
\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^l \arrow[r, "\Phi \times 1_{\mathbb{R}^l}"] \arrow[ddr, "\Psi"'] & \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^l \arrow[d, "(\widetilde{\Phi}, \widetilde{\phi})"] \\
& \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^k \arrow[d, "\Phi^{-1} \times 1_{\mathbb{R}^k}"] \\
& \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^k
\end{tikzcd}
$$

where $(\widetilde{\Phi}, \widetilde{\phi})(x', \lambda', \mu', \beta) = (\widetilde{X}(x', \lambda', \mu', \beta), \widetilde{\Lambda}(\lambda', \mu', \beta), \widetilde{M}(\mu', \beta), \widetilde{\phi}(\beta))$. It is not difficult to see that we can write Ψ in the form

$$\Psi(x, \lambda, \mu, \beta) = (X_1(x, \lambda, \mu, \beta), \Lambda_1(\lambda, \mu, \beta), M_1(\mu, \beta), \psi_1(\beta)),$$

and clearly $\psi_1(\beta) = \widetilde{\phi}(\beta)$. Therefore we can rewrite (16) as

$$K(x, \lambda, \mu, \beta) = S_1(x, \lambda, \mu, \beta) H(X_1(x, \lambda, \mu, \beta), \Lambda_1(\lambda, \mu, \beta), M_1(\mu, \beta), \psi_1(\beta)).$$

Thus to prove K factors through H we only need to verify that $S_1, X_1, \Lambda_1, M_1, \psi_1$ satisfy the conditions corresponding to (6)-(8). Because S, $\Phi = (X, \Lambda, M)$ are determined by u, S, X, X', S^{-1} and $\widetilde{S}, \widetilde{X}$ satisfy the equivariant conditions corresponding to (8) hence so does S_1. Since $\widetilde{S}, \widetilde{X}, \widetilde{\Lambda}, \widetilde{M}$ satisfy (6),(7), $S_1(x, \lambda, \mu, 0) = I_n$. By

$$
\begin{aligned}
& X_1(x, \lambda, \mu, \beta) = \\
& X'(\widetilde{X}(X(x, \lambda, \mu), \Lambda(\lambda, \mu), M(\mu), \beta), \widetilde{\Lambda}(\Lambda(\lambda, \mu), M(\mu), \beta), \widetilde{M}(M(\mu), \beta)), \\
& \Lambda_1(\lambda, \mu, \beta) = \Lambda'(\widetilde{\Lambda}(\Lambda(\lambda, \mu), M(\mu), \beta), \widetilde{M}(M(\mu), \beta)), \\
& M_1(\mu, \beta) = M'(\widetilde{M}(M(\mu), \beta)),
\end{aligned}
$$

we have

$$
\begin{aligned}
& X_1(\gamma x, \lambda, \mu, \beta) = \gamma X_1(x, \lambda, \mu, \beta), \forall \gamma \in \mathit{\Gamma}, \\
& X_1(x, \lambda, \mu, 0) = X'(X(x, \lambda, \mu), \Lambda(\lambda, \mu), M(\mu)) = x, \\
& \Lambda_1(\lambda, \mu, 0) = \Lambda'(\Lambda(\lambda, \mu), M(\mu)) = \lambda, \\
& M_1(\mu, 0) = M'(M(\mu)) = \mu
\end{aligned}
$$

and Fact 1 is proved.

Fact 2. H and G are equivalent.
Suppose $l = dim_{\mathbb{R}} \frac{\mathbf{E}_{x,\lambda,\mu}(\mathit{\Gamma})}{T(g, \mathfrak{U})}$, it is clear by Theorem 5 in §3 that $l \leqslant k$. By Lemma 1 we can take a l-parameter versal unfolding K of g, $K \in \mathbf{E}_{x,\lambda,\mu,\beta}(\mathit{\Gamma}), \beta \in \mathbb{R}^l$. Hence there are $S_i, X_i, \Lambda_i, M_i, \psi_i (i = 1, 2)$ which satisfy the conditions corresponding to (6)-(8) shch that

