Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISBN 1083-0464
Issue 1 July 2000, pp. 55 – 73
Submitted: May 27, 1999. Published: July 15, 2000

Periodic Solutions of Quasilinear Equations with Discontinuous Perturbations

Janete Crema and José Luiz Boldrini

Abstract

The present study deals with the existence of periodic solutions of equation $u_t - \Delta_p u = F(t,u)$, where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacean, with Dirichlet boundary condition, and when F undergoes abrupt changes depending on u. An abstract result generalizing this situation is also presented.

1991 Mathematics Subject Classifications: 35K55, 35B10
Key words: Quasi-Parabolic Equations, Periodic Solutions, p-Laplacean, Discontinuous Perturbations.

Janete Crema, USP - ICMC; CP 668; 13560-970 São Carlos, SP, Brazil
José Luiz Boldrini, UNICAMP-IMECC; CP 6065
13083-970 Campinas, SP, Brazil
E-mail: boldrini@ime.unicamp.br

1 Introduction

Monotone operators, in particular the ones that are subdifferentials of convex functions, like the p-Laplacean, appear in several situations. For instance, in LeTallec [6], a problem modelling the behaviour of viscoelastic materials, which includes the p-Laplacean, is studied.

These nonlinear operators may be accompanied by other nonlinear terms or operators that may complicate the analysis of the equations and the prediction of the behaviour of their solutions. This is specially true when these acompanying operators correspond to situations in which it is possible to impart energy into the system by means of the interaction between the material and external actions (by changes in the external or internal conditions, for instance).

This means in particular that the overall operator associated to the problem is no longer always monotone and not even always dissipative. Thus, due to this possible imbalance between the gain and the dissipation of energy, it is not clear that, when acted upon periodic external forces, the internal dissipation will be enough to guarantee that the system also responds in a periodic way. In other words, there is no guarantee that the corresponding mathematical problem will have a time periodic solution.

If, in addition, the system is submitted to some kind of abrupt change, implying in some sort of discontinuity in the characteristics of the problem, the question of the existence of periodic solution is even harder to answer.

In this paper we intend to address this question and furnish a partial answer to it by giving sufficient conditions for the existence of periodic solutions of problems of form:

$$\begin{cases} u_t + Au = F(t,u), \\ u(0) = u(T), \end{cases} \tag{1.1}$$

where A is a monotone operator having certain properties. In particular, this could be a nonlinear quasi parabolic boundary value problem in $\Omega \subset \mathbb{R}^N$, $N \geq 1$. $(t,u) \mapsto F(t,u)$ is a certain operator, which may have discontinuities in u, and which are not necessarily either monotone or bounded in any region of the space of u variable. On the other hand, the absence of such conditions will be compensated by the requirement of certain restriction in the growth of the nonmonotone perturbation F with respect to u. To cope with the discontinuity, the corresponding solution will be understood in the sense of differential inclusions.

Thus, to simplify the notation and ease the exposition, conditions for the existence of solutions of (1.1) will be obtained by initially studying in detail the following model problem:

$$\begin{cases} u_t(t,x) - \Delta_p u(t,x) \in F(u)(t,x), & (t,x) \in (0,T)\times\Omega, \\ u(t,x)|_{\partial\Omega} = 0, \quad t \in (0,T), \\ u(0,x) = u(T,x), \quad x \in \Omega. \end{cases} \tag{1.2}$$

In this problem Ω is a bounded and regular open set in $\mathbb{R}^N$, $N \geq 1$; $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p \geq 2$, is the p-Laplacean operator; R is a positive constant. Moreover, the set-valued mapping F is defined as

$$F(u)(t,x) = \begin{cases} \{f_1(t,x,u(t,x))\} & \text{if } u(t,x) < R, \\ \{f_2(t,x,u(t,x))\} & \text{if } u(t,x) > R, \\ \mathcal{A}_{12} & \text{if } u(t,x) = R. \end{cases} \tag{1.3}$$

where the set $\mathcal{A}_{12}$ is defined as

$$\mathcal{A}_{12} = \{\sigma f_1(t,x,u(t,x)) + (1-\sigma) f_2(t,x,u(t,x)); \sigma \in [0,1]\}.$$

Here, for each $i = 1,2$, $f_i : [0,T]\times\Omega\times\mathbb{R} \to \mathbb{R}$ is a measurable function, continuous in its third variable and satisfying

$$|f_i(t,x,\mu)| \leq m_i(t,x)\,|\mu|^{s_i} + h_i(t,x), \tag{1.4}$$

with $m_i \in L^\infty([0,T]\times\Omega)$, $s_i \geq 0$, and $h_i \in L^{p'}(0,T;L^{q'}(\Omega))$, p' and q' are the respecting conjugates of numbers p and q satisfying $1 - N/p > -N/q$.

An approach based on Boldrini and Crema [2], will be used to give, in Section 3, sufficient conditions for the existence of the weak periodic solutions for the problem (1.2)-(1.3).

Theorem 3.1 shows the existence of periodic solution for the subcritical case, i.e, for $0 \leq s_i < p-1$, $i = 1,2$. In Theorems 3.2 and 3.3, we analyze the critical and supercritical cases, respectivelly, and we obtain periodic solutions since $\sup\limits_{(t,x)\in(0,T)\times\Omega} m(t,x)$ is small enough and either $p < N$ and $p-1 \leq s_i < p-1+2p/N$ or $p \geq N$ and $p-1 \leq s_i < p+1$.

Based on the analysis of this problem, in Section 4 we will state a result containing conditions guaranteeing the existence of solutions of the abstract problem (1.1) for classes of operators A including the p-Laplacean operator as a particular case.

The results of this paper should be compared with previous results of similar problems. Thus we remark that in [8], Ôtani presents sufficient conditions for the existence of strong periodic solutions of problems in a class that includes equations like (1.2).

