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A Generalization of 2-Periodic Trigonometric Interpolation

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Abstract

It is well-known that the interpolation theory plays an important role in many fields of computer vision, especially in surface reconstruction and Image processing. In this paper, we introduce a new kind of 2-period interpolation of functions with period 2π *at the nodes* $x_k = x_{k,n} = k\pi/n, \quad (k = 0, 1, \cdots, 2n-1)$. *We find out the necessary and sufficient conditions for regularity of this new interpolation problem. Moreover, a closed form expression for the interpolation polynomial is given. Our interpolation is of more practical significance. Our results provide the theoretical basis for using our interpolation in practical problems.*

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1 Introduction

It is well-known that the interpolation theory plays an important role in many fields of computer vision, especially in surface reconstruction and Image processing. In many practical issues of computer vision, we do not know if the interpolated function is differentiable or we only know the values of the interpolated function at nodes. We often use divided difference to replace derivative because divided difference is the descretization of the derivative of functions. In this paper, we introduce a new kind of 2-periodic interpolation of functions with period 2π at the nodes $x_k = x_{k,n} = k\pi/n, \quad (k = 0, 1, \cdots, 2n-1)$. That is, we would replace the conditions of derivatives for 2-periodic interpolation by the those of differences at the nodes. Similar idea was used by Jiang[1] for Birkhoff-type interpolation.

For $f \in C_{2\pi}$ and $0 < h < \pi/n$, we define

$$\delta_h f(x) = \delta_h^1 f(x) = f(x+h) - f(x-h),$$
$$\delta_h^m f(x) = \delta_h(\delta_h^{m-1} f(x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(x + (m-2k)h), \quad m \geq 2.$$

We shall say that $t(x) \in \mathcal{T}_n$, if $t(x)$ is a polynomial of the form

$$t_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

If

$$t_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx) + a_n \cos(nx + \varepsilon\pi/2), \quad (\varepsilon = 0 \quad \text{or} \quad 1),$$

we shall say that $t_n(x) \in \mathcal{T}_{n,\varepsilon} (\varepsilon = 0 \quad \text{or} \quad 1)$.

Our problems are

P_1: For any two given sets of complex numbers $\{\alpha_k\}_0^{n-1}$ and $\{\beta_k\}_0^{n-1}$, whether or not there exists a unique trigonometric polynomial $t_n(x) \in \mathcal{T}_{n,\varepsilon} (\varepsilon = 0 \quad \text{or} \quad 1)$ satisfying the conditions

$$t_n(x_{2k}) = \alpha_k, \quad \delta^m(t_n)(x_{2k+1})/(2h)^m = \beta_k, \quad (k = 0, 1, \cdots, n-1).$$

P_2: If the answer of Problem P_1 is positive, usually, we say the interpolation problem is regular. Find the necessary and sufficient conditions on m and n such that Problem P_1 is regular.

P_3: Find the fundamental polynomials of the interpolation when Problem P_1 is regular.

The rest of this paper is organized as follows. Section 2 describes some lemmas needed for the subsequent parts. Section 3 is devoted to the regularity of this new interpolation. Section 4 will give the fundamental polynomials of the new interpolation. Final section is the conclusion and the future works.

2 Lemmas

In order to prove our main results we need the following lemmas. Among them, Lemma 2.1 and Lemma 2.2 come from Sharma and Sun [3]

Lemma 2.1 ([6, Lemma 2.1]) *If*

$$K_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx), \tag{1}$$

then

(1) $K_n(x_{2k}) = \delta_{0,k}, (k = 0, 1, \cdots, n-1)$ *if and only if*

$$\begin{cases} \frac{1}{2}a_0 + a_n = \frac{1}{n} \\ a_k + a_{n-k} = \frac{2}{n} \\ b_k - b_{n-k} = 0 \\ k = 1, \cdots, n-1. \end{cases}$$

(2) $K_n(x_{2k}) = 0, (k = 0, 1, \cdots, n-1)$ *if and only if*

$$\begin{cases} \frac{1}{2}a_0 + a_n = 0, \\ a_k + a_{n-k} = 0, \\ b_k - b_{n-k} = 0, \\ k = 1, \cdots, n-1. \end{cases}$$

Proof It is an easy consequence of the following two known identities.

$$\frac{\sin\frac{nx}{2}}{n\sin\frac{x}{2}} = \frac{1}{n}\left[1 + 2\sum_{j=1}^{(n-1)/2}\cos jx\right], \qquad n \quad \text{odd},$$

and

$$\frac{\cos\frac{x}{2}\sin\frac{nx}{2}}{n\sin\frac{x}{2}} = \frac{1}{n}\left[1 + 2\sum_{j=1}^{(n-2)/2}\cos jx + \cos\frac{nx}{2}\right], \qquad n \quad \text{even}.$$