$$G(x, \lambda, \mu, \alpha) = S_1(x, \lambda, \mu, \alpha) K(X_1(x, \lambda, \mu, \alpha), \Lambda_1(\lambda, \mu, \alpha), M_1(\mu, \alpha), \psi_1(\alpha)), \tag{17}$$

$$H(x, \lambda, \mu, \alpha) = S_2(x, \lambda, \mu, \alpha) K(X_2(x, \lambda, \mu, \alpha), \Lambda_2(\lambda, \mu, \alpha), M_2(\mu, \alpha), \psi_2(\alpha)). \tag{18}$$

We claim: There is a diffeomorphism germ $\phi : (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$ such that

$$\psi_2 = \psi_1 \circ \phi. \tag{19}$$

In fact, since G and H are both versal unfoldings of g, theorem 5 implies that the real linear space $\frac{\mathbf{E}_{x,\lambda,\mu}(\Gamma)}{T(g,\mathfrak{U})}$ is generated by $\{ \frac{\partial G}{\partial \alpha_1}|_{\alpha=0}, \cdots, \frac{\partial G}{\partial \alpha_k}|_{\alpha=0} \}$ and also by $\{ \frac{\partial H}{\partial \alpha_1}|_{\alpha=0}, \cdots, \frac{\partial H}{\partial \alpha_k}|_{\alpha=0} \}$. Hence by differentiating (17) and (18) we have

$$\frac{\partial G}{\partial \alpha_j}|_{\alpha=0} \equiv \frac{\partial K}{\partial \beta} \cdot \frac{\partial \psi_1}{\partial \alpha_j}|_{\alpha=0} (mod\, T(g, \mathfrak{U})), j = 1, \cdots, k;$$

$$\frac{\partial H}{\partial \alpha_j}|_{\alpha=0} \equiv \frac{\partial K}{\partial \beta} \cdot \frac{\partial \psi_2}{\partial \alpha_j}|_{\alpha=0} (mod\, T(g, \mathfrak{U})), j = 1, \cdots, k.$$

It follows that ψ_1 and ψ_2 are both submersion germs. So there is a diffeomorphism $\phi : (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$ such that $\psi_2 = \psi_1 \circ \phi$. Now let $K_1(x, \lambda, \mu, \alpha) = K(x, \lambda, \mu, \psi_1(\alpha))$, then

$$G(x, \lambda, \mu, \alpha) = S_1(x, \lambda, \mu, \alpha) K_1(X_1(x, \lambda, \mu, \alpha), \Lambda_1(\lambda, \mu, \alpha), M_1(\mu, \alpha), \alpha).$$

Let $\Psi(x, \lambda, \mu, \alpha) = (X_1(x, \lambda, \mu, \alpha), \Lambda_1(\lambda, \mu, \alpha), M_1(\mu, \alpha), \alpha)$, then Ψ is a Γ-equivariant diffeomorphism germ, and it's inverse can be written in the form

$$\Psi^{-1}(x', \lambda', \mu', \alpha) = (X_1'(x', \lambda', \mu', \alpha), \Lambda_1'(\lambda', \mu', \alpha), M_1'(\mu', \alpha), \alpha).$$

Thus by (17) we have

$$K_1(x', \lambda', \mu', \alpha) = S_1^{-1}(\Psi^{-1}(x', \lambda', \mu', \alpha)) G(\Psi^{-1}(x', \lambda', \mu', \alpha)). \tag{20}$$

From (18),(19),(20) we get

$$\begin{aligned} H(x, \lambda, \mu, \alpha) &= S_2(x, \lambda, \mu, \alpha) K(X_2(x, \lambda, \mu, \alpha), \Lambda_2(\lambda, \mu, \alpha), M_2(\mu, \alpha), \psi_1(\phi(\alpha))) \\ &= S_2(x, \lambda, \mu, \alpha) K_1(X_2(x, \lambda, \mu, \alpha), \Lambda_2(\lambda, \mu, \alpha), M_2(\mu, \alpha), \phi(\alpha)) \\ &= S_2(x, \lambda, \mu, \alpha) S_1^{-1}(\Psi^{-1}(X_2(x, \lambda, \mu, \alpha), \Lambda_2(\lambda, \mu, \alpha), M_2(\mu, \alpha), \phi(\alpha))) \\ &\quad G(\Psi^{-1}(X_2(x, \lambda, \mu, \alpha), \Lambda_2(\lambda, \mu, \alpha), M_2(\mu, \alpha), \phi(\alpha))). \end{aligned} \tag{21}$$