However, his results require more regularity and a certain smallness condition on h_i. This implies that the amplitude of the periodic solutions in Ôtani [8] are also small (in suitable norms). On the other hand, in our case h_i can be less regular (where a weak solution is obtained), and no smallness condition on h_i is imposed instead, a smallness condition on $|m|_\infty$ is imposed. Thus, our solutions (weak or strong) are found in a large ball and, in fact, it is easy to prove that when h_i is large, our solutions are also large (in suitable norms). Consequently our solutions are different from the ones obtained by the specialization of Ôtani's results.

Recently Avgerinos and Papageorgiou [1] have showed the existence of periodic solutions of equations involving monotone nonlinear parabolic operators perturbed by a discontinuous term of form $F(t,u)(x) = -a(t,x,u) + f(u)$, where $x \in \Omega \subset \mathbb{R}^N$, $N \leq 3$. In this term, f is a function of bounded variation, and $u \to a(t,x,u)$ is a bounded lipschitzian monotone function in $u \in [\phi(t,x), \psi(t,x)]$, where ϕ and ψ are respectively upper and lower solutions of the problem. We observe that for the autonomous case, i.e., when a does not depends on t, it is easy to show the existence of lower and upper solutions, which in general will be constant functions.This is no longer true for the nonautonomous case. For reference see also Kandilakis and Papageorgiou [4].

Boldrini and Crema established in [2] enough conditions to obtain periodic solutions to problems which undergo sudden time-dependent changes in their intrinsic behavior.

It should also be mentioned that investigations concerning the problem involving the p-Laplacean perturbed by a maximal monotonic operator, with periodic boundary conditions, were made by Yamada in [10]. The existence of periodic solutions was proved, but we remark that the problem considered in [10] is always dissipative as opposed to the one considered here.

Kawohl and Ruhl in [5] treated a problem involving the Laplacean, with monotonic boundary conditions changing abruptly in time, perturbed by other monotonic operators. Again the problem considered in [5] is dissipative.

Finally, it should be observed that for simplicity of exposition, all results in this paper will be stated with perturbations with just one discontinuity. However, as it is easy to see from the proofs, the results are still true for perturbations with a finite number of discontinuities.

2 Preliminaries

For the sake of fixing notations and easing the reading, in this section we will recall results to be used later on.

Let V be a Banach Reflexive space and V' be its topological dual. Let H be a Hilbert space such that $V \subset H \subset V'$, with continuous and dense inclusions. We will denote by $|\cdot|_V$, $|\cdot|_{V'}$, and $|\cdot|_H$ respectively the norms of V, V' and H; by $(\ ,\ )_{V',V}$ we denote the duality pairing between V' and V, and by $(\ ,\ )_H$ the inner product of H is represented.

For $T > 0$, we consider the Banach space $L^p(0,T;V) = \{u : (0,T) \to V; \int_0^T |u(t)|_V^p\ dt < \infty\}$, with norm $|u|_{L^pV} = (\int_0^T |u(t)|_V^p\ dt)^{1/p}$. Let p' be defined by $1/p + 1/p' = 1$ and denote $u_t = \dfrac{du}{dt}$; we recall that the Banach space $\{u \in L^p(0,T;V);\ u_t \in L^{p'}(0,T;V')\}$, with norm given by $|u|_{L^pV} + |u_t|_{L^{p'}V'}$ is a subspace of $C([0,T];H)$, the class of continuous functions defined on $[0,T]$ with values in H. Moreover, for u and v in this space, there holds

$$\int_0^T (u_t(t), v(t))_{V'V} + (v_t(t), u(t))_{V'V} dt = (u(T), v(T))_H - (u(0), v(0))_H. \quad (2.1)$$

For a proof of these results, see Lions [7], pp. 156 and 321.

The following compactness criterion given by Aubin-Lions (see Lions [7], p. 58, Theorem 5.1 and Strauss [9], p. 34, Theorem 2) will be necessary:

Lemma 2.1 *Let X, Y and Z be three Banach spaces, X and Z be reflexive, such that $X \subset Y \subset Z$ with continuous inclusions. Moreover, assume that the inclusion $X \subset Y$ is compact and that $1 \le p \le \infty$ and $1 < q \le \infty$. Then, the space $\{u \in L^p(0,T;X);\ u_t \in L^q(0,T;Z)\}$ is compactly included in $L^p(0,T;Y)$.*

$L^q(\Omega)$ will denote the usual Banach space of real valued function defined on Ω, with norm $|u|_q = (\int_\Omega |u(x)|^q dx)^{1/q}$. When $q = 2$, $(\cdot,\cdot)_2$ denotes the usual inner product in the Hilbert space $L^2(\Omega)$.

The norm of $u \in W_0^{1,p}(\Omega)$, which is the usual Sobolev space, will be denoted by

$$|u|_{1,p} = \left(\int_\Omega |\nabla u(x)|^p dx\right)^{1/p}.$$

Here we only recall that if $p \geq 2$, then $W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'}(\Omega)$, with continuous (compact) and dense inclusions.

To rigorously define the p-Laplacean, we first consider the functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$, defined for any $u \in W_0^{1,p}(\Omega)$ by $\Phi(u) = \frac{1}{p}|u|_{1,p}^p$.

Concerning this functional, it is known that it is of class C^1 and that, for any u and v belonging to $W_0^{1,p}(\Omega)$, we have:

$$(\Phi'(u), v)_{W^{-1,p'}, W_0^{1,p}} = \int_\Omega |\nabla u(x)|^{p-2} \langle \nabla u(x), \nabla v(x) \rangle dx.$$

When $\Delta_p u \equiv \text{div}\,(|\nabla u|^{p-2} \nabla u) \in L^2(\Omega)$, we have

$$(\Phi'(u), v)_{W^{-1,p'}, W_0^{1,p}} = -\int_\Omega \Delta_p u(x) \cdot v(x) dx$$

Thus, from now on we will denote Φ' by $-\Delta_p$, and, for simplicity of notation, the duality map between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$ just by $(\cdot,\cdot)$.