Lemma 2.2 ([5, Lemma 2.2]) *If* $K_n(x)$ *is given by (1) , then*

(1) $K_n(x_{2k+1}) = \delta_{0,k}, (k = 0, 1, \cdots, n-1)$ *if and only if*

$$\begin{cases} \frac{1}{2}a_0 - a_n = \frac{1}{n} \\ a_k - a_{n-k} = \frac{2}{n}\cos jx_1 \\ b_k + b_{n-k} = \frac{2}{n}\sin jx_1 \\ k = 1, \cdots, n-1. \end{cases}$$

(2) $K_n(x_{2k+1}) = 0, (k = 0, 1, \cdots, n-1)$ *if and only if*

$$\begin{cases} \frac{1}{2}a_0 - a_n = 0, \\ a_k - a_{n-k} = 0, \\ b_k + b_{n-k} = 0, \\ k = 1, \cdots, n-1. \end{cases}$$

The proof depends upon using Lemma 2.1 after the observation that

$$\begin{aligned}
&K_n(x_{2k+1}) = K_n(x_{2k} + \tfrac{\pi}{n}) \\
&= \tfrac{a_0}{2} - a_n + \textstyle\sum_{j=1}^{n-1}(P_j \cos jx_{2k} + Q_j \sin jx_{2k}), \\
&(k = 0, 1, \cdots, n-1).
\end{aligned}$$

where

$$P_j = a_j \cos jx_1 + b_j \sin jx_1,$$

and

$$Q_j = b_j \cos jx_1 - a_j \sin jx_1.$$

Lemma 2.3 *For $m = 1, 2, \cdots,$ we have*

$$\delta^m \cos jx = 2^m \sin^m jh \cos(jx + \frac{m\pi}{2}),$$
$$\delta^m \sin jx = 2^m \sin^m jh \sin(jx + \frac{m\pi}{2}).$$

Proof It is easy to prove it by induction.

3 Regularity

Theorem 3.1 *(1) For $\varepsilon = 0$, Problem P_1 is regular if and only if m is even.*
(2) For $\varepsilon = 1$, Problem P_1 is regular if and only if m and n are both odd.

Proof We only prove part (1) because the proof of part (2) is similar. Let $Q_n(x) \in \mathcal{T}_{n,0}$ have the form

$$Q_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n-1}(a_k \cos kx + b_k \sin kx) + a_n \cos nx,$$

and let

$$\begin{cases} Q_n(x_{2k}) = 0 \\ \delta^m(Q_n)(x_{2k+1}) = 0. \\ k = 0, 1, \cdots, n-1. \end{cases}$$

Taking $K_n(x) = Q_n(x)$ by the part (2) of Lemma 2.1, we have

$$\begin{cases} \frac{1}{2}a_0 + a_n = 0 \\ a_j + a_{n-j} = 0 \\ b_j - b_{n-j} = 0 \\ j = 1, \cdots, n-1. \end{cases} \tag{2}$$

Applying Lemma 2.3, we have

$$\begin{aligned}
\delta^m(Q_n)(x) &= \sum_{j=1}^{n-1}(a_j 2^m \sin^m jh \cos(jx + \frac{m\pi}{2}) + b_j 2^m \sin^m jh \sin(jx + \frac{m\pi}{2})) \\
&\quad + a_n 2^m \sin^m nh \cos(nx + \frac{m\pi}{2})
\end{aligned}$$

If $m = 2s$, we have

$$\begin{aligned}\delta^m(Q_n)(x) &= \sum_{j=1}^{n-1}(a_j(-1)^{m/2}2^m\sin^m jh\cos jx + b_j(-1)^{m/2}2^m\sin^m jh\sin jx)\\ &\quad +a_n(-1)^{m/2}2^m\sin^m nh\cos nx\end{aligned}$$

Taking $K_n(x) = \delta^m(Q_n)(x)$ by the part (2) of Lemma 2.2, we have

$$\begin{cases} a_n(-1)^{m/2}2^m\sin^m nh = 0\\ a_j(-1)^{m/2}2^m\sin^m jh - a_{n-j}(-1)^{m/2}2^m\sin^m(n-j)h = 0\\ b_j(-1)^{m/2}2^m\sin^m jh + b_{n-j}(-1)^{m/2}2^m\sin^m(n-j)h = 0\\ j = 1, 2, \cdots, n-1.\end{cases}$$

That is

$$\begin{cases} a_n\sin^m nh = 0\\ a_j\sin^m jh - a_{n-j}\sin^m(n-j)h = 0\\ b_j\sin^m jh + b_{n-j}\sin^m(n-j)h = 0\\ j = 1, 2, \cdots, n-1.\end{cases} \tag{3}$$