Let

$$\begin{aligned} &\widetilde{S}(x, \lambda, \mu, \alpha) = S_2(x, \lambda, \mu, \alpha) S_1^{-1}(\Psi^{-1}(X_2(x, \lambda, \mu, \alpha), \Lambda_2(\lambda, \mu, \alpha), M_2(\mu, \alpha), \phi(\alpha))), \\ &\widetilde{X}(x, \lambda, \mu, \alpha) = X_1'(X_2(x, \lambda, \mu, \alpha), \Lambda_2(\lambda, \mu, \alpha), M_2(\mu, \alpha), \phi(\alpha)), \\ &\widetilde{\Lambda}(\lambda, \mu, \alpha) = \Lambda_1'(\Lambda_2(\lambda, \mu, \alpha), M_2(\mu, \alpha), \phi(\alpha)), \\ &\widetilde{M}(\mu, \alpha) = M_1'(M_2(\mu, \alpha), \phi(\alpha)),\ \widetilde{\phi} = \phi, \end{aligned}$$

then (21) can be written in the form of (9). Note the conditions which S_i,X_i,Λ_i,M_i and ψ_i $(i = 1, 2)$ satisfy, $\Psi|_{\alpha=0}$ is the identity germ on $(\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t, 0)$, thus

$$\widetilde{S}(x, \lambda, \mu, 0) = I_n, \widetilde{X}(x, \lambda, \mu, 0) = x, \widetilde{\Lambda}(\lambda, \mu, 0) = \lambda, \widetilde{M}(\mu, 0) = \mu, \tag{22}$$

and $\widetilde{S}, \widetilde{X}$ also satisfy the equivariant condintions (8). This completes the proof of Fact 2.

Finally, we prove F and G are equivalent. Interchange G and H in the proof above, then G can be written in the form:

$$G(x,\lambda,\mu,\alpha) = \widetilde{S}(x,\lambda,\mu,\alpha)H(\widetilde{X}(x,\lambda,\mu,\alpha),\widetilde{\Lambda}(\lambda,\mu,\alpha),\widetilde{M}(\mu,\alpha),\alpha),$$

where $\widetilde{S}, \widetilde{X}$ satisfy (8) and (22). By (12), we have

$$G(x,\lambda,\mu,\alpha) = \widetilde{S}(x,\lambda,\mu,\alpha)S(\widetilde{X},\widetilde{\Lambda},\widetilde{M})F(X(\widetilde{X},\widetilde{\Lambda},\widetilde{M}),\Lambda(\widetilde{\Lambda},\widetilde{M}),M(\widetilde{M}),\widetilde{\phi}(\alpha)). \qquad (23)$$

Therefore F and G are equivalent, this completes the proof of the theorem.

Remark. The Theorem 2 is fundamental in classification of two-parameter bifurcation problems, we will discuss classification in another paper. In addition, if $f = g$, then theorem 2 implies that F and G are Γ-isomorphic.Thus by the following corollary, theorem 2 is a generalization of Theorem 7.4 in [8].

Corollary 2 *Suppose F and G are versal k-parameter, l-parameter unfoldings of f respectively. Then F and G are Γ-isomorphic if and only if $l = k$.*

proof. We need only prove the "if" part. At the begining of the proof of theorem 2, let u be the unit of the group $\mathfrak{U}$, so H=F there. By the proof of Fact 2, see (21), H namely F can be written in the form

$$F(x,\lambda,\mu,\alpha) = \hat{S}(x,\lambda,\mu,\alpha)G(\hat{X}(x,\lambda,\mu,\alpha),\hat{\Lambda}(\lambda,\mu,\alpha),\hat{M}(\mu,\alpha),\phi(\alpha)),$$

where $\hat{S}, \hat{X}, \hat{\Lambda}, \hat{M}$ satisfy (22) and $\hat{S}, \hat{X}$ satisfy (8). Since ϕ is a diffeomorphism here, so F and G are Γ-isomorphic.