It is also known that $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ has the following properties:

(i) *Strong Monotonicity:* there is $\alpha > 0$ such that for any $u, v \in W_0^{1,p}(\Omega)$ it holds

$$((-\Delta_p)u - (-\Delta_p)v, u - v) \geq \alpha |u - v|_{1,p}^p. \tag{2.2}$$

(ii) *Hemicontinuity:* for any $\lambda \in \mathbb{R}$ and any $u, v, w \in W_0^{1,p}(\Omega)$, the following function is continuous:

$$\lambda \to (-\Delta_p(u + \lambda w), v). \tag{2.3}$$

(iii) *Coercivity:* for any $u \in W_0^{1,p}(\Omega)$, it holds that

$$(-\Delta_p u, u) = |u|_{1,p}^p \tag{2.4}$$

(iv) *Boundness:* for any $u \in W_0^{1,p}(\Omega)$, it is true that

$$|-\Delta_p u|_{W^{-1,p'}} \leq |u|_{1,p}^{p-1} \tag{2.5}$$

However, the meaning of weak and strong solutions need to be clarified:

Definitions:
(i) Let $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $u_0 \in L^2(\Omega)$. A function u is a *weak solution* of

$$\begin{cases} u_t - \Delta_p u = f, \\ u|_{\partial\Omega} = 0, \\ u(0) = u_0, \end{cases} \tag{2.6}$$

when $u \in W = \{w \in L^p(0,T;W_0^{1,p}(\Omega)); w_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))\}$, u satisfies (2.6) in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $u(0) = u_0$ (which makes sense because $W \subset C([0,T];L^2(\Omega))$).
(ii) In the case where $f \in L^2(0,T;L^2(\Omega))$, u is called a *strong solution* of (2.6) if it is a weak solution , $u_t \in L^2(0,T;L^2(\Omega))$, and (2.6) is satisfied in $L^2(0,T;L^2(\Omega))$.

3 Existence Results for the Model Problem

The proof of existence of periodic solutions of problem (1.2) will be accomplished by firstly establishing the existence of a solution u_ε, for $\varepsilon > 0$ sufficiently small, of the approximate problem:

$$\begin{cases} (u_\varepsilon)_t - \Delta_p(u_\varepsilon) = F_\varepsilon(u_\varepsilon), \\ u_\varepsilon|_{\partial\Omega} = 0 \\ u_\varepsilon(0) = u_\varepsilon(T) \end{cases} \tag{3.1}$$

where

$$F_\varepsilon(u)(t,x) = \begin{cases} f_1(t,x,u(t,x)) & \text{if } u(t,x) < R\,, \\ f_2(t,x,u(t,x)) & \text{if } u(t,x) > R+\varepsilon\,, \\ \left[1 - \left(\dfrac{u(t,x)-R}{\varepsilon}\right)\right] f_1(t,x,u(t,x)) & \\ \qquad + \left(\dfrac{u(t,x)R}{\varepsilon}\right) f_2(t,x,u(t,x)) & \text{if } u(t,x) \in [R, R+\varepsilon] \end{cases} \tag{3.2}$$

and then by showing the existence of a sequence $\{u_{\varepsilon_i}\}_{i\in N}$, of solutions of (3.1)-(3.2), which by its turn converges to a solution of (1.2)-(1.3).

According to the value of $s = \max\{s_1, s_2\}$, as compared to the value of $p-1$, there are differences in the argument and conditions to be imposed, so the exposition will be split into three instances: the subcritical $(s < p-1)$, the critical $(s = p-1)$ and the supercritical $(s > p-1)$ cases.

Theorem 3.1 (Subcritical Case) *Let $2 \le p \le q < \infty$ satisfy $1 - N/p > -N/q$, and let the functions in (1.4) be such that for each $i = 1, 2$, $0 \le s_i < p-1$, $m_i \in L^\infty([0,T] \times \Omega)$ and $h_i \in L^{p'}(0,T; L^{q'}(\Omega))$, with p' and q' being the conjugates of p and q, respectively. Then problem (1.2)-(1.3) has a weak periodic solution.*

Proof: By our hypotheses on f_1, f_2, see(1.4), and by the definition of F_ε, it is easy to verify that

$$|F_\varepsilon(u)(t,x)| \le |m|_\infty |u(t,x)|^s + h(t,x) \tag{3.3}$$

with, $0 \le s = \max\{s_1, s_2\} < p-1$, $|m|_\infty = \max\{|m_1|_\infty, |m_2|_\infty\}$, with $|m_i|_\infty = |m_i|_{L^\infty([0,T]\times\Omega)}$, and $h(t,x) = \max_{i=1,2}\{h_i(t,x)\} + 1$.

Moreover, since $p \le q$, with the help of the Lebesgue Convergence Theorem, we conclude that

$$F_\varepsilon : L^p(0,T; L^q(\Omega)) \to L^{p'}(0,T; L^{q'}(\Omega)) \tag{3.4}$$

is a continuous map such that for any $\varepsilon \in (0,R)$ and $u \in L^p(0,T; L^q(\Omega))$ we find nonnegative constants $a = |m|_\infty$ and b such that

$$|F_\varepsilon(u)|_{L^{p'}L^{q'}} \le a|u|^s_{L^pL^q} + b. \tag{3.5}$$

Then it is possible to apply Theorem 4.1 (i) in Boldrini and Crema [2] and conclude that, for each $\varepsilon \in (0,R)$ there is a weak solution $u_\varepsilon \in W$ of (3.1)-(3.2). Here the space W is defined as

$$W = \left\{ w \in L^p(0,T; W_0^{1,p}(\Omega))\,;\, w_t \in L^{p'}(0,T; W^{1,p'}(\Omega)) \right\}.$$

The next step is to look for uniform estimates in the norm of W for the sequence $\{u_\varepsilon\}_{\varepsilon\in(0,R)}$. This will allow showing that there is a subsequence $\{u_{\varepsilon_i}\}_{i\in\mathbb{N}}$ which converges in some sense to a solution of (1.2)-(1.3). For that, we observe that uniform estimates for u_ε in $L^p(0,T; W_0{}^{1,p}(\Omega))$ are easily obtained using the boundness of F_ε, given by (3.5), and the usual energy estimates. With the help of these estimates, we can use the equation in (3.1) and (2.5) to conclude the uniform boundness of $(u_\varepsilon)_t$ in $L^{p'}(0,T;W^{-1,p'}(\Omega))$.