From the first equations (2) and (3) we have $a_0 = a_n = 0$.
From other equations of (2) and (3) we have

$$\begin{cases} a_j(\sin^m jh + \sin^m(n-j)h) = 0\\ b_j(\sin^m jh + \sin^m(n-j)h) = 0\\ j = 1, 2, \cdots, n-1.\end{cases} \tag{4}$$

From $m = 2s$ we have

$$\sin^m jh + \sin^m(n-j)h > 0 \quad (j = 1, 2, \cdots, n-1).$$

Hence $a_j = b_j = 0 \quad (j = 1, 2, \cdots, n-1)$. This implies that Problem P_1 is regular.
If $m = 2s+1$, we have

$$\begin{aligned}\delta^m(Q_n)(x) &= \sum_{j=1}^{n-1}(b_j(-1)^{m/2}2^m\sin^m jh\cos jx - a_j(-1)^{m/2}2^m\sin^m jh\sin jx)\\ &\quad -a_n(-1)^{m/2}2^m\sin^m nh\sin nx\end{aligned}$$

Taking $K_n(x) = \delta^m(Q_n)(x)$ by the part (2) of Lemma 2.2, we have

$$\begin{cases} b_j\sin^m jh - b_{n-j}\sin^m(n-j)h = 0\\ a_j\sin^m jh + a_{n-j}\sin^m(n-j)h = 0\\ j = 1, 2, \cdots, n-1.\end{cases} \tag{5}$$

It is clear that equation systems (2) and (5) does not have a unique solution. Hence, Problem P_1 is not regular. This completes the proof of Theorem 3.1.

4 Fundamental Polynomials of Interpolation

If we denote the fundamental polynomials of $(0, \delta^m)$ interpolation by $r_{j,\varepsilon}(x)$ and $\rho_{j,\varepsilon}(x)$, respectively, then it is clear that

$$r_{j,\varepsilon}(x) = r_{0,\varepsilon}(x - x_j), \quad \rho_{j,\varepsilon}(x) = \rho_{0,\varepsilon}(x - x_j), j = 1, 2, \cdots, n-1.$$

We come to give the explicit forms of the fundamental polynomials $r_{0,\varepsilon}(x)$ and $\rho_{0,\varepsilon}(x)$. They determined by the following conditions.

$$r_{0,\varepsilon}(x_{2k}) = \delta_{0,k}, \quad \delta^m(r_{0,\varepsilon})(x_{2k+1})/(2h)^m = 0. \tag{6}$$

$$\rho_{0,\varepsilon}(x_{2k}) = 0, \quad \delta^m(\rho_{0,\varepsilon})(x_{2k+1})/(2h)^m = \delta_{0,k}. \tag{7}$$

$k = 1, 2, \cdots, n-1$.

Theorem 4.1 *(1) If $\varepsilon = 0$ and m is even, then the fundamental polynomials are given by*

$$\begin{cases} r_{0,0} = \frac{1}{n} + \frac{2}{n}\sum_{j=1}^{n-1} \frac{\sin^m(n-j)h\cos jx}{\sin^m(n-j)h+\sin^m jh} \\ \rho_{0,0} = \frac{2(-1)^{m/2}h^m}{n}\left\{-\frac{1}{2\sin^m nh} + \sum_{j=1}^{n-1} \frac{\cos jx_1\cos jx+\sin jx_1 \sin jx}{\sin^m(n-j)h+\sin^m jh} + \frac{\sin nx}{\sin^m nh}\right\} \end{cases} \tag{8}$$

(2) If $\varepsilon = 1$ and m and n are odd, then the fundamental polynomials are given by

$$\begin{cases} r_{0,1} = \frac{1}{n} + \frac{2}{n}\sum_{j=1}^{n-1} \frac{\sin^m(n-j)h\cos jx}{\sin^m(n-j)h-\sin^m jh} \\ \rho_{0,1} = \frac{2(-1)^{m/2}h^m}{n}\left\{\sum_{j=1}^{n-1} \frac{\cos jx_1\cos jx+\sin jx_1 \sin jx}{\sin^m(n-j)h-\sin^m jh} - \frac{\sin nx}{2\sin^m nh}\right\} \end{cases} \tag{9}$$

Proof We only prove part (2) because the proof of part (1) is similar. For $\varepsilon = 1$, we denote

$$Q_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n-1}(a_k \cos kx + b_k \sin kx) + b_n \sin nx. \tag{10}$$

Since m is odd, we have

$$\begin{aligned} \delta^m(Q_n)(x) &= \sum_{j=1}^{n-1}(b_j(-1)^{m/2}2^m \sin^m jh \cos jx - a_j(-1)^{m/2}2^m \sin^m jh \sin jx) \\ &\quad + b_n(-1)^{m/2}2^m \sin^m nh \cos nx \end{aligned}$$