3 Stability and Versal Unfolding Theorem

Definition 2 *(1) Let F be an unfolding of a bifurcation problem f, the unfolding parameter is $\omega \in \mathbb{R}^m$. Suppose H is a k-parameter unfolding of F, the unfolding parameter is $\alpha \in \mathbb{R}^k$ (where H=F when k=0), and G is a l-parameter unfolding of H, the unfolding parameter is $\beta \in \mathbb{R}^l$. If G can be written in the form*

$$\begin{aligned} G(x,\lambda,\mu,\omega,\alpha,\beta) = S(x,\lambda,\mu,\omega,\alpha,\beta)H(X(x,\lambda,\mu,\omega,\alpha,\beta),\\ \Lambda(\lambda,\mu,\omega,\alpha,\beta),M(\mu,\omega,\alpha,\beta),\phi(\omega,\alpha,\beta)), \end{aligned} \qquad (24)$$

where $\phi : (\mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^l, 0) \to (\mathbb{R}^m \times \mathbb{R}^k, 0)$,and $\forall \gamma \in \Gamma$,

$$S(\gamma x, \cdots) = \gamma S(x, \cdots)\gamma^{-1}, X(\gamma x, \cdots) = \gamma X(x, \cdots). \qquad (25)$$

$$S(\cdots,0) = I_n, X(\cdots,0) = x, \Lambda(\cdots,0) = \lambda, M(\cdots,0) = \mu, \phi(\omega,\alpha,0) = (\omega,\alpha), \qquad (26)$$

then we say G is **isomorphic** *to the l-parameter trivial unfolding of H.*
(2) If any l-parameter unfolding of F is isomorphic to the l-parameter trivial unfolding of F, $\forall l \in \mathbb{N}$, then F is called **stable**. *Where F is also understood as the 0-parameter unfolding of itself.*

Notation 1 *Suppose G is a unfolding of H, H is a unfolding of K. If G is isomorphic to a l_1-parameter trivial unfolding of H, H is isomorphic to a l_2-parameter trivial unfolding of K, then G is isomorphic to a (l_1+l_2)-parameter trivial unfolding of K.*

Notation 2 *Suppose $G(x,\lambda,\mu,\omega,\beta)$ is isomorphic to the 1-parameter trivial unfolding of $F(x,\lambda,\mu,\omega)$, $\beta \in \mathbb{R}$. Then G factor through F.*

Proof. By Definition 2, G can be written in the form

$$G(x,\lambda,\mu,\omega,\beta) = S(x,\lambda,\mu,\omega,\beta)F(X(x,\lambda,\mu,\omega,\beta),\Lambda(\lambda,\mu,\omega,\beta),M(\mu,\omega,\beta),\phi(\omega,\beta)),$$

where ϕ, S, X, Λ, M satisfy (25),(26) thus they satisfy (6)-(8). So G factor through F.

We suppose F is of finite codimension in this section, namely $dim_{\mathbb{R}}\mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma})/\tilde{T}(F) < +\infty$, where

$$\tilde{T}(F) = \vec{\mathbf{E}}_{x,\lambda,\mu,\omega}(\mathit{\Gamma}) \cdot F + (D_xF)\mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma}),$$

and the definitions of $\vec{\mathbf{E}}_{x,\lambda,\mu,\omega}(\mathit{\Gamma}), \mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma})$ are similar to $\vec{\mathbf{E}}_{x,\lambda,\mu}(\mathit{\Gamma})$ and $\mathbf{E}_{x,\lambda,\mu}(\mathit{\Gamma})$ in §1, they are also finitely generated modules over the ring $\varepsilon_{x,\lambda,\mu,\omega}(\mathit{\Gamma})$ of $\mathit{\Gamma}$-invariant function germs. $\tilde{T}(F)$ is a submodule of $\mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma})$.