Therefore, $\{u_\varepsilon\}_{\varepsilon\in(0,R)}$ is a bounded set of W .

Since by Lemma 2.1, W is compactly embedded in $L^p(0,T;L^q(\Omega))$ for $1-N/p > -N/q$, we can extract a sequence $\{u_{\varepsilon_i}\}_{i\in\mathbb{N}}$ and find $u \in W \subset L^p(0,T;L^q(\Omega))$ such that when $i\to\infty$ then

$$u_{\varepsilon_i} \rightharpoonup u \quad \text{in} \quad W\,, \tag{3.6}$$

$$u_{\varepsilon_i} \to u \quad \text{in} \quad L^p(0,T,L^q(\Omega))\,. \tag{3.7}$$

Here $\rightharpoonup$ denotes weak convergence, while $\to$ denotes convergence in norm.

Moreover, $\{F_\varepsilon(u_\varepsilon)\}_{\varepsilon\in(0,R)}$ is bounded in $L^{p'}(0,T;L^{q'}(\Omega))$ and $\{\Delta_p u_\varepsilon\}_{\varepsilon\in(0,R)}$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega))$. Thus, there are $\xi \in L^{p'}(0,T;L^{q'}(\Omega))$ and $\eta \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ such that along a subsequence

$$-\Delta_p\, u_{\varepsilon_i} \rightharpoonup \eta \quad \text{in} \quad L^{p'}(0,T;W^{-1,p'}(\Omega))\,, \tag{3.8}$$

$$F_{\varepsilon_i}(u_{\varepsilon_i}) \rightharpoonup \xi \quad \text{in} \quad L^{p'}(0,T;L^{q'}(\Omega))\,. \tag{3.9}$$

Then, taking the weak limit in (3.1), we conclude that u satisfies

$$u_t + \eta = \xi.$$

Now let us verify that $u(0) = u(T)$. Note that if $v \in W_0^{1,p}(\Omega)$ and $\varphi \in C_0^\infty([0,T])$ we have $v\varphi \in L^p(0,T;W_0^{1,p}(\Omega))$. Let us take φ such that $\varphi(0) = \varphi(T)$; then, using (2.1) and the fact of $u_\varepsilon(0) = u_\varepsilon(T)$ we conclude that

$$\int_0^T \Big(\frac{d}{dt}u_\varepsilon(t), v\Big)\varphi(t)dt = -\int_0^T (u_\varepsilon(t), v)\varphi'(t)dt.$$

Passing to the limit in the last equality and using the density of $W_0^{1,p}(\Omega)$ in $L^2(\Omega)$ we conclude that $u(0) = u(T)$. Hence,

$$\begin{cases} u_t + \eta = \xi\,, \\ u(0) = u(T)\,. \end{cases} \tag{3.10}$$

Now we must show that $\eta = -\Delta_p u$ and $\xi(t,x) \in F(u(t,x))$ a.e. in $(0,T)\times\Omega$. To prove the first equality, let us fix an arbitrary element $v \in L^p(0,T;W_0^{1,p}(\Omega))$ and denote

$$S_i := \int_0^T (-\Delta_p u_{\varepsilon_i}(t) + \Delta_p v(t), u_{\varepsilon_i}(t) - v(t))dt$$

where $S_i \geq 0$ by (2.2).

Due to the fact that u_{ε_i} is solution of (3.1), by using (3.5), (3.6) and (3.7), we conclude that $S_i \to S$, as $i \to \infty$, where

$$S = \int_0^T (\eta(t) + \Delta_p v(t), u(t) - v(t))dt \geq 0 \ .$$

Replacing v with $u - \lambda\omega$ in S, for $\lambda > 0$ and $\omega \in L^p(0,T; W_0^{1,p}(\Omega))$ and after taking the limit when $\lambda \to 0$ we can use the hemicontinuity of Δ_p, (2.3), to conclude that for any ω

$$\int_0^T (\eta(t) + \Delta_p u(t), \omega(t)) \geq 0.$$

Hence $\eta = -\Delta_p u$ and, therefore,

$$\left\{ \begin{array}{l} u_t - \Delta_p(u) = \xi, \\ u(0) = u(T). \end{array} \right. \tag{3.11}$$

Finally let us show that for a.e. in $(0,T)\times\Omega$, we have $\xi(t,x) \in F(u)(t,x)$, which is given by (1.3).

By (3.7), we have in particular that $\{u_{\varepsilon_i}\}$ converges in $L^p((0,T)\times\Omega)$ for $p \leq q$. Thus, there is a new subsequence, still denoted by $\left\{u_{\varepsilon_i}\right\}$, such that

$$u_{\varepsilon_i}(t,x) \to u(t,x) \qquad \text{a.e.} \qquad (0,t)\times\Omega \ , \tag{3.12}$$

In particular this convergence occurs for any measurable subset of $(0,T)\times\Omega$. Then, let us consider the following measurable sets:

$$\begin{array}{rcl} I &=& \{(t,x) \in (0,T)\times\Omega\,;\; u(t,x) < R\} \\ J &=& \{(t,x) \in (0,T)\times\Omega\,;\; u(t,x) > R\} \\ K &=& \{(t,x) \in (0,T)\times\Omega\,;\; u(t,x) = R\} \ . \end{array}$$

Firstly we will show that for a.e. I we have that

$$F_{\varepsilon_i}(u_{\varepsilon_i})(t,x) \to f_1(t,x,u(t,x))\,. \tag{3.13}$$

This is easy to prove because if $(t,x) \in I$ then $u(t,x) < R$ and from (3.12), for i large enough, we also have $u_{\varepsilon_i}(t,x) < R$. Consequently, $F_{\varepsilon_i}(u_{\varepsilon_i})(t,x) =$

$f_1(t,x,u_{\varepsilon_i}(t,x))$. Thus (3.13) follows from the continuity of f_1 in its third variable.