If $Q_n(x)$ satisfies conditions (7), then taking $K_n(x) = Q_n(x)$ by the part (1) of Lemma 2.1 and $K_n(x) = \delta^m(Q_n)(x)$ by the part (2) of Lemma 2.2 respectively, we have

$$\begin{cases} \frac{1}{2}a_0 = \frac{1}{n} \\ a_j + a_{n-j} = \frac{2}{n} \\ b_j - b_{n-j} = 0 \\ j = 1, \cdots, n-1, \end{cases} \tag{11}$$

and

$$\begin{cases} b_n \sin^m nh = 0 \\ b_j \sin^m jh - b_{n-j} \sin^m (n-j)h = 0 \\ a_j sin^m jh + a_{n-j} \sin^m (n-j)h = 0 \\ j = 1, 2, \cdots, n-1. \end{cases} \tag{12}$$

From the first equations (1) and (2) we have $a_0 = 2/n$ and $b_n = 0$.

From other equations of (4) and (7) we have

$$\begin{cases} a_j(\sin^m (n-j)h - \sin^m jh) = \frac{2}{n} \sin^m (n-j)h \\ b_j(sin^m jh - \sin^m (n-j)h) = 0 \\ j = 1, 2, \cdots, n-1. \end{cases} \tag{13}$$

Since n is odd, $\sin^m (n-j)h - \sin^m jh \neq 0$. Hence, from (14) we have

$$\begin{cases} a_j = \frac{2 \sin^m (n-j)h}{n(\sin^m (n-j)h - \sin^m jh)} \\ b_j = 0 \\ j = 1, 2, \cdots, n-1. \end{cases}$$

This implies the first equality of (10).

If $Q_n(x)$ satisfies conditions (8), then taking $K_n(x) = Q_n(x)$ by the part (2) of Lemma 2.1 and $K_n(x) = \delta^m(Q_n)(x)/(2h)^m$ by the part (1) of Lemma 2.2 respectively, we have

$$\begin{cases} \frac{1}{2} a_0 = 0 \\ a_j + a_{n-j} = 0 \\ b_j - b_{n-j} = 0 \\ j = 1, \cdots, n-1, \end{cases} \tag{14}$$

and

$$\begin{cases} -b_n(-1)^{m/2} 2^m \sin^m nh/(2h)^m = \frac{1}{n} \\ b_j(-1)^{m/2} 2^m \sin^m jh/(2h)^m - b_{n-j}(-1)^{m/2} 2^m \sin^m (n-j)h/(2h)^m = \frac{2}{n} \cos jx_1 \\ a_j(-1)^{m/2} 2^m \sin^m jh/(2h)^m + a_{n-j}(-1)^{m/2} 2^m \sin^m (n-j)h/(2h)^m = \frac{2}{n} \sin jx_1 \\ j = 1, 2, \cdots, n-1. \end{cases} \tag{15}$$

From (15) and (16) we have

$$a_0 = 0, \quad b_n = -\frac{(-1)^{m/2} h^m}{n \sin^m nh} \tag{16}$$

and

$$\begin{cases} a_j = \frac{2(-1)^{m/2} h^m \sin jx_1}{n(\sin^m (n-j)h - \sin^m jh)} \\ b_j = \frac{2(-1)^{m/2} h^m \cos jx_1}{n(\sin^m (n-j)h - \sin^m jh)} \\ j = 1, 2, \cdots, n-1. \end{cases}$$

This implies the Second equality of (10).

5 Conclusion

It is well-known that interpolation theory plays an important role in many fields of computer vision, especially in surface reconstruction. In many practical issues of computer vision, we do not know if the interpolated function is differentiable or we only know the values of the interpolated function at nodes. From computational view, divided difference is a natural candidate for replacing derivative because divided difference is the descretization of the derivative of functions. In this paper, we would replace the conditions of derivatives for Hermite interpolation and Birkhoff interpolation by those of symmetric differences at the nodes. First, we introduce a new kind of 2-period interpolation of functions with period 2π. Second, we find out the necessary and sufficient conditions for regularity of this new interpolation problem. Moreover, a closed form expression for the interpolation polynomial is given. Our interpolation is of more practical significance. Our results provide the theoretical basis for using our interpolation in practical problems. Here we only consider the interpolation of 2π-periodic functions at the nodes $x_k = x_{k,n} = 2k\pi/n, \quad (k = 0, 1, \cdots, n-1)$. In future, we will try to get the convergence of this new interpolation. We would also like to extend our idea to algebraic polynomial interpolation case and multivariate interpolation cases.

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