Theorem 3 *(Stability Theorem) Suppose $F \in \mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma})$ is a m-parameter unfolding of bifurcation problem $f \in \mathbf{E}_{x,\lambda,\mu}(\mathit{\Gamma})$. Then F is stable if and only if*

$$\begin{aligned}\mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma}) &= \vec{\mathbf{E}}_{x,\lambda,\mu,\omega}(\mathit{\Gamma}) \cdot F + (D_xF)\mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma}) \\ &+ (D_\lambda F)\mathbf{E}_{\lambda,\mu,\omega} + (D_\mu F)\mathbf{E}_{\mu,\omega} + (D_\omega F)\mathbf{E}_\omega \end{aligned} \tag{27}$$

Proof. Necessity is obvious: for any $h \in \mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma})$, consider the 1-parameter unfolding G of F, $G(x,\lambda,\mu,\omega,\beta) = F + \beta h$, $\beta \in \mathbb{R}$. Thus $F + \beta h = G$ isomorphic to the 1-parameter trivial unfolding of F implies $h \in$ the right side of (27).

Remark. Before the proof of sufficiency, we give an importent lemma which will be proved at the end of this section, it will makes the proofs of the sufficiency in this theorem and in versal unfolding theorem very simple.

Lemma 1 *Suppose G is a k-parameter unfolding of F, the unfolding parameter is $\alpha \in \mathbb{R}^k$. If (27) holds and G_0 satisfies*

$$G_0 \in \mathbf{M}_{x,\lambda,\mu,\omega,\alpha,\tau}(\mathit{\Gamma}), G_0(x,\lambda,\mu,\omega,\alpha,0) \equiv 0, (\tau \in \mathbb{R}), \tag{28}$$

then $\hat{G} \equiv G + G_0$ is isomorphic to the 1-parameter trivial unfolding of G.

Now we assume the lemma 1 is proved. Let H be a l-parameter unfolding of F, the unfolding parameter is $\beta \in \mathbb{R}^l$. Define

$$
\begin{aligned}
&H_0(x,\lambda,\mu,\omega) = F(x,\lambda,\mu,\omega),\\
&H_i(x,\lambda,\mu,\omega,\beta_1,\cdots,\beta_i) = H(x,\lambda,\mu,\omega,\beta_1,\cdots,\beta_i,0,\cdots,0), i=1,\cdots,l;\\
&\hat{G}_0(x,\lambda,\mu,\omega) = F = H_0,\\
&\hat{G}_1(x,\lambda,\mu,\omega,\beta_1) = \hat{G}_0 + [H_1 - H_0] = H_1,\\
&\cdots\\
&\hat{G}_l(x,\lambda,\mu,\omega,\beta_1,\cdots,\beta_l) = \hat{G}_{l-1} + [H_l - H_{l-1}] = H_l = H.
\end{aligned}
$$

Let $\hat{G} = \hat{G}_1(x,\lambda,\mu,\omega,\beta_1)$, $G = \hat{G}_0(x,\lambda,\mu,\omega) = F$, and $G_0(x,\lambda,\mu,\omega,\alpha,\beta_1) = H_1 - H_0$. For F, (27) holds; and for $G_0 = H_1 - H_0$ (28) holds. Thus it follows from Lemma 1 that $\hat{G}_1$ is isomorphic to the 1-parameter unfolding of $\hat{G}_0$. By induction principle we can prove that $\hat{G}_i$ is isomorphic to the 1-parameter trivial unfolding of $\hat{G}_{i-1}$ because of Lemma 1, where $i = 1,2,\cdots,l$. Thus $H = \hat{G}_l$ is isomorphic to the l-parameter trivial unfolding of $\hat{G}_0 = F$ by Notation 1. Therefore the theorem is proved.