Now using Lebesgue Convergence Theorem, with the help of (3.3), we conclude that

$$F_{\varepsilon_i}(u_{\varepsilon_i}) \to f_1(\cdot,\cdot,u) \quad \text{in} \quad L^{p'}(I).$$

However, $L^{p'}((0,T)\times\Omega) = L^{p'}(0,T;L^{p'}(\Omega)) \subset L^{p'}(0,T;L^{q'}(\Omega))$. Then, by (3.9), we obtain

$$\xi(t,x) = f_1(t,x,u(t,x)) \in F(u)(t,x) \ \text{ a.e. } \ \text{in } I\ .$$

Analogously we conclude that

$$\xi(t,x) = f_2(t,x,u(t,x)) \in F(u)(t,x) \ \text{ a.e. } \ \text{in } J\ .$$

Now, we will show that when $(t,x) \in K$, $\xi(t,x)$ is a pontual convex combination of $f_1(t,x,u(t,x))$ and $f_2(t,x;u(t,x))$.

Recall that, by continuity and boundness of f_1, f_2 and f_ε, we have that if

$$u_{\varepsilon_i}(t,x) \to u(t,x) \quad \text{ a.e. } \ \text{in} \quad K\ ,$$

then

$$\begin{aligned}
&F_{\varepsilon_i}(u_{\varepsilon_i}) \rightharpoonup \xi \quad \text{in} \ \ L^{p'}(0,T;L^{q'}(\Omega)) \text{ (in particular weakly in } L^{q'}(K)),\\
&f_k(\cdot,\cdot,u_{\varepsilon_i}) \to f_k(\cdot,\cdot,u) \quad \text{in} \ \ L^{p'}(0,T;L^{q'}(\Omega)),\ \text{for } k=1,2, \qquad (3.14)\\
&f_k(t,x,u_{\varepsilon_i}(t,x)) \to f_k(t,x,u(t,x)) \quad \text{a.e. in } \ (0,T)\times\Omega\,. \qquad (3.15)
\end{aligned}$$

By the weak convergence of $F_{\varepsilon_i}(u_{\varepsilon_i})$, we find a sequence of convex combinations of these elements, strongly converging to ξ, i.e., there are functions

$$\xi_i = \sum_{k=1}^{k_i} a_k^{(i)}\, F_{\varepsilon_k}(u_{\varepsilon_k}) \ \text{ with } \ \sum_{k=1}^{k_i} a_k^{(i)} = 1\,,\ 0 \le a_k^{(i)} \le 1 \qquad (3.16)$$

such that

$$\xi_i \to \xi \quad \text{strongly in} \quad L^{q'}(K) \qquad (3.17)$$

With the help of these facts, a special sequence will be constructed as follows.

Let us take $(t,x) \in K$ such that (3.15) is satisfied. Then, for $m = 1$, there is $N(1) > 0$ such that for $i \geq N(1)$ and $j = 1,2$, we have

$$|f_j(t,x,u_\varepsilon(t,x)) - f_j(t,x,u(t,x))| < 1 \ .$$

Now, as in (3.17), there is $k(1) \geq N(1)$ and $\xi_{i,1}(t,x)$ such that

$$\xi_{i,1}(t,x) = \sum_{k=N(1)}^{k(1)} a_k^{(1)} F_{\varepsilon_k}(u_{\varepsilon_k})(t,x)$$

with $a_k^{(1)}$ as in (3.16), and

$$|\xi_{i,1}(t,x) - \xi(t,x)| < 1 \ .$$

Now, for $m = 2$ there is $N(2) > k(1)$ such that for $i \geq N(2)$ and $j = 1,2$, we have

$$\left| f_j(t,x,u_{\varepsilon_i}(t,x)) - f_j(t,x,u(t,x)) \right| < \frac{1}{2} \ .$$

Again as in (3.17), there is $k(2) \geq N(2)$ and $\xi_{i,2}(t,x)$ such that

$$\xi_{i,2}(t,x) = \sum_{k=N(2)}^{k(2)} a_k^{(2)} F_{\varepsilon_k}(u_{\varepsilon_k})(t,x) \ ,$$

with $a_k^{(2)}$ as in (3.16), and

$$|\xi_{1,2}(t,x) - \xi(t,x)| < \frac{1}{2} \ .$$

Successively, given $m \in \mathbb{N}$, there is $N(m) > k(m-1)$ such that for $i \geq N(m)$ and $j = 1,2$ it holds that

$$\left| f_j(t,x,u_{\varepsilon_i}(t,x)) - f_j(t,x,u(t,x)) \right| < \frac{1}{m} \ . \tag{3.18}$$

Moreover, there is $k(m) \geq N(m)$ and $\xi_{i,m}(t,x)$ such that

$$\xi_{i,m}(t,x) = \sum_{k=N(m)}^{k(m)} a_k^{(m)} F_{\varepsilon_k}(u_{\varepsilon_k})(t,x) \ ,$$

with $a_k^{(m)}$ as in (3.16), and

$$|\xi_{i,m}(t,x) - \xi(t,x)| < \frac{1}{m} \ . \tag{3.19}$$

Now recall that by the definition of F_ε there is $0 \le \alpha_k(t,x) \le 1$ such that

$$\xi_{i,m}(t,x) = \sum_{k=N(m)}^{k(m)} a_k^{(m)} \left[\alpha_k(t,x) f_1(t,x,u_{\varepsilon_k}(t,x)) + (1-\alpha_k(t,x)) f_2(t,x,u_{\varepsilon_k}(t,x))\right],$$

which implies that $\xi_{i,m}(t,x)$ is also a convex combination of $f_1(t,x,u_{\varepsilon_k}(t,x))$ and $f_2(t,x,u_{\varepsilon_k}(t,x))$. Changing u_{ε_k} by u in $\xi_{i,m}(t,x)$ we can define

$$w_m(t,x) = \sum_{k=N(m)}^{k(m)} a_k^{(m)} \left[\alpha_k(t,x) f_1(t,x,u(t,x)) + (1-\alpha_k(t,x)) f_2(t,x,u(t,x))\right] .$$