Theorem 4 *Suppose f is a bifurcation problem of finite codimension, F is a k-parameter unfolding of f, the unfolding parameter is $\alpha = (\alpha_1,\cdots,\alpha_k) \in \mathbb{R}^k$. Then F is stable if and only if*

$$
T(g,\mathfrak{U}) + \mathbb{R}\{\frac{\partial F}{\partial \alpha_1}|_{\alpha=0},\cdots,\frac{\partial F}{\partial \alpha_k}|_{\alpha=0}\} = \mathbf{E}_{\mathbf{x},\lambda,\mu}(\Gamma). \tag{29}
$$

Proof. Necessity. Let $T(F) = \tilde{T}(F) + (D_\lambda F)(\mathbf{E}_{\lambda,\mu,\alpha}) + (D_\mu F)(\mathbf{E}_{\mu,\alpha}) + (D_\alpha F)(\mathbf{E}_\alpha)$, where $\tilde{T}(F) = \vec{\mathbf{E}}_{x,\lambda,\mu,\alpha}(\Gamma)\cdot F + (D_x F)\mathbf{E}_{x,\lambda,\mu,\alpha}(\Gamma)$. By Theorem 3, $T(F) = \mathbf{E}_{x,\lambda,\mu,\alpha}(\Gamma)$. Restricting $\alpha = 0$ we can get a map π from $\mathbf{E}_{x,\lambda,\mu,\alpha}(\Gamma)$ onto $\mathbf{E}_{x,\lambda,\mu}(\Gamma)$. But $\pi(T(F)) \subset T(g,\mathfrak{U}) + \mathbb{R}\{\frac{\partial F}{\partial \alpha_1}|_{\alpha=0},\cdots,\frac{\partial F}{\partial \alpha_k}|_{\alpha=0}\}$, this implies (29). Sufficiency. By Theorem 3, it is enough to prove $\mathbf{E}_{x,\lambda,\mu,\alpha}(\Gamma) \subset T(F)$. The proof for this is just like that in the first part of the proof of Lemma 1 hence we omit the details here.

Remark. The following is a simple form of Lemma 1, it will be used to prove versal unfolding theorem next.

Corollary 3 *Suppose G is a k-parameter unfolding of bifurcation problem g, the unfolding parameter is $\alpha \in \mathbb{R}^k$. If $\mathbf{E}_{x,\lambda,\mu}(\Gamma) = T(g,\mathfrak{U}) + \mathbb{R}\{\frac{\partial G}{\partial \alpha_1}|_{\alpha=0},\cdots,\frac{\partial G}{\partial \alpha_k}|_{\alpha=0}\}$. and G_0 satisfies*

$$
G_0 \in \mathbf{M}_{x,\lambda,\mu,\alpha,\tau}(\Gamma), G_0(x,\lambda,\mu,\alpha,0) \equiv 0, (\tau \in \mathbb{R}), \tag{30}
$$

then $\hat{G} \equiv G+G_0$ factor through G.

Proof. In fact, kill ω (or let $\omega \equiv 0$ at all time) in (27) and Lemma 1, then Lemma 1 implies that $\hat{G}$ is isomorphic to the 1-parameter trivial unfolding of G, thus Notation 2 implies $\hat{G}$ factor through G.

Theorem 5 *(Versal Unfolding Theorem,cf.[2][8][5]) Let bifurcation problem g be of finite codimension, G be a k-parameter unfolding of g. Then G is a versal unfolding of g if and only if* $\mathbf{E}_{x,\lambda,\mu}(\mathit{\Gamma}) = T(g,\mathfrak{U}) + \mathbb{R}\{\frac{\partial G}{\partial \alpha_1}|_{\alpha=0}, \cdots, \frac{\partial G}{\partial \alpha_k}|_{\alpha=0}\}$.

Proof. Necessity. Consider the 1-parameter unfolding of g $H = g + \beta p$ for $\forall p \in \mathbf{E}_{x,\lambda,\mu}(\mathit{\Gamma})$.It is easy to see $p \in T(g,\mathfrak{U}) + \mathbb{R}\{\frac{\partial G}{\partial \alpha_1}|_{\alpha=0}, \cdots, \frac{\partial G}{\partial \alpha_k}|_{\alpha=0}\}$. Sufficiency. By Corollary 3, it is not difficult to see that the proof is similar to that in Theorem 3.

Corollary 4 *Suppose f is a bifurcation problem of finite codimension, F is a k-parameter unfolding of f. Then F is stable iff F is a versal unfolding.*

*Proof.*Use Theorem 4 and Theorem 5 directly.

At last we prove Lemma 1.

Proof of Lemma 1. Since F is of finite codimension, so from condition (27) there exists

$$Y_1, \cdots, Y_a \in \mathbf{E}_{\lambda,\mu,\omega}; Z_1, \cdots, Z_b \in \mathbf{E}_{\mu,\omega}; W_1, \cdots, W_c \in \mathbf{E}_{\omega}$$

(where a,b,c are nonnegative integers) such that

$$\begin{aligned} &\mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma}) = \vec{\mathbf{E}}_{x,\lambda,\mu,\omega}(\mathit{\Gamma}) \cdot F + (D_x F)\mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma}) + \mathbb{R}\{(D_\lambda F)Y_1, \\ &\cdots, (D_\lambda F)Y_a, (D_\mu F)Z_1, \cdots, (D_\mu F)Z_b, (D_\omega F)W_1, \cdots, (D_\omega F)W_c\} \end{aligned} \tag{31}$$

Consider the finitely generated modules $\vec{\mathbf{E}}_{x,\lambda,\mu,\omega,\alpha,\tau}(\mathit{\Gamma}), \mathbf{E}_{x,\lambda,\mu,\omega,\alpha,\tau}(\mathit{\Gamma})$ over the ring $\varepsilon_{x,\lambda,\mu,\omega,\alpha,\tau}(\mathit{\Gamma})$ of $\mathit{\Gamma}$-invariant function germs (they are similar to $\vec{\mathbf{E}}_{x,\lambda,\mu}(\mathit{\Gamma})$ and $\mathbf{E}_{x,\lambda,\mu}(\mathit{\Gamma})$ in §1). Let $\hat{G} \in \mathbf{E}_{x,\lambda,\mu,\omega,\alpha,\tau}(\mathit{\Gamma})$ and

$$\begin{aligned} \tilde{T}(\hat{G}) &= \vec{\mathbf{E}}_{x,\lambda,\mu,\omega,\alpha,\tau}(\mathit{\Gamma}) \cdot \hat{G} + (D_x\hat{G})\mathbf{E}_{x,\lambda,\mu,\omega,\alpha,\tau}(\mathit{\Gamma}), \\ T(\hat{G}) &= \tilde{T}(\hat{G}) + (D_\lambda\hat{G})\mathbf{E}_{\lambda,\mu,\omega,\alpha,\tau} + (D_\mu\hat{G})\mathbf{E}_{\mu,\omega,\alpha,\tau} + (D_\omega\hat{G})\mathbf{E}_{\omega,\alpha,\tau}, \end{aligned} \tag{32}$$

then $\tilde{T}(\hat{G})$ is a submodule of $\mathbf{E}_{x,\lambda,\mu,\omega,\alpha,\tau}(\mathit{\Gamma})$. We claim:

$$\mathbf{E}_{x,\lambda,\mu,\omega,\alpha,\tau}(\mathit{\Gamma}) \subset T(\hat{G}). \tag{33}$$