Then it follows from (3.18) that

$$|w_m(t,x) - \xi_{i,m}(t,x)| < \frac{1}{m} \cdot \sum_{k=N(m)}^{k(m)} a_k^{(m)} = \frac{1}{m} \ .$$

From this and (3.19) we conclude that

$$w_m(t,x) \to \xi(t,x) \ , \qquad \text{a.e. in } \ K \ .$$

However,

$$w_m(t,x) \in \sum_{(t,x)} = \{\sigma\, f_1(t,x,u(t,x)) + (1-\sigma) f_2(t,x,u(t,x));\ \sigma \in [0,1]\} \, ,$$

which is a closed set. Therefore, there is $\sigma(t,x) \in [0,1]$ such that

$$\xi(t,x) = \sigma(t,x) f_1(t,x,u(t,x)) + (1-\sigma(t,x)) f_2(t,x,u(t,x)) \in F(u)(t,x),$$

for a.e. $(t,x) \in K$, which concludes the proof. $\square$

Now let us cosider the critical case $s = p-1$, for which we have the following result:

Theorem 3.2 (Critical Case) *Let $2 \le p \le q < \infty$ satisfy $1 - N/p > -N/q$, and let the functions in (1.4) be such that for each $i = 1, 2$, $0 \le \max\{s_1, s_2\} = p-1$, $m_i \in L^\infty([0,T] \times \Omega)$ and $h_i \in L^{p'}(0,T; L^{q'}(\Omega))$, with p' and q' being the conjugates of p and q, respectively. Then when $|m|_\infty = \max\{|m_1|_\infty, |m_2|_\infty\}$ is small enough, problem (1.2)-(1.3) has a weak periodic solution.*

Proof: It is enough to proceed as in the previous theorem, with the remark that constants $a = |m|_\infty = \max\{|m_1|_\infty, |m_2|_\infty\}$ and b in estimate (3.5) are independent of ε. Then, Theorem 4.1 (ii) in [2] guarantees that when $|m|_\infty$ is small enough, independently of the value of $\varepsilon \in (0, R)$, (3.1)-(3.2) has a solution u_ε, for each ε. The same theorem also furnishes that $\{u_\varepsilon\}$ is uniformly bounded in W.

The rest of the proof follows exactly the same arguments as in the proof of Theorem 3.1 and we conclude that problem (1.2)-(1.3) has a weak periodic solution. $\square$

Our next result will provide sufficient conditions for the existence of time periodic solutions of (1.2)-(1.3) when the growth-rate of one of f_i (given by s_i in (1.4)) is supercritical, that is, it is bigger than $p - 1$.

Before stating and proving it, it is necessary to recall the following result of [2].

Lemma 3.1 *Let $f \in L^{p'}(0,T; W^{-1,p'}(\Omega)$ and v be the solution of*

$$\begin{cases} v_t - \Delta_p v = f, \\ v|_{\partial\Omega} = 0, \\ v(0) = v(T). \end{cases}$$

Then $v \in L^\infty(0,T, L^2(\Omega))$ and satisfies, for some $c > 0$ independent of f,

$$|v|^2_{L^\infty L^2} \le c \left(|f|^{2/(p-1)}_{L^{p'} W^{-1,p'}} + |f|^{p'}_{L^{p'} W^{-1,p'}} \right) \tag{3.20}$$

We also have the following:

Lemma 3.2 *Problem (3.1)-(3.2) has a weak solution if $|f_i(t,x,\mu)| \le |m|_\infty |\mu|^{s_i} + |h(t,x)|$ with $h_i \in L^{p'}(0,T; L^{q'}(\Omega))$, $|m|_\infty$ small enough and either*

$$p - 1 \le s < p - 1 + 2p/N \quad \textit{for} \quad N > p\ , \tag{3.21}$$

or

$$p-1 \leq s < p+1 \quad for \quad N \leq p \; , \tag{3.22}$$

Proof: By the hypotheses on f_i we conclude that $|F_\varepsilon(u)(t,x)| \leq |m|_\infty |u|^s + |h(t,x)|$ where $s = max\{s_1, s_2\}$ and $|h(t,x)| = max|h_i(t,x)|$. Then by Theorem 4.1 of [2], the result follows. □

We briefly sketch here the proof of Theorem 4.1 of [2] because certain of its details and notations will be refered to in the arguments that follow.

Theorem 4.1 of [2] is proved by showing that there are numbers $k, r \geq 1$ and $0 < R = R(|m|_\infty \, , \, s \, , \, |h|_{L^{p'}W^{-1,p'}})$ such that the solution operator S defined by

$$\begin{array}{rcl} S : \overline{B_R(0)} \subset L^k(0,T;L^r(\Omega)) & \longrightarrow & \overline{B_R(0)} \subset L^k(0,T;L^r(\Omega)) \; , \\ v & \longmapsto & S(v) = u. \end{array}$$

where u is the solution of

$$\begin{cases} u_t - \Delta_p u = f(\cdot,\cdot,v) \\ u|_{\partial\Omega} = 0 \\ u(0) = u(T), \end{cases} \tag{3.23}$$

has a fixed point u with $|u|_{L^kL^r} \leq R$. Moreover k and r are obtained by interpolation of $L^p(0,T;L^q(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$, with $1 - N/p > -N/q$. They are such that:

$$\begin{array}{rcl} W \subset X & = & L^p(0,T;L^q(\Omega)) \cap L^\infty(0,T,L^2(\Omega)) \subset L^k(0,T,L^r(\Omega)) \\ & \subset & L^{p's}(0,T;L^{q's}(\Omega)) \end{array} \tag{3.24}$$

$$|u|_{L^kL^r} \leq |u|^{1-\theta}_{L^\infty L^2} \, |u|^{\theta}_{L^pL^q} \qquad \forall \, u \in X \tag{3.25}$$

$$|f(\cdot,\cdot,u)|_{L^{p'}L^{q'}} \leq c|m|_\infty \, |u|^s_{L^kL^r} + |h|_{L^{p'}W^{-1,p'}} \tag{3.26}$$

with $\theta \in (0,1)$ and $c > 0$.

Now it is possible to state the following result:

Theorem 3.3 *(*Supercritical Case*) If in (1.4) we have nonegative functions $h_i \in L^{p'}(0,T;L^{q'}(\Omega))$, $i = 1,2$, $|m|_\infty$ is small enough and $s = \max\{s_1, s_2\}$ satisfying either (3.21) or (3.22), then (1.2)-(1.3) has a weak solution.*

Proof: As in Theorem 3.1, we will construct a solution for (1.2), (1.3) with the help of the aproximate problem (3.1),(3.2). Remember that according to (3.3), F_ε has uniform estimates in $\varepsilon \in (0,R)$. Then by Lemma 3.2 we conclude that problem (3.1),(3.2) has a solution u_ε satisfying:

$$|u_\varepsilon|_{L^kL^r} \le R \qquad \forall\, \varepsilon \in (0,R)\,. \tag{3.27}$$

This fact and (3.26) give us

$$|F_\varepsilon(u_\varepsilon)|_{L^{p'}L^{q'}} \le c|m|_\infty R^s + |h|_{L^{p'}L^{q'}} = \overline{R} \qquad \forall\, \varepsilon \in (0,R)\,. \tag{3.28}$$

Then, using energy estimates and recalling the inclusions $W_0^{1,p}(\Omega) \subset L^q(\Omega) \subset L^2(\Omega) \subset L^{q'}(\Omega) \subset W^{-1,p'}(\Omega)$, we conclude, as in Theorem 3.1 that u_ε is bounded in W.

Still following the proof of Theorem 3.1 we conclude that there is a subsequence $\left\{u_{\varepsilon_i}\right\}_{i\in\mathbb{N}}$, $u \in W$ and $\xi \in L^{p'}(0,T;L^{q'}(\Omega))$ such that $u_{\varepsilon_i} \to u$, in the sense of (3.6) and (3.7). Moreover, u satisfies:

$$\begin{cases} u_t - \Delta_p u = \xi\ , \\ u|_{\partial\Omega} = 0 \\ u(0) = u(T)\,. \end{cases}$$

However, to conclude that $\xi(t,x) \in F(u)(t,x)$ a.e. in $(0,T)\times\Omega$, we need to show that $u_{\varepsilon_i} \to u$ in $L^k(0,T,L^r(\Omega))$ and then follow the proof of Theorem 3.1 .

To obtain this convergence observe that by using (3.24) and (3.25), u and u_{ε_i} belong to $W \cap L^\infty(0,T,L^2(\Omega)) \subset L^k(0,T,L^r(\Omega))$ and there is $\theta \in (0,1)$ such that

$$|u_{\varepsilon_i} - u|_{L^kL^r} \le |u_{\varepsilon_i} - u|_{L^\infty L^2}^{1-\theta}\, |u_{\varepsilon_i} - u|_{L^pL^q}^{\theta}\ . \tag{3.29}$$

Thus, we only need to show that $\left\{|u_{\varepsilon_i} - u|_{L^\infty L^2}\right\}_{i\in\mathbb{N}}$ is bounded.

However, according to Lemma 3.1 and (3.28) we have the estimate

$$\begin{aligned} \left|u_{\varepsilon_i}\right|^2_{L^\infty L^2} &\le c\left(\left|F_{\varepsilon_i}(u_{\varepsilon_i})\right|^{2/(p-1)}_{L^{p'}W^{-1,p'}} + \left|F_{\varepsilon_i}(u_{\varepsilon_i})\right|^{p'}_{L^{p'}W^{-1,p'}}\right) \\ &\le \tilde{c}\left(\left|F_{\varepsilon_i}(u_{\varepsilon_i})\right|^{2/(p-1)}_{L^{p'}L^{q'}} + \left|F_{\varepsilon_i}(u_{\varepsilon_i})\right|^{p'}_{L^{p'}L^{q'}}\right) \\ &\le \tilde{c}\left(\overline{R}^{2/(p-1)} + \overline{R}^{p'}\right)\ , \quad \text{for } \ i \in \mathbb{N}\ , \end{aligned}$$

which gives us the boundness of $(u_{\varepsilon_i} - u)$ in $L^\infty(0,T,L^2(\Omega))$. Thus following the proof of Theorem 3.1, from (3.17) on, we conclude the required result. $\square$

Remark:(Strong Solutions) If we take $h_i \in L^2(0,T;L^2(\Omega))$ and either $0 \leq s = max\{s_1, s_2\} \leq p/2$ or $|m|_\infty$ small enough and $p/2 \leq s \leq q/2$, with q so that $1 - N/p > -N/q$, then strong solutions of (1.2)-(1.3) are obtained.

To prove this result, firstly we use Theorem 6.1 of [2] to show the existence of a bounded sequence of solutions u_ε of (3.1)-(3.2) such that $u_\varepsilon \in \tilde{W} = \{w \in L^p(0,T;W_0^{1,p}(\Omega));\ w_t \in L^2(0,T;L^2(\Omega))\}$. We then follow the proof of the previous theorem to conclude the required result.

4 An Abstract Result

Following the ideas in the proofs of the theorems of the previous section, it is easy to obtain the following abstract result.

Theorem 4.1 *Let V and X be reflexive Banach spaces, with V' and X' being their respective topological duals, and H being a Hilbert space such that $V \subset X \subset H \subset X' \subset V'$, with continuous and dense inclusions and $V \subset X$ compactly. Let also $A : V \to V'$ be a monotone and hemicontinuous operator for which there are constants $\alpha_1, \alpha_2 > 0$, $\beta, \gamma_1, \gamma_2 \geq 0$ and $p \geq 2$ such that for any $u, v \in V$ the following holds:*

$$(Au - Av\, ,\, u - v)_{V',V} \geq \alpha_1 |u - v|_V^p\ ,$$

$$(Au\, ,\, u)_{V',V} \geq \alpha_2 |u|_V^p - \beta\ ,$$

$$|A(u)|_{V'} \geq \gamma_1 |u|_V^{p-1} + \gamma_2\ .$$

Also consider a set-valued operator $F(t,\cdot) : X \to X'$, with $t \in (0,T)$ defined by