In fact, let $A = \mathbf{E}_{x,\lambda,\mu,\omega,\alpha,\tau}(\mathit{\Gamma})$, $A' = \tilde{T}(\hat{G})$, $B = \mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma})$, $B' = \tilde{T}(F) = \vec{\mathbf{E}}_{x,\lambda,\mu,\omega}(\mathit{\Gamma}) \cdot F + (D_x F)\mathbf{E}_{x,\lambda,\mu,\omega}(\mathit{\Gamma})$, $\pi : A \to B$ be defined by $\pi(p)(x,\lambda,\mu,\omega) = p(x,\lambda,\mu,\omega,0,0)$ and

$$N = \{(D_\lambda\hat{G})y_1, \cdots, (D_\lambda\hat{G})y_a; (D_\mu\hat{G})z_1, \cdots, (D_\mu\hat{G})z_b; (D_\omega\hat{G})w_1, \cdots, (D_\omega\hat{G})z_c\},$$

where $(D_\lambda\hat{G})y_i \in \pi^{-1}((D_\lambda F)Y_i), (D_\mu\hat{G})z_i \in \pi^{-1}((D_\mu F)Z_i), (D_\omega\hat{G})w_i \in \pi^{-1}((D_\omega F)W_i)$, thus (31) can be written in the form $B = B' + \mathbb{R} \cdot \pi(N)$. So the Theorem 2.5 in [9] implies that $A = A' + \varepsilon_{\alpha,\tau} \cdot N$, hence the claim follows $\varepsilon_{\alpha,\tau} \cdot N \subset T(\hat{G})$. Thus $-\frac{\partial\hat{G}}{\partial\tau}$ can be written in the form

$$\begin{aligned} -\tfrac{\partial\hat{G}}{\partial\tau} &= -\hat{S}(x,\lambda,\mu,\omega,\alpha,\tau)\hat{G} + \sum_{i=1}^{n} X_i(x,\lambda,\mu,\omega,\alpha,\tau) \cdot \frac{\partial\hat{G}}{\partial x_i} + \sum_{i=1}^{s} \Lambda_i(\lambda,\mu,\omega,\alpha,\tau) \cdot \frac{\partial\hat{G}}{\partial\lambda_i} \\ &+ \sum_{i=1}^{t} M_i(\mu,\omega,\alpha,\tau) \cdot \frac{\partial\hat{G}}{\partial\mu_i} + \sum_{i=1}^{m} W_i(\omega,\alpha,\tau) \cdot \frac{\partial\hat{G}}{\partial\omega_i}, \end{aligned} \tag{34}$$

where $\hat{S}$ and $\hat{X} = (X_1(x,\lambda,\mu,\omega,\alpha,\tau),\cdots,X_n(x,\lambda,\mu,\omega,\alpha,\tau)$ satisfy appropriate equivariance conditions. Define a vector field germ on $(\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^t \times \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}, 0)$:

$$V = (\hat{X}, \hat{\Lambda}, \hat{M}, \hat{W}, 0, 1), \tag{35}$$

where $\hat{X}$, $\hat{\Lambda} = (\Lambda_1(\lambda,\mu,\omega,\alpha,\tau),\cdots,\Lambda_s(\lambda,\mu,\omega,\alpha,\tau))$, $\hat{M} = (M_1(\mu,\omega,\alpha,\tau),\cdots,M_t(\mu,\omega,\alpha,\tau))$, $\hat{W} = (W_1(\omega,\alpha,\tau),\cdots,W_m(\omega,\alpha,\tau))$ are given by (34).
Further, by the argument similar to that in [8] (cf. p.239-241) or see the proof of Lemma 2.2 in [9] we can construct $S(x,\lambda,\mu,\omega,\alpha,\tau), X(x,\lambda,\mu,\omega,\alpha,\tau), \Lambda(\lambda,\mu,\omega,\alpha,\tau), M(\mu,\omega,\alpha,\tau)$ and $\phi(\omega,\alpha,\tau)$ such that

$$\begin{aligned}\hat{G}(x,\lambda,\mu,\omega,\alpha,\tau) = S(x,\lambda,\mu,\omega,\alpha,\tau)G(X(x,\lambda,\mu,\omega,\alpha,\tau),\\ \Lambda(\lambda,\mu,\omega,\alpha,\tau), M(\mu,\omega,\alpha,\tau), \phi(\omega,\alpha,\tau)),\end{aligned} \tag{36}$$

and (25)(26) hold. Therefore $\hat{G}$ is isomorphic to the 1-parameter trivial unfolding of G by Definition 2.

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