$$F(t,u) = \begin{cases} \{F_1(t,u)\} & \text{if } \ |u|_X < R, \\ \{F_2(t,u)\} & \text{if } \ |u|_X > R, \\ \{\sigma F_1(t,u) + (1 - \sigma F_2(t,u);\ \sigma \in [0,1]\} & \text{if } \ |u|_X = R, \end{cases} \tag{4.1}$$

where for each $j = 1, 2$, $F_j(t,\cdot) : X \to X'$ is a continuous and bounded operator satisfying, for some positive functions $m \in L^\infty(0,T)$ and $h_j \in L^{p'}(0,T)$ and any $u \in X$, the following

$$|F_j(t,u)|_{X'} \leq m(t)|u|_X^{s_j} + h_j(t)\ .$$

Then, if either $0 \leq s = \max\{s_1, s_2\} < p-1$ or $s = p-1$ and $|m|_\infty$ is small enough, there is $u \in \{\omega \in L^p(0,T;V);\ \omega_t \in L^{p'}(0,T;V')\}$ and $\xi \in L^{p'}(0,T;X')$ such that

$$\begin{cases} u_t + Au = \xi, \\ u(0) = u(T), \end{cases} \tag{4.2}$$

where

$$\xi(t) \in F(t, u(t)) \quad \text{a.e} \quad \text{in } (0,T).$$

(Here A is naturally seeing as an operator from $L^p(0,T;V)$ to $L^{p'}(0,T;V')$)

Remarks: Observe that in this last theorem the abstract result corresponding to the supercritical case of the previous section was not stated . The reason for this is that in Theorem 3.3 either conditions on $s = \max\{s_1, s_2\}$ are directly related to Sobolev functional spaces on a set Ω; more specifically, to the dimension N where $\Omega \subset \mathbb{R}^N$. If we allow the spaces be concrete Sobolev spaces, in Theorem 4.1 for instance, $V = W_0^{1,p}(\Omega)$, $H = L^2(\Omega)$ and $X = L^r(\Omega)$ (with r as in (3.25)), s satisfies either (3.21) or (3.22), and $|m|_\infty$ is small enough, then (4.2) has a weak periodic solution for any operator A as described above. That is, for such concrete Sobolev spaces a partially abstract version of Theorem 3.3 is true .

In addition to $-\Delta_p$ with Dirichlet boundary condition the following operators are examples to which the results in Theorem 4.1 (and in the previous remark) apply.

Example 1: $Au = -\mathrm{div}\,(a(|\nabla u|^p)\,|\nabla u|^{p-2}\,\nabla u)$ with $D(A) = W_0^{1,p}(\Omega)$, $p \geq 2$, $a \in C^1(\mathbb{R})$, $a' \geq 0$ and $\delta\,\sigma^{p-1} \leq a(\sigma^p)\sigma^{p-1} \leq \alpha\,\sigma^{p-1} + \beta$ where α, β and δ are positive constants, $\beta \geq 2$.
Example 2: $Au = -\lambda\Delta_p u + \lambda_2\Delta_{p_2} u - \lambda_1\Delta_{p_1} u$ with $DA = W_0^{1,p}(\Omega)$, $p > p_2 > p_1 \geq 2$, $p = 2p_2 - p_1$, $\lambda, \lambda_1, \lambda_2 > 0$, $\lambda_2^2 \leq 4\,\lambda\,\lambda_1$ and $[\lambda_2(p_2-1)]^2 - 4\,\lambda\,\lambda_1(p-1)(p_1-1) \leq 0$.
Example 3: $Au = \sum_{i=1}^{N} \dfrac{\partial}{\partial x_i}\left(\left|\dfrac{\partial u}{\partial x_i}\right|^{p-2}\dfrac{\partial u}{\partial x_i}\right)$ with $D(A) = W_0^{1,p}(\Omega)$, $p \geq 2$.

Example 4: $Bu = Au + \lambda|u|^{p-2}u$, $\lambda > 0$ with A any of the operator described above and $D(B) = \left\{u \in W^{1,p}(\Omega); \dfrac{\partial u}{\partial \eta} = 0\right\}$.

References

[1] E.P. Avgerinos, N.S. Papageorgiou, On the Existence of Extremal Periodic Solutions for Nonlinear Parabolic Problems with Discontinuities. Yokohama Math. J. 45 (1998).no.1,39-60.

[2] J.L. Boldrini, J. Crema, On Forced Periodic Solutions of Superlinear Problems, Eletronic J.D.E.,vol.1998 (1998),no.14,pp.1-18; URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu; ftp (login:ftp) 147.26.103.110 or 129.120.3.113 .

[3] H. Brezis, Operateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert, Notes of Mathematics 50, North Holland Mathematics Studies, 1973.

[4] D.A.Kandilakis, N.S.Papageorgiou, Nonlinear Periodic Parabolic Problems with Nonmonotone Discontinuities, Proceedings of Edinburg Mathematical Society(1998), 41, 117-132.

[5] E. Kawohl, R. Rühl, Periodic Solutions of Nonlinear Heat Equations under Descontinuous Boundary Conditions, Lecture Notes in Math., 1017, Springer Verlag, 1990.

[6] P. LeTallec, Numerical Analysis of ViscoelasticProblems, RMA 15, Masson, Springer Verlag, 1990.

[7] J.L. Lions, Quelques Méthodes de Resolution de Problèmes aux Limites Non Lineaires, Dunot Gauthier-Villars, Paris, 1969.

[8] M. Ôtani, Nonmonotone Perturbations for Nonlinear Parabolic Equations Associated with Subdifferential Operators, Periodic Problems, J.D.E. 54,(1984), 248-273.

[9] W.A. Strauss, The Energy Method in Nonlinear Partial Differential Equations, Notas de Matemática No. 47, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, Brazil, 1969.

[10] Y. Yamada, Periodic Solutions of certain Nonlinear Parabolic Equations in Domains with Periodically Moving Boundaries, Nagoya Math. J. 70 (1978), 111-